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Treatise on Algebra

VOLUME II

ON SYMBOLICAL ALGEBRA, AND ITS APPLICATIONS TO THE GEOMETRY OF POSITION

GEORGE PEACOCK

A TREATISE ON ALGEBRA

Volume II: On Symbolical Algebra, and Its Applications to the Geometry of Position

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A TREATISE ON ALGEBRA

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ERRATA.

Page	Line	From	Error	Correction
87	9	bottom	$a+\frac{(1-\sqrt{5})\,a}{2}$	$x=a+\frac{(1-\sqrt{5})\,a}{2}$.
—	10	bottom	$a+\frac{(1+\sqrt{5})\,a}{2}$	$x=a+\frac{(1+\sqrt{5})\,a}{2}$.
108	15, 17	top	$(-1)^{\frac{1}{3}}$	$(1)^{\frac{1}{3}}$.
197	11	bottom	$\cos\theta$	$\cos-\theta$.
206	7	top	$-2\sqrt[6]{(a^2+b^2)}\cos\left(\frac{2\pi-\theta}{3}\right)$	$2\sqrt[6]{(a^2+b^2)}\cos\left(\frac{2\pi-\theta}{3}\right)$.
—	8	bottom	$-2\sqrt{\frac{q}{3}}\cos\left(\frac{2\pi-\theta}{3}\right)$	$2\sqrt{\frac{q}{3}}\cos\left(\frac{2\pi-\theta}{3}\right)$.
289	3	bottom	$\log\left(1-\frac{b}{a}e^{C\sqrt{-1}}\right)$	$\log\left(1-\frac{b}{a}e^{-C\sqrt{-1}}\right)$.
344	4	top	$2\cos\frac{2\pi-\theta}{2}$	$2\cos\left(\frac{2\pi-\theta}{3}\right)$.
357	11	bottom	$=\frac{R}{2}$	$=r$.

It is feared that very few of the errors have been noticed, no sufficient time having been devoted to the revision of the work.

PREFACE.

I HAVE endeavoured, in the present volume, to present the principles and applications of Symbolical, in immediate sequence to those of Arithmetical, Algebra, and at the same time to preserve that strict logical order and simplicity of form and statement which is essential to an elementary work. This is a task of no ordinary difficulty, more particularly when the great generality of the language of Symbolical Algebra and the wide range of its applications are considered; and this difficulty has not been a little increased, in the present instance, by the wide departure of my own views of its principles from those which have been commonly entertained.

It is true that the same views of the relations of the principles of Arithmetical and Symbolical Algebra formed the basis of my first publication on Algebra in 1830: but not only was the nature of the dependance of Symbolical upon Arithmetical Algebra very imperfectly developed in that work, but no sufficient attempt was made to reduce its principles and their applications to a complete and regular system, all whose parts were connected with each other: they have consequently been sometimes controverted upon grounds more or less erroneous; and notwithstanding a very general acknowledgment of their theoretical authority, they have hitherto exercised very little influence upon the views of elementary writers on Algebra.

It may likewise be very reasonably contended that the reduction of such principles, as those which I have ventured to put forward, to an elementary form, in which they may be fully understood by an ordinary student, is the only practical and decisive test, I will not say of their correctness, but of their value: for we are very apt to conclude that the most difficult theories and researches which have become familiar to us from long study and contemplation, may be made equally clear and intelligible to others as well as to ourselves: and though I will not say that I feel perfectly secure that I may not have been, in some degree, under the influence of this very common source of self-deception and error, yet I have adopted the only course which was open to me, in order to bring this question to an issue, by embodying my own views in an elementary work, and by suppressing as much as possible any original or other researches, which might be considered likely to interfere with its complete and systematic developement.

