LINEAR PROGRAMMING
Basic Theory and Applications

Leonard W. Swanson

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PREFACE

This text has been written primarily for those college students who have no previous background in linear programming and are planning careers as administrators or managers in business, nonprofit organizations, or government. The text can be used in its entirety in either a semester or two-quarter course at the advanced undergraduate or early graduate level in business administration, economics, or engineering. A sound background in college algebra is necessary; any further mathematics that is needed is provided within the text.

OBJECTIVES

The principal objectives are

- ★ To introduce the important ideas in linear programming at a fundamental level
- ★ To assist students in understanding a wide range of linear programming models
- ★ To motivate students to continue their study of linear programming applications
- ★ To provide students with the fundamentals of computer use to solve linear programming problems
- ★ To assist the student in the formulation, solution, and interpretation of linear programming models

COVERAGE AND EMPHASIS

A course in linear programming can be given at various levels, ranging from the cookbook variety, with little understanding of the theory, to the highly theoretical, with little understanding of the practical aspects. This text attempts to follow neither of these extremes, but to include the best features of each.

I am a firm believer in using the computer to solve linear programming problems. Students must therefore become quite familiar with particular linear programming codes. I have found that early introduction of computer methods enhances the overall success of the course. My philosophy is to present the ideas in matrix notation, to give a fuller understanding of what actually happens in the process of obtaining a solution to a problem.

Throughout the text I have attempted to use as small a model as possible to illustrate the principles being covered while solving these problems by manual methods. Consequently, it is difficult to approach real-world problems until you get to the chapters on modeling and computer solutions. Some of the early chapters do call for the setting up of problems without actually carrying out the solutions.

It is important to cover the topics in sequence rather than to skip around in the text. Where readers are knowledgeable on certain topics, they should still cover enough of them to maintain the thread of ideas that weaves through the text.

The book is organized in the following fashion. Chapter 1 introduces the linear programming concept by presenting a number of problems. These problems are solved by the graphical method in order to produce a mental image of the solution process together with difficulties introduced whenever more than two variables appear in the original statement of the problem.

Chapter 2 introduces a specific maximization problem with all equal-to-or-less-than constraints and begins their solution by means of some basic algebra. The solution progresses until the algebra becomes burdensome and gives some indication that a good systematic way of handling the problem is desirable. (The same example is used later as a vehicle to present other ideas.) It is then stated that matrix algebra provides the basis for this system, and the remainder of the chapter is devoted to basic concepts of matrix algebra. The principles developed lead to the solution of linear systems of equations by matrix methods without the burden of dealing with situations in which the matrix of coefficients is of less than full rank. The concepts of linear independence of vectors, basis set of vectors, and spanning set of vectors are developed. These are then applied to systems of linear equations in order to discuss basic solutions to linear systems in terms of basic and nonbasic variables. The material concentrates on the idea that one should be able to recognize independent sets of vectors and basic solutions to linear systems of equations without placing emphasis on either redundant or inconsistent systems, since these offer no great difficulty in the solution of linear programming problems.

Chapter 3 returns to the sample problem started at the beginning of Chapter 2 and uses matrix algebra developed earlier to derive the simplex solution. Detailed computations are given to make clear such concepts as slack variables, the test for optimality, determination of entering and leaving variables, transformations to shift from one basic solution to another, the inverse of a basis set of vectors, and the use of the objective function as just another constraint. Thorough understanding of this chapter should enable one to solve in detail any linear programming problem whose solution involves maximization with equal-to-or-less-than constraints.

Chapter 4 extends the ideas of Chapter 3 to include problems with equal-to or greater-than-and-equal-to constraints by introducing negative slack or surplus and artificial variables when appropriate. In addition, minimization problems are handled by changing such problems into maximization problems, solving, and returning the solution to results applicable to the original problem. In order to handle the artificial variables, both the big-M method and the two-phase method are described and analyzed, with detailed tableaux given for each sample problem. At this point, such concepts as redundancy, inconsistency, and unbounded solutions are discussed, but not in theoretical detail.

Chapter 5 deals with sensitivity analysis resulting from changes in the **b** vector, changes in the **c** vector, addition of a new activity level, changes in the coefficients of the constraints, addition of a new constraint, and the removal of a constraint—all applied to an already optimal solution. In order to study these concepts, the principle of duality is first presented and three very basic theorems are proved using matrix algebra. The results of these theorems, together with the matrix algebra previously developed, are then used for the sensitivity analysis. Much computational detail is shown for the sample problems of Chapters 3 and 4, to make the concept of duality clear.

