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Methods for Electromagnetic Field Analysis

ISMO V. LINDELL

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Preface

The present monograph discusses a number of mathematical and conceptual methods applicable in the analysis of electromagnetic fields. The leading tone is dyadic algebra. It is applied in the form originated by J.W. Gibbs more than one hundred years ago, with new powerful identities added, making coordinate-dependent operations in electromagnetics all but obsolete. The chapters on complex vectors and dyadics are independent of the rest of the book, actually independent of electromagnetics, so they can be applied in other branches of physics as well. It is claimed that by memorizing about five basic dyadic identities (similar to the well-known bac-cab rule in vector algebra), a working knowledge of dyadic algebra is obtained. To save the memory, a collection of these basic dyadic identities, together with their most important special cases, is given as an appendix. In different chapters the dyadics are seen in action. It is shown how simply different properties can be expressed in terms of dyadics: boundary and interface conditions, medium equations, solving Green functions, generalizing circuit theory to vector field problems with dyadic impedances, finding transformations between field problems and, finally, working on multipole and image sources for different problems.

Dyadic algebra is seen especially to aid in solving electromagnetic problems involving different linear media. In recent years, the chiral medium with its wide range of potential applications has directed theoretical interest to new materials. The most general isotropic medium, the bi-isotropic medium, has made electromagnetic theory a fresh subject again, with new phenomena being looked for. The medium aspect is carried along in this text. What is normally analysed in isotropic media is done here for bi-isotropic or sometimes for bianisotropic media, if possible. Especially new is the duality transformation, which actually exists as a pair of transformations. It is seen to shed new light on the plus and minus field decomposition, which has proved useful for analysing fields in chiral media, by showing that they are nothing more than self-dual fields with respect to each of the two transformations.

In Chapter 5, Green dyadics for different kinds of media are discussed and a systematic method for their solution, without applying the Fourier transformation, is given. In Chapter 6, source equivalence and its relation to non-radiating sources is discussed, together with certain equivalent sources: point sources (multipoles) and surface sources (Huygens' sources). Everywhere in the text the main emphasis is not on specific results but methods of analysis.

The final chapter gives a summary of the work done by this author and colleagues on the EIT, exact image theory. This is a general method for solving problems involving layered media by replacing them by image sources which are located in complex space. The EIT is presented here for the first time in book form.

The contents of this monograph reflect some of the work done and courses given by this author during the last two decades. The results should be of interest to scientists doing research work in electromagnetics, as well as to graduate students. For classroom use, there are numerous possibilities for homework problems requiring the student to fill in steps which have been omitted to keep the size of this monograph within certain limits. The EIT can also be studied independently and additional material, not found in this text, exists in print (see referece lists at section ends of Chapter 7).

The text has been typed and figures drawn by the author alone, leaving no-one else to blame. On the other hand, during graduate courses given on the material, many students have helped in checking a great number of equations. Also, the material of Chapters 1 and 2 has been given earlier as a laboratory report and a few misprints have been pointed out by some international readers. For all these I am thankful. The rest of the errors and misprints are still there to be found.

This book is dedicated to my wife Liisa. A wise man is recognized for having a wife wiser than himself. I have the pleasure to consider myself a wise man.

Helsinki I.V.L.
July 1991

Contents

Chapter 1

Complex vectors

Complex vectors are vectors whose components can be complex numbers. They were introduced by the famous American physicist J. WILLARD GIBBS, sometimes called the 'Maxwell of America', at about the same period in the 1880's as the real vector algebra, in a privately printed but widely circulated pamphlet *Elements of vector analysis.* Gibbs called these complex extensions of vectors 'bivectors' and they were needed, for example, in his analysis of time-harmonic optical fields in crystals. In a later book compiled by Gibbs's student WILSON in 1909, the text reappeared in extended form, but with only few new ideas (GIBBS and WILSON 1909). Thenceforth, complex vectors have been treated mainly in books on electromagnetics in the context of time-harmonic fields. Instead of a full application of complex vector algebra, the analyses, however, mostly made use of trigonometric function calculations. As will be seen in this chapter, complex vector algebra offers a simple method for the analysis of time-harmonic fields. In fact, it is possible to use many of the rules known from real vector algebra, although not all the conclusions. Properties of the ellipse of time-harmonic vectors can be seen to be directly obtainable through operations on complex vectors.

1.1 Notation

As mentioned above, complex vector formalism is applied in electromagnetics when dealing with time-harmonic field quantities. A time-harmonic field vector $\mathbf{F}(t)$, or 'sinusoidal field' is any real vector function of time t that satisfies the differential equation

$$\frac{d^2}{dt^2}\mathbf{F}(t) + \omega^2\mathbf{F}(t) = 0. \tag{1.1}$$

A general solution can be expressed in terms of two constant real vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ in the form

$$\mathbf{F}(t) = \mathbf{F}_1 \cos\omega t + \mathbf{F}_2 \sin\omega t. \tag{1.2}$$

The complex vector formalism can be used to replace the time-harmonic vectors provided the angular frequency ω is constant. There are certain advantages to this change in notation and, of course, the disadvantage that some new concepts and formulas must be learned. The main bulk of formulas, however, is the same as for real vectors. As an advantage, in using complex vector algebra, work with trigonometric formulas can be avoided, and the formulas look much simpler.