It is from the relations of space that Symbolical Algebra derives its largest range of interpretations, as well as the chief sources of its power of dealing with those branches of science and natural philosophy which are essentially connected with them: it is for this reason that I have endeavoured to associate Algebra with Geometry thoughout the whole course of its developement, beginning with the geometrical interpretation of the signs + and − when considered with reference to each other, and advancing to that of the various other signs which are symbolized by the roots of 1: we are thus enabled, in the present volume, to bring the Geometry of Position, embracing the whole theory of lines considered both in relative position and magnitude and the properties of rectilineal figures, under the dominion of Algebra: in a subsequent volume this application will be further extended to the Geometry of Situation, (where lines are considered in their absolute as well as in their relative position with respect to each other), and also to the theory of curves.

The theory of the roots of 1 is so important, not merely with reference to the signs of affection which they symbolize, but likewise in the exposition of the general theory of equations and in all the higher branches of Symbolical Algebra, that I have thought it expedient to give it with unusual fulness and detail: such roots may be considered as forming the connecting link between Arithmetical and Symbolical Algebra, without whose aid the two sciences could be very imperfectly separated from each other.

I have not entered further into the general theory of equations than was necessary to enable me to exhibit the theory of their general solution, as far it can be carried by existing methods, reserving the more complete exposition of their properties and of the methods employed for their numerical solution, to a subsequent volume.

The plan which I have adopted necessarily brings Trigonometry, or to speak more properly, Goniometry, within the compass of the present volume, not merely as forming the most essential element in the application of Algebra to the Geometry of Position, but as intimately connected with many important analytical theories.

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CHAPTER XI.

ON THE OPERATIONS OF ADDITION AND SUBTRACTION IN SYMBOLICAL ALGEBRA.

543. THE symbols in Arithmetical Algebra represent numbers, whether abstract or concrete, whole or fractional, and the operations to which they are subject are assumed to be identical in meaning and extent with the operations of the same name in common Arithmetic: the only distinction between the two sciences consisting in the substitution of general symbols for digital numbers. Distinction between Arithmetic and Arithmetical Algebra.

Thus, if a be added to b, as in the expression $a+b$, it is assumed that a and b are either abstract numbers or concrete numbers of the same kind: if b be subtracted from a, as in the expression $a-b$, it is assumed that a is greater than b, which implies likewise that they are numbers of the same kind: if a be multiplied by b, as in the expression ab (Art. 34), or if a be divided by b, as in the expression $\frac{a}{b}$ (Art. 71), it is assumed that b is an abstract number. In all these cases, the operation required to be performed, whether it be addition or subtraction, multiplication or division, is clearly defined and understood, and the result which is obtained, is a necessary consequence of the definition: the same observation applies to all the results of Arithmetical Algebra.

544. But the symbols, which are thus employed, do not convey, either to the eye or to the mind, in the same manner as digital numbers and geometrical lines, the limitations of value to which they are subject in Arithmetical Algebra: for they are equally competent to represent quantities of all kinds, and of all relations of magnitude. But if we venture to ascribe to them a perfect generality of value, (upon which a conventional limitation was imposed in Arithmetical Algebra), it will be found to involve, as an immediate and necessary consequence, the The assumption of the unlimited values of the symbols employed involves the necessary recognition of the independent use of the signs + and −.

recognition of the use of symbols preceded by the signs + and −, without any direct reference to their connection with other symbols.

Thus, in the expression $a - b$, if we are authorized to assume a to be either *greater* or *less* than b, we may replace a by the equivalent expression $b + c$ in one case, and by $b - c$ in the other: in the first case, we get $a - b = b + c - b = b - b + c$ (Art. 22), $= 0 + c$ (Art. 16) $= + c = c$: and in the second $a - b = b - c - b$ $= b - b - c = 0 - c = - c$. The first result is recognized in Arithmetical Algebra (Art. 23): *but there is no result in Arithmetical Algebra which corresponds to the second:* inasmuch as it is assumed that no operation can be performed and therefore no result can be obtained, when a is less than b, in the expression $a - b$ (Art. 13).*

Positive and negative quantities.