Chapter 6 develops the basic concepts of the revised simplex method, since this is so important for many computational methods, in particular for problems handled by the decomposition principle. Details are given for examples used previously.

Chapter 7 discusses and develops the method of solution of the transportation model. The principle of duality is used to develop the algorithm.

Chapter 8 discusses use of the computer for the solution of linear programming problems. It is discussed with reference to a particular program, MPOS, developed at Northwestern University. Even though the discussion gives information pertaining to MPOS, it is sufficiently general that one should be able to apply the methods to any linear programming code that might be available. Many details are given for the interpretation of the output, and it is related particularly to the material on sensitivity analysis given in Chapter 5.

Chapter 9 emphasizes the importance of proper modeling of a linear programming problem to an in-depth understanding of its solution. Since the principles involved have an element of heuristics, the material in this chapter has not been stated in the form of an algorithm. Two very enlightening examples are discussed in great detail in order to emphasize modeling.

The text can be used for courses of varying length. Some of the material could be omitted if the course is no more than one quarter in length. The first five chapters are essential for a course of any length, and the remaining chapters add considerable depth of understanding. Chapter 6 could be omitted if no further work in linear programming is contemplated. Chapter 7 is particularly important for readers who are planning to study production or operations management. Chapters 8 and 9 add conceptionally important material to the first five chapters. In any case, the first five chapters should be studied as a unit, and after that, any of the next four chapters can be studied independently in any order.

DEVELOPMENT OF THE TEXT

The material as presented has been used in draft form in several classes taught by the author during the last three years. It has gone through several revisions during that time and has been very well received by students. Although matrix algebra concepts are new to a great majority of students studying linear programming for the first time, the concepts have been understood almost without exception. I have found unusual success in teaching linear programming through the use of matrix algebra and strongly feel that it is the only way in which it should be done if one is to get any depth of understanding.

ACKNOWLEDGMENTS

My thanks are due to many of my colleagues at Northwestern University for their valuable comments during our many discussions concerning the way in which I have taught the material on linear programming.

Particular thanks are due Professor Martin Starr of Columbia University for his critical review of the manuscript. He was able to offer many suggestions that have been incorporated without changing my basic philosophy.

To our daughter, Sue, I express thanks for her suggestions for getting the material across to her generation.

To my wife, Winifred, I express particular thanks and appreciation for encouraging me and for allowing me to use so much of the time that was rightfully hers. She has experienced many lonesome moments while I was laboring over many details, torn between getting the text into print and enjoying her company.

Any errors that exist are solely mine. I trust that there will be few.

Leonard W. Swanson

CONTENTS

CHAPTER

ONE

LINEAR PROGRAMMING ORIGIN AND DEFINITION

1-1 ORIGIN

The methods of linear programming are essentially new, having been developed during the last 30 years. Although many references can be made to work along this general line, the initial work was developed by George Dantzig in about 1947. George Dantzig was a member of an Air Force Group that developed the simplex method of solution, but his work was not generally available until 1951. Even though the theory had been devised before 1950, very little use was made of it until the computer came into being, since the computer is so essential to the solution of problems of even moderate size.

The basic problem of linear programming is one of either maximizing or minimizing a function of several variables, with the variables being subject to a number of constraints. The constraints, as well as the function being optimized, are linear. Although calculus is helpful in many types of optimization problems, it is of no value in this type of problem—hence the necessity for developing an entirely new procedure.

In the development of the material that follows, we shall be concerned with the theory as well as the techniques for the solution of linear programming problems, but a great deal of emphasis will be placed upon the practical approach to the solution of problems and to a full understanding of its important aspects. The general linear programming problem deals with the allocation of resources in seeking the optimization. The fact that the resources are limited means that a choice must be made in assigning the use of these resources. With only a few constraints, the problem becomes complicated enough that intuitional choices or those based upon experience alone are seldom correct.

1-2 GRAPHICAL ILLUSTRATIONS—MAXIMIZATION

To introduce linear programming, we shall use a few graphical illustrations. Since it is easy to find graphical solutions and give geometric interpretations in two dimensions, these illustrations will be limited to two dimensions. Initially these examples will be elementary.

