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EDITOR'S PREFACE

The papers that were presented during the Third Symposium in Applied Mathematics of the American Mathematical Society are published in this volume. The Symposium was held at the University of Michigan from June 14 to 16, 1949. The subject of the Symposium was *Elasticity;* it included plasticity. The Symposium was cosponsored by the Applied Mechanics Division of the American Society of Mechanical Engineers, and joint sessions of the Symposium and that Division's Fifteenth Applied Mechanics Conference were held on June 15.

The papers in this volume are grouped roughly according to subjects. Owing to previous arrangements for publication, two of the papers appear here as abstracts containing references to the complete papers. Four papers from the program of the Applied Mechanics Conference were presented at the joint sessions with the Symposium. The place of publication of those four papers is announced at the end of this volume.

On behalf of the American Mathematical Society, the Editorial Committee, consisting of Eric Reissner, A. H. Taub, and the undersigned, wishes to express its gratitude to the McGraw-Hill Book Company, Inc., for undertaking the publication of this volume. The Committee extends its thanks to R. C. Gibbs of the National Research Council for his assistance in initiating these arrangements for publication and to the officers of McGraw-Hill and of the Society for carrying out this arrangement. In the editorial processing of this volume, the undersigned wishes to acknowledge the work of the New York Office of the Society as well as the assistance of the other two members of the Editorial Committee and the help given by G. E. Hay and P. G. Hodge, Jr.

R. V. Churchill
Chairman, Editorial Committee
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CONTENTS

APPROXIMATE METHODS OF SOLUTION OF TWO-DIMENSIONAL PROBLEMS IN ANISOTROPIC ELASTICITY

BY

I. S. SOKOLNIKOFF

1. The scope of research. This paper contains an account of one phase of research in the domain of two-dimensional problems of anisotropic elasticity initiated at the University of California (Los Angeles) this year. The object of this research is to develop practical methods for the explicit solution of a wide class of problems involving the states of plane stress and plane strain in anisotropic elastic media, and to study the problems of deflection of thin anisotropic elastic plates. The investigation hinges on an approximate determination of appropriate stress functions for the anisotropic problems from certain known stress functions in the corresponding isotropic cases.

A systematic use of Airy's stress function in the solution of two-dimensional problems of anisotropic elasticity, following the pattern laid down by N. I. Muschelisvili for the isotropic case, was made by S. G. Lechnitzky. An outline of this mode of attack on the two-dimensional boundary-value problems of elasticity is contained in an address given by the author [1] before the meeting of the American Mathematical Society in 1941. Since numerous references to original sources are given in that address, they will not be duplicated here, but to facilitate the understanding of the perturbation procedures discussed in Sec. 3, the essential concepts, in so far as they bear on the boundary-value problems of anisotropic elasticity, are sketched in the following section.

2. Formulation of boundary-value problems. We consider a two-dimensional anisotropic elastic medium, having at least one plane of elastic symmetry, which we take as the XY plane of our rectangular coordinate system. The region R occupied by the medium is, in general, multiply connected. We denote the boundary of the region by C, where C may consist of the exterior contour C_0 and several interior contours C_i $(i = 1, \cdots, n)$. The region R (if multiply connected) may be regarded either as representing a thin plate with holes or the cross section of an infinitely long cylinder with longitudinal cavities.

We suppose that either the distribution of stresses along C (the first boundary-value problem) or the displacement of points of the boundary C (the second boundary-value problem), is known. The problem is to determine stresses and strains in the region.

It is well known[1] that the solution of these boundary-value problems hinges on the determination of Airy's stress function $U(x, y)$, satisfying the differential equation of the form

[1] See Bibliography on pp. 553–555 of the address cited in [1].

$$c_1 \frac{\partial^4 U}{\partial x^4} + c_2 \frac{\partial^4 U}{\partial x^3 \partial y} + c_3 \frac{\partial^4 U}{\partial x^2 \partial y^2} + c_4 \frac{\partial^4 U}{\partial x \partial y^3} + c_5 \frac{\partial^4 U}{\partial y^4} = 0, \tag{1}$$

where the constants c_i are real functions of the known elastic moduli of the medium.