545. Symbols, preceded by the signs + or −, without any connection with other symbols, are called *positive* and *negative* (Art. 32) symbols, or *positive* and *negative* quantities: such symbols are also said to be *affected* with the signs + and −. *Positive* symbols and the numbers which they represent, form the subjects of the operations both of Arithmetical and Symbolical Algebra: but *negative* symbols, whatever be the nature of the quantities which the *unaffected* symbols represent, belong exclusively to the province of Symbolical Algebra.

Assumptions made in symbolical addition and subtraction.

546. The following are the assumptions, upon which the rules of operation in Symbolical Addition and Subtraction are founded.

1st. Symbols, which are general in form, are equally general in representation and value.

2nd. The rules of the operations of addition and subtraction in Arithmetical Algebra, when applied to symbols which are general in form though restricted in value, are applied, without alteration, in Symbolical Algebra, where the symbols are general in their value as well as in their form.

It will follow from this second assumption, as will be afterwards more fully shewn, that all the results of the operations of addition and subtraction in Arithmetical Algebra, will be results likewise of Symbolical Algebra, but not conversely.

* If we assume symbols to be capable of all values, from *zero* upwards, we may likewise include *zero* in their number: upon this assumption, the expressions $a + b$ and $a - b$ will become $0 + b$ and $0 - b$, or $+ b$ and $- b$, or b and $- b$ respectively, when a becomes equal to *zero*: this is another mode of deriving the conclusion in the text.

547. Proceeding upon the assumptions made in the last Article, the rule for Symbolical Addition may be stated as follows: Rule for symbolical addition.

"Write all the *addends* or *summands* (Art. 24, Ex. 1.) in the same line, preceded by their proper signs, collecting *like* terms (Art. 28) into one (Art. 29): and arrange the terms of the result or *sum* in any order, whether alphabetical or not, which may be considered most symmetrical or most convenient."

It will be understood that *negative* (Art. 545) as well as *positive* symbols or expressions may be the subjects of this operation, and it is therefore not necessary, as in Arithmetical Algebra, that the first term of the final result should be positive (Art. 22). Negative terms may occupy the first place in the results of Symbolical Algebra.

548. The following are examples of Symbolical Addition. Examples.

(1) Add together $3a$ and $5a$.

$$3a + 5a = 8a \quad \text{(Art. 29)}.$$

(2) Add together $3a$ and $-5a$.

$$3a - 5a = -2a \quad \text{(Art. 31)}.$$

This is exclusively a result of Symbolical Algebra.

(3) Add together $-3a$ and $5a$.

$$-3a + 5a = 2a.$$

This result, which is obtained by the Rule, is equivalent to that which would arise from the subtraction of $3a$ from $5a$: or, in other words, the addition of $-3a$ to $5a$ in Symbolical Algebra, is equivalent to the subtraction of $3a$ from $5a$ in Arithmetical Algebra.

(4) Add together $-3a$ and $-5a$.

$$-3a - 5a = -8a \quad \text{(Art. 31)}.$$

This is exclusively a result of Symbolical Algebra: in contrasting it however with Ex. 1, it merely differs from it in the use of the sign $-$ throughout, instead of the sign $+$.

(5)		(6)		(7)	
	$3a$		$3x^2$		$-abc$
	$-5a$		$-x^2$		$12abc$
	$+7a$		$-7x^2$		$13abc$
	$-4a$		$-4x^2$		$-20abc$
	a		$-9x^2$		$4abc$

In these examples, the coefficients of the like terms, which have the same sign, are added together, and to the difference of the sums, preceded by the sign of the greater, is subjoined the symbolical part of the several like terms: it is the rule given in Art. 31, applied without any reference to the signs of the first term.