Example 1-1 Jim and Jane have a small furniture workshop in their garage, where they assemble from purchased parts and finish the furniture in preparation for sale. They are presently limiting their production to tables and chairs. Each chair requires 4 h to assemble and 2 h to finish, but each table requires 2 h to assemble and 4 h to finish. Jim works only on the assembly operation, Jane works only on the finishing operation, and each puts in no more than an 8-h day. It is known that each chair can be sold for $300 and each table for $200. What should the average daily production of tables and chairs be to maximize the total income?

Mathematical Model

To obtain a solution to this problem, it is necessary first to write a mathematical model of the problem. Some simplifying assumptions are made to do this, but one must keep in mind that it is necessary to make realistic assumptions if the solution is to have any meaning. For this solution, let

x_1 = number of chairs to be produced
x_2 = number of tables to be produced

Consider the income to be measured in units of *hundreds of dollars*, so that the total income to be maximized will be given by

$$3x_1 + 2x_2$$

Note that the income on chairs is directly proportional to the number being produced and that the income on tables is directly proportional to the number being produced; the income to be maximized is a *linear* function of the number of chairs being produced and the number of tables being produced. This linear property is one of the basic tenets of linear programming.

Since each chair requires 4 h for assembly and each table requires 2 h for assembly, the total amount of time in the assembly operation is given by

$$4x_1 + 2x_2$$

Since assembly time is limited to 8 h, we have the constraint

$$4x_1 + 2x_2 \leqq 8$$

Since each chair requires 2 h of finishing and each table requires 4 h of finishing, the total amount of time in the finishing operation is

$$2x_1 + 4x_2$$

and since Jane performs the finishing task during her day, we have the constraint

$$2x_1 + 4x_2 \leqq 8$$

With the further restriction that negative solution values are not allowed, the mathematical model becomes:

Find

$$x_1, x_2 \geqq 0$$

such that

$$4x_1 + 2x_2 \leqq 8$$

$$2x_1 + 4x_2 \leqq 8$$

and such that

$$z = 3x_1 + 2x_2 \text{ is a } maximum$$

To solve this problem graphically, we first construct the set of points (x_1, x_2) which represents a feasible solution to the problem. This is the set of points that satisfies all the constraints. This set of points, called the *solution space*, is shown by the shaded section in Fig. 1-1.

The solution space is bounded by four straight lines and lies entirely within the first quadrant. This is explained by the fact that the restriction

$$4x_1 + 2x_2 \leqq 8$$

requires that all points be either on or below the line

$$4x_1 + 2x_2 = 8$$

Similarly, the restriction

$$2x_1 + 4x_2 \leqq 8$$

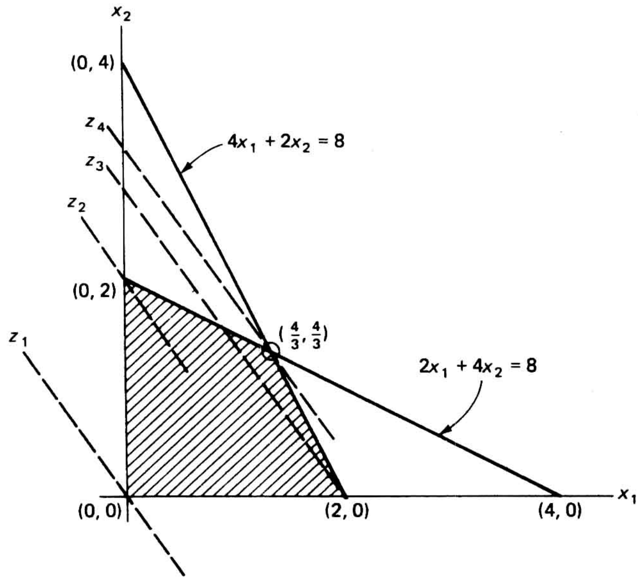

Figure 1-1

requires that all points be on or below the line

$$2x_1 + 4x_2 = 8$$

The constraints

$$x_1, x_2 \geqq 0$$

restrict the points to the first quadrant.

For any selected value of z the function

$$z = 3x_1 + 2x_2$$

is a straight line. For each value of z we have a different straight line, but all the lines in the system are parallel simply because the slope of any one of these lines is completely independent of the value of z.

We wish to find the line from the entire system of parallel lines with the largest value of z while having at least one point in or on the boundary of the solution space. On the graph:

z_1 passes through (0, 0) and has the value zero.
z_2 passes through (0, 2) and has the value 4.
z_3 passes through (2, 0) and has the value 6.
z_4 passes through $(\frac{4}{3}, \frac{4}{3})$ and has the value $\frac{20}{3}$.