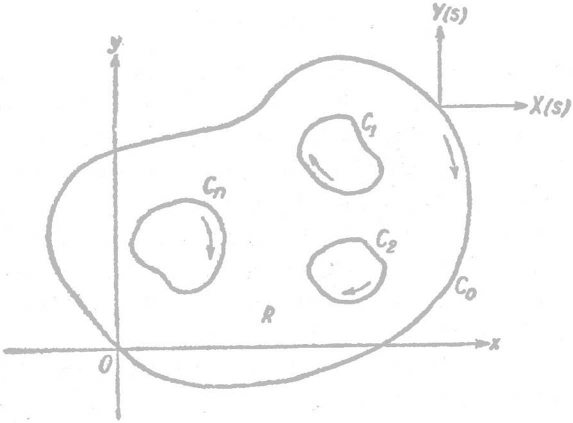

FIG. 1

The components τ_{xx}, τ_{xy}, τ_{yy} of the stress tensor τ are related to U by the formulas

$$\tau_{xx} = \frac{\partial^2 U}{\partial y^2}, \qquad \tau_{xy} = -\frac{\partial^2 U}{\partial x \partial y}, \qquad \tau_{yy} = \frac{\partial^2 U}{\partial x^2}. \tag{2}$$

The general solution of equation (1) has the form

$$U = \sum_{i=1}^{4} F_i(x + \mu_i y), \tag{3}$$

where the functions F_i are of class C^4 and the μ's are the *distinct roots* of the characteristic equation

$$c_5\mu^4 + c_4\mu^3 + c_3\mu^2 + c_2\mu + c_1 = 0. \tag{4}$$

The case of multiple roots of the characteristic equation is of relatively trivial interest because in that case the problem can be reduced to an isotropic one by a simple linear transformation.[2]

From the fact that the energy of deformation is nonnegative, it is easy to show that the roots μ_i of equation (4) are complex numbers. This was first demonstrated by S. G. Lechnitzky.[3]

[2] This follows from remarks made immediately after equation (5) below.

[3] See [2], a paper which is concerned with the problem of deflection of thin anisotropic plates.

Since the c_i in equation (4) are real, it follows that its roots are conjugate complex numbers, and we can write

$$\mu_1 = \bar{\mu}_3, \qquad \mu_2 = \bar{\mu}_4.$$

We can rewrite the general solution (3) in a more convenient form by introducing two complex variables z_1 and z_2 defined as follows:

$$\begin{aligned} z_1 &= x + \mu_1 y \\ &\equiv x + (\alpha_1 + i\beta_1)y \\ &\equiv x_1 + iy_1, \end{aligned}$$

where

$$x_1 = x + \alpha_1 y, \qquad y_1 = \beta_1 y;$$

$$\begin{aligned} z_2 &= x + \mu_2 y \\ &\equiv x + (\alpha_2 + i\beta_2)y \\ &\equiv x_2 + iy_2, \end{aligned}$$

with

$$x_2 = x + \alpha_2 y, \qquad y_2 = \beta_2 y.$$

Since $U(x, y)$ is a real function, the solution (3) assumes the form

$$U = F_1(z_1) + F_2(z_2) + \overline{F_1(z_1)} + \overline{F_2(z_2)}, \tag{5}$$

where we use bars to denote the conjugate complex values.

If the characteristic equation (4) has multiple roots, then $z_1 = z_2$, and the solution of equation (1) has the form

$$U = F_1(z_1) + \bar{z}_1 F_2(z_1) + \overline{F_1(z_1)} + \overline{z_1 F_2(z_1)},$$

which becomes identical with the solution of the biharmonic equation when the variable z_1 is replaced by $z = x + iy$. Thus the study of the behavior of anisotropic media, whose elastic properties are such that the roots of the associated characteristic equation are multiple, is reducible to an isotropic case.

It follows from the definition of the variables z_1 and z_2 that their domains R_1 and R_2 are obtainable from R by the homogeneous deformations:

$$T_1: \quad \begin{cases} x_1 = x + \alpha_1 y, \\ y_1 = \beta_1 y, \end{cases}$$

and

$$T_2: \quad \begin{cases} x_2 = x + \alpha_2 y, \\ y_2 = \beta_2 y. \end{cases}$$

From the differentiability of Airy's stress function $U(x, y)$, it follows that $F_1(z_1)$ and $F_2(z_2)$ are analytic functions defined in the regions R_1 and R_2, respectively.