(8)
$$\begin{array}{r} 3a - 4b \\ -7a + 8b \\ a - 3b \\ \hline -3a + b \end{array}$$

(9)
$$\begin{array}{r} -7x^2 + 6xy - 7y^2 \\ 8x^2 - 4xy - y^2 \\ -3x^2 - xy + 10y^2 \\ \hline -2x^2 + xy + 2y^2 \end{array}$$

In these examples, the sets of like terms are severally combined into one (Art. 31), and arranged, in the result, in alphabetical order, no regard being paid to the placing a positive term, when any exists, in the first place.

(10)
$$\begin{array}{l} -3a - 4b + 5c \\ -\ a + 2b - 3d \\ 3b - 4c + 6e \\ 7c - 8d - 9e \\ \hline -4a + b + 8c - 11d - 3e \end{array}$$

The several sets of like terms are collected together out of the several addends.

(11)
$$\begin{array}{r} 7x^2 - 4ax + a^2 \\ -10x^2 - 11ax - 4a^2 \\ -3x^2 + 13ax + 10a^2 \\ \hline -6x^2 - 2ax + 7a^2 \end{array}$$

The alphabetical order of the symbols is, in this case, reversed. It should be kept in mind in this and in all other cases that the arrangement of the terms in the final result, does not affect its value or signification, but is merely adopted as an aid to the eye or to the memory, or with reference to peculiar circumstances connected with some one or more of the symbols involved: see the Examples in Art. 33.

Rule for Subtraction in Symbolical Algebra.

549. The rule for subtraction in Symbolical Algebra is derived, by virtue of the assumptions in Art. 546, from the corresponding rule in Arithmetical Algebra: it may be stated as follows.

"To the minuend or minuends, add (Art. 547) the several terms of the subtrahend or subtrahends with their signs changed from + to − and from − to +."

The following are examples of Symbolical subtraction. Examples.

(1) From a subtract $-b$.

The result is $a+b$: or the symbolical *difference* of a and $-b$ is equivalent to the *sum* of a and b.

$$\begin{array}{rr@{\qquad}rr@{\qquad}rr@{\qquad}rr}
(2) & 7a & (3) & -7a & (4) & 7a & (5) & -7a \\
 & 3a & & -3a & & -3a & & 3a \\ \hline
 & 4a & & -4a & & 10a & & -10a
\end{array}$$

In these examples, the minuend and subtrahend are written underneath each other, as in common Arithmetic: the results are severally the same as in the following examples of addition. (Art. 547)

$$\begin{array}{r@{\qquad}r@{\qquad}r@{\qquad}r}
7a & -7a & 7a & -7a \\
-3a & 3a & 3a & -3a \\ \hline
 & & &
\end{array}$$

$$\begin{array}{rr@{\qquad}rr@{\qquad}rr@{\qquad}rr}
(6) & a+b & (7) & a-b & (8) & a+b & (9) & -a-b \\
 & a-b & & a+b & & -a+b & & a-b \\ \hline
 & 2b & & -2b & & 2a & & -2a
\end{array}$$

These examples are respectively equivalent to

(6) $a+b-(a-b)$, or $a+b-a+b=2b$.

(7) $a-b-(a+b)$, or $a-b-a-b=-2b$.

(8) $a+b-(-a+b)$, or $a+b+a-b=2a$.

(9) $-a-b-(a-b)$, or $-a-b-a+b=-2a$.

The terms of the several subtrahends are included between brackets, and, when the brackets are removed, all their signs are changed (Art. 24).

$$\begin{array}{rc}
(10) & a^3+3a^2x+3ax^2+x^3 \\
 & a^3-3a^2x+3ax^2-x^3 \\ \hline
 & 6a^2x+2x^3
\end{array}$$

$$\begin{array}{rl}
(11) & 3a-\ \ 4b+\ \ 7c-\ \ 9d \\
 & 2b-10c-\ \ 6d+14e \\ \hline
 & 3a-\ \ 6b+17c-\ \ 3d-14e
\end{array}$$