It is evident that the value of z increases as the line is moved farther and farther away from the origin. Thus for the two-dimensional case it is necessary first to determine the direction of the line representing the function to be maximized. Then the line is moved as far away from the origin as possible while keeping at least one point in the feasible region. For this case it is easily seen that the point $(\frac{4}{3}, \frac{4}{3})$ is the only point meeting these conditions and hence one needs only to evaluate z for this point to find the maximum value.

Interpretation of the Solution

Consider next the meaning of the optimal solution just found. It shows that Jim and Jane should make $\frac{4}{3}$ chairs and $\frac{4}{3}$ tables each day, with a resulting income of $\frac{20}{3}$ hundreds of dollars, or $667 per day. You might say that one does not make one and one-third chairs or one and one-third tables, but considering that one would have essentially a continuous production line, 4 chairs and 4 tables could be produced every 3 days, so the results merely tell us that Jim and Jane make equal numbers of chairs and tables and that the rate of production is $1\frac{1}{3}$ of each per day.

At this point you might wonder whether there exists a scheduling problem related to making sure that there is always something assembled for Jane to finish and always a supply of parts for Jim to assemble. Actually, at this stage we ignore that problem by suggesting that we are interested here only in the product-mix problem, so that we always assume we have a modest stockpile of parts on hand and in turn have a few tables and chairs assembled so that Jim and Jane both have work to do. With these assumptions we can concentrate on the product-mix problem, as we have done above. Note that the solution is *unique*.

1-3 GRAPHICAL SOLUTION—ALTERNATE OPTIMA

Example 1-2 Suppose now that we modify the income for Example 1-1 so that each chair will bring $200 and each table will bring $100. All other facets of the problem will remain the same.

Mathematical Model

The function to be maximized is now

$$z = 2x_1 + x_2$$

With all constraints remaining as before so that the feasible solution space is unaltered, Fig. 1-2 shows that the line z is parallel to one of the boundaries of the solution space. That boundary is marked PQ.

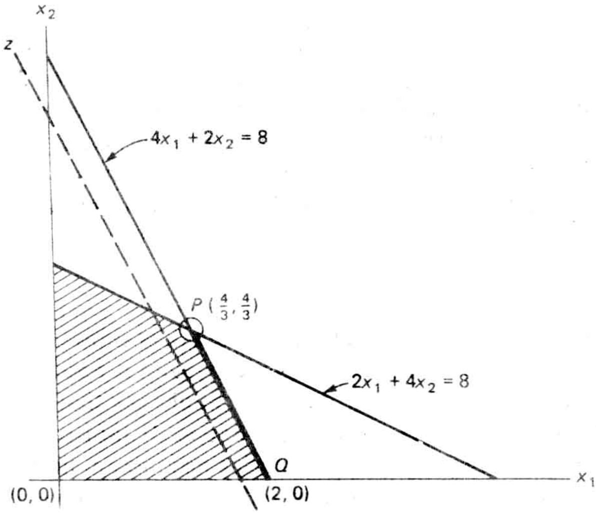

Figure 1-2

In this case, in moving the line z as far from the origin as possible while keeping at least one point from the feasible region on the line, it is found that all the points on the line segment PQ qualify. The point $x_1 = \frac{4}{3}$, $x_2 = \frac{4}{3}$ produces an optimal solution value of $z = \$400$. Similarly, $x_1 = 2$, $x_2 = 0$ produces an optimal income of \$400, so the solution is *not* unique. There are, in fact, an infinite number of solutions for x_1 and x_2 that produce the optimal income of \$400. The managerial implications of this are that one must choose the best solution influenced by factors other than those included in the linear programming formulation. For example, one might want to make $x_1 = x_2$ in order to have a balanced product line. This is acceptable, since there is a point on the line PQ where $x_1 = x_2$.

1-4 GRAPHICAL SOLUTION—MINIMIZATION

Example 1-3 Let us now consider Jim and Jane's operation in Examples 1-1 and 1-2 from a slightly different point of view. Income no longer seems to be of primary concern, since there are a number of fixed-cost operations. There is a variable cost of \$30 for each chair and \$20 for each table. We now wish to minimize the variable cost subject to a number of different types of restrictions. The sale of chairs cannot exceed *three units* and the sale of tables cannot exceed *four units*; furthermore, the sale of chairs and tables together cannot exceed *five units*. With the new approach, a drying room is used after the gluing has been