We readily deduce from the representation (5) and relations (2) that

$$\begin{aligned}\tau_{xx} &= 2\mathcal{R}[\mu_1^2F_1''(z_1) + \mu_2^2F_2''(z_2)],\\ \tau_{yy} &= 2\mathcal{R}[F_1''(z_1) + F_2''(z_2)],\\ \tau_{xy} &= -2\mathcal{R}[\mu_1F_1''(z_1) + \mu_2F_2''(z_2)],\end{aligned}$$

where $\mathcal{R}$ denotes the real part of expressions contained in the brackets.

If we insert these expressions in the boundary conditions:

$$\begin{aligned}\tau_{xx}\cos(x, n) + \tau_{xy}\cos(y, n) &= X(s),\\ \tau_{xy}\cos(x, n) + \tau_{yy}\cos(y, n) &= Y(s), \qquad \text{on } C,\end{aligned}$$

in which $X(s)$ and $Y(s)$ are the components of prescribed external forces acting on C (s being the arc parameter along C) and integrate along the contours from some initial value $s = s_0$ to a variable point $s = s$, we get[4]

$$(6)\qquad \begin{aligned}2\mathcal{R}[\mu_1F_1'(z_1) + \mu_2F_2'(z_2)] &= \int_{s_0}^{s} X(s)\,ds + c_1\\ &\equiv f_1(s),\\ 2\mathcal{R}[F_1'(z_1) + F_2'(z_2)] &= -\int_{s_0}^{s} Y(s)\,ds + c_2\\ &\equiv f_2(s),\end{aligned}$$

where f_1 and f_2 are known functions along C, and c_1 and c_2 are constants. These can be fixed arbitrarily on only one of the contours, say C_0.

If, on the other hand, the components of the displacement vector are specified along C as functions of the arc-parameter s, so that

$$\begin{aligned}u &= g_1(s),\\ v &= g_2(s),\end{aligned}$$

then a somewhat less obvious computation utilizing the stress-strain relations yields [3]

$$(7)\qquad \begin{aligned}2\mathcal{R}[a_1F_1'(z_1) + a_2F_2'(z_2)] &= g_1(s),\\ 2\mathcal{R}[b_1F_1'(z_1) + b_2F_2'(z_2)] &= g_2(s), \qquad \text{on } C,\end{aligned}$$

where the a_i and b_i are known rational functions of the elastic moduli of the medium.

It is clear from these formulas that the solution of the first and second boundary-value problems of anisotropic elasticity is reduced to the determination of analytic functions $F_1'(z_1)$ and $F_2'(z_2)$ from functional equations (6) and (7). The questions of existence and uniqueness of solution of these equations are settled by reducing them to certain equivalent systems of integral equations of the standard

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[4] Note that if n denotes the exterior unit normal, then $dx/ds = -\cos(y, n)$ and $dy/ds = \cos(x, n)$.

types.[5] The integral equations can serve as a point of departure for the actual determination of unknown functions, but there are also methods of solution of equations (6) and (7) that are based on an extension of the scheme of N. I. Muschelisvili [4] which proved so successful in the treatment of numerous boundary-value problems of isotropic elasticity. However, in the application of these methods to problems in anisotropic elasticity one frequently runs into serious computational difficulties connected with the construction of suitable conformal mapping functions.[6] It is natural therefore to attempt to reduce the solution of such problems to the isotropic case. This can be done, for certain types of anisotropic media, in the manner outlined in Sec. 3.

The results summarized in this section have been obtained during the past decade or so by several Russian investigators, notably S. G. Lechnitzky. However, many Soviet publications in which these researches appeared were not made accessible to workers in the English-speaking countries. As a consequence, results obtained by the Soviet investigators have been extensively duplicated in this country and in England.[7]

3. Perturbation methods. The fundamental idea underlying the reduction of solution of the boundary-value problems of anisotropic elasticity to the solution of a sequence of isotropic problems is simple. It is a variant of the perturbation procedure often used to obtain approximate solutions of nonlinear differential equations.

In order to simplify the exposition, we consider, instead of equation (1), the equation

$$c_1 \frac{\partial^4 U}{\partial x^4} + c_3 \frac{\partial^4 U}{\partial x^2 \, \partial y^2} + c_5 \frac{\partial^4 U}{\partial y^4} = 0, \tag{8}$$

which corresponds to the case of orthotropic elastic medium.[8] The corresponding treatment for the more general case typified by equation (1) presents no complication and merely leads to the more involved recursion relations.