The complex vector $\mathbf{f}$ is defined as a combination of two real vectors, $\mathbf{f}_{\rm re}$ the real part, and $\mathbf{f}_{\rm im}$ the imaginary part of $\mathbf{f}$:

$$\mathbf{f} = \mathbf{f}_{\rm re} + j\mathbf{f}_{\rm im}. \tag{1.3}$$

The subscripts re and im can be conceived as operators, giving the real and, respectively, the imaginary parts of a complex vector.

The essential point in the complex vector formalism lies in the one-to-one correspondence with the time-harmonic vectors $\mathbf{f} \leftrightarrow \mathbf{F}(t)$. In fact, there are two mappings which give a unique time-harmonic vector for a given complex vector and vice versa. They are:

$$\mathbf{f} \to \mathbf{F}(t): \quad \mathbf{F}(t) = \Re\{\mathbf{f}e^{j\omega t}\} = \mathbf{f}_{\rm re}\cos\omega t - \mathbf{f}_{\rm im}\sin\omega t, \tag{1.4}$$

$$\mathbf{F}(t) \to \mathbf{f}: \quad \mathbf{f} = \mathbf{F}(0) - j\mathbf{F}(\pi/2\omega) = \mathbf{F}_1 - j\mathbf{F}_2. \tag{1.5}$$

Thus, for the two representations (1.2) and (1.3) we can see the correspondences $\mathbf{f}_{\rm re} = \mathbf{F}_1$ and $\mathbf{f}_{\rm im} = -\mathbf{F}_2$.

The mappings (1.4), (1.5) are each other's inverses, as is easy to show. For example, let us insert (1.4) into (1.5):

$$\mathbf{f} = \Re\{\mathbf{f}e^0\} - j\Re\{\mathbf{f}e^{j\pi/2}\} = \mathbf{f}_{\rm re} + j\mathbf{f}_{\rm im}, \tag{1.6}$$

which results in the identity $\mathbf{f} = \mathbf{f}$.

It is important to note that there always exists a time-harmonic counterpart to a complex vector whatever its origin. In fact, in analysis, there arise complex vectors, which do not represent a time-harmonic field quantity, for examle the wave vector $\mathbf{k}$ or the Poynting vector $\mathbf{P}$. We can, however, always define a time-harmonic vector through (1.4), maybe lacking physical content but helpful in forming a mental picture.

A time-harmonic vector $\mathbf{F}(t) = \mathbf{F}_1\cos\omega t + \mathbf{F}_2\sin\omega t$ traces an ellipse in space, which may reduce to a line segment or a circle. This is seen from the following reasoning.

- If $\mathbf{F}_1 \times \mathbf{F}_2 = 0$, the vectors are parallel or at least one of them is a null vector. Hence, $\mathbf{F}(t)$ is either a null vector or moves along a line and is called *linearly polarized* (LP).

- If $\mathbf{F}_1 \times \mathbf{F}_2 \neq 0$, the vectors define a plane, in which the vector $\mathbf{F}(t)$ rotates. Forming the auxiliary vectors $\mathbf{b} = \mathbf{F}_1 \times (\mathbf{F}_1 \times \mathbf{F}_2)$ and $\mathbf{c} = \mathbf{F}_2 \times (\mathbf{F}_1 \times \mathbf{F}_2)$, we can easily see that the equation $(\mathbf{b} \cdot \mathbf{F}(t))^2 + (\mathbf{c} \cdot \mathbf{F}(t))^2 = |\mathbf{F}_1 \times \mathbf{F}_2|^4$ is satisfied. This is a second order equation, whose solution $\mathbf{F}(t)$ is obviously finite for all t, whence the curve it traces is an ellipse.

- The special case of a *circularly polarized* (CP) vector is obtained, when $|\mathbf{F}(t)|^2 = \mathbf{F}_1^2 \cos^2 \omega t + \mathbf{F}_2^2 \sin^2 \omega t + \mathbf{F}_1 \cdot \mathbf{F}_2 \sin 2\omega t$ is constant for all t. Taking $t = 0$ and $t = \pi/2$ gives $\mathbf{F}_1^2 = \mathbf{F}_2^2$, which leads to the second condition $\mathbf{F}_1 \cdot \mathbf{F}_2 = 0$.

Thus, to every complex vector $\mathbf{f}$ there corresponds an ellipse just as for every real vector there corresponds an arrow in space. The real and imaginary parts $\mathbf{f}_{\rm re}$, $\mathbf{f}_{\rm im}$ both lie on the ellipse. $\mathbf{f}_{\rm re}$ equals the time origin value and is called *the phase vector of the ellipse*. The direction of rotation of $\mathbf{F}(t)$ on the ellipse equals that of $\mathbf{f}_{\rm im}$ turned towards $\mathbf{f}_{\rm re}$ in the shortest way. A complex vector which is *not linearly polarized* (NLP) has a handedness of rotation, which depends on the direction of aspect. The rotation is right handed when looked at in a direction $\mathbf{u}$ (a real vector) such that $\mathbf{f}_{\rm im} \times \mathbf{f}_{\rm re} \cdot \mathbf{u}$ is a positive number and, conversely, left handed if it is negative.

An LP vector must be represented by a double-headed arrow (infinitely thin ellipse), which is in distinction with the one-headed arrow representation of real vectors. The difference is of course due to the fact that the time-harmonic vector (1.2) oscillates between its two extremities.

The complex conjugate of a complex vector $\mathbf{f}$, denoted by $\mathbf{f}^*$, is defined by

$$\mathbf{f}^* = (\mathbf{f}_{\rm re} + j\mathbf{f}_{\rm im})^* = \mathbf{f}_{\rm re} - j\mathbf{f}_{\rm im}. \tag{1.7}$$

From (1.4) we can see that $\mathbf{f}^*$ corresponds to the time-dependent vector $\mathbf{F}(-t)$, or it rotates in the opposite direction along the same ellipse as $\mathbf{f}(t)$.

The complex vector $\mathbf{f}$ is LP if and only if $\mathbf{f}_{\rm re} \times \mathbf{f}_{\rm im} = 0$. This is equivalent with the following condition:

$$\mathbf{f} \text{ is LP} \quad \Leftrightarrow \quad \mathbf{f} \times \mathbf{f}^* = 0. \tag{1.8}$$

The corresponding condition for the CP vector is

$$\mathbf{f} \text{ is CP} \quad \Leftrightarrow \quad \mathbf{f} \cdot \mathbf{f} = 0. \tag{1.9}$$

In fact, (1.9) implies that $\mathbf{f}_{\rm re}^2 = \mathbf{f}_{\rm im}^2$ and $\mathbf{f}_{\rm re} \cdot \mathbf{f}_{\rm im} = 0$, which is equivalent with the CP property of the corresponding time-harmonic vector, as was seen above.

Every LP vector can be written as a multiple of a real unit vector $\mathbf{u}$ in the form $\mathbf{f} = \alpha\mathbf{u}$. Every CP vector $\mathbf{f}$ can be written in terms of two orthogonal real unit vectors $\mathbf{u}$, $\mathbf{v}$ in the form $\mathbf{f} = \alpha(\mathbf{u} + j\mathbf{v})$. In these expressions α is a complex scalar, in general.

1.2 Complex vector identities

The algebra of complex vectors obeys many of the rules known from the real vector algebra, but not all. For example, the implication $\mathbf{a}\cdot\mathbf{a} = 0 \;\Rightarrow\; \mathbf{a} = 0$ is not valid for complex vectors. To be more confident in using identities of real vector algebra, the following theorem appears useful:

> *all multilinear identities valid for real vectors are also valid for complex vectors.*

A multilinear function F of vector arguments $\mathbf{a}_1,\ \mathbf{a}_2, ...$ is a function which is linear in every argument, or the following is valid:

$$F(\mathbf{a}_1, \mathbf{a}_2, ..., (\alpha\mathbf{a}_i' + \beta\mathbf{a}_i''), ..., \mathbf{a}_n) =$$

$$\alpha F(\mathbf{a}_1, ..., \mathbf{a}_i', ..., \mathbf{a}_n) + \beta F(\mathbf{a}_1, ..., \mathbf{a}_i'', ...\mathbf{a}_n). \tag{1.10}$$

A multilinear identity is of the form

$$F(\mathbf{a}_1, ..., \mathbf{a}_n) = 0 \quad \text{for all } \mathbf{a}_i,\ i = 1...n. \tag{1.11}$$

Now, if the identity is valid for real vectors $\mathbf{a}_i$ and the function does not involve a conjugation operation, from the linearity property (1.10) we can show that it must be valid for complex vectors $\mathbf{a}_i$ as well. In fact, taking $\alpha = 1, \beta = j$, the identity is obviously valid if the real vector $\mathbf{a}_i$ is replaced by the complex vector $\mathbf{a}_i' + \alpha\mathbf{a}_i''$. This can be repeated for every i and, thus, all vectors $\mathbf{a}_i$ can be complex in the identity (1.11). As an example of a trilinear identity we might write

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) - (\mathbf{a}\cdot\mathbf{c})\mathbf{b} + (\mathbf{a}\cdot\mathbf{b})\mathbf{c} = 0 \quad \text{for all} \ \ \mathbf{a},\ \mathbf{b},\ \mathbf{c}. \tag{1.12}$$

Also, all non-linear identities which can be derived from multilinear identities are valid for complex vectors, like $\mathbf{a}\times\mathbf{a} = 0$ for all vectors $\mathbf{a}$. The conjugation operation can be introduced by inserting conjugated complex vectors in multilinear identities. Thus, the identity

$$|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}\cdot\mathbf{b}^*|^2, \tag{1.13}$$

can be obtained from the real quadrilinear identity

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}), \tag{1.14}$$