We observe that, if the medium is isotropic, equation (8) reduces to the biharmonic equation

$$\nabla^4 U = 0,$$

[5] These matters have been dealt with by S. G. Michlin, Publications de l'institut séismologique, Acad. Sci. URSS, No. 76 (1936). D. I. Sherman, ibid., No. 86 (1938).

[6] See, for example, [5].

[7] See, in particular, numerous papers by A. E. Green, S. Holgate, A. C. Stevenson, and G. I. Taylor in the Royal Proceedings for the year 1945.

[8] It may help to recall that if equation (8) refers to a problem dealing with the state of plane stress, the constants are related to the elastic moduli as follows: $c_1 = 1/E_y$, $c_3 = 1/\mu_{xy} - (2\sigma_{xy})/E_x$, $c_5 = 1/E_x$, where E_x and E_y are Young's moduli in the principal directions indicated by the subscripts, σ_{xy} is Poisson's ratio representing the contraction in the Y direction by a tensile stress acting in the X direction, μ_{xy} is the shearing modulus characterizing the change in angle between the principal directions, and $E_x\sigma_{yx} = E_y\sigma_{xy}$. Similar relations, involving elastic moduli associated with the Z direction, can be written for the c_i's in problems concerned with the state of plane strain.

in which case the roots of characteristic equation (4) are $\mu_1 = \mu_2 = i$. We also note that the μ_i characterize the nature of anisotropy of the medium inasmuch as the coefficients c_i are symmetric functions of the roots μ_i. Thus, if we define the numbers ϵ_i (in general complex) by the formulas

$$\mu_1 = i(1 + \epsilon_1), \qquad \mu_3 = \bar{\mu}_1,$$
$$\mu_2 = i(1 + \epsilon_2), \qquad \mu_4 = \bar{\mu}_2,$$

we see that the parameters ϵ_i can be taken as the measure of deviation of the medium from isotropic state. For many structural materials, the deviation parameters ϵ_i are less than 1 in absolute value, and this suggests that the solution of equation (8) might be represented in the form

$$U = \sum_{i,j=0}^{\infty} U_{ij}(x, y)\epsilon_1^i\epsilon_2^j, \tag{9}$$

where the U_{ij} are unknown functions of x and y to be determined so that U satisfies equation (8). If $|\epsilon_i| \ll 1$, a few terms of the series (9) might be expected to yield a good approximation to the desired solution.

Replacing the coefficients c_i in equation (8) by their values in terms of the ϵ_i gives

$$(1 + \epsilon_1)^2(1 + \epsilon_2)^2 \frac{\partial^4 U}{\partial x^4} + [(1 + \epsilon_1)^2 + (1 + \epsilon_2)^2] \frac{\partial^4 U}{\partial x^2\, \partial y^2} + \frac{\partial^4 U}{\partial y^4} = 0,$$

and the substitution of the series (9) in this equation leads to the following system of equations to be satisfied by the functions U_{ij}:

$$\begin{aligned} &\nabla^4 U_{00} = 0, \\ &\nabla^4 U_{ij} = -\frac{\partial^2}{\partial x^2}\nabla^2(2U_{i-1\,j} + 2U_{i\,j-1} + U_{i-2\,j} + U_{i\,j-2}) \\ &\qquad\qquad - \frac{\partial^4}{\partial x^4}(2U_{i-1\,j-2} + 2U_{i-2\,j-1} + 4U_{i-1\,j-1} + U_{i-2\,j-2}), \\ &U_{ij} = 0 \qquad \text{(if } i \text{ or } j \text{ is negative).} \end{aligned} \tag{10}$$

We see that these equations are of the type

$$\nabla^4 U_{ij} = f_{ij}(U_{00}, U_{01}, \cdots, U_{i-1\,j-1}) \qquad (i, j = 1, 2, \cdots), \tag{11}$$

in which the f_{ij} are known functions of the indicated arguments and $U_{ij} = U_{ji}$.

The general solution of the first equation in the system (10) has the form

$$U_{00} = \mathcal{R}[\varphi_{000}(z) + \bar{z}\varphi_{001}(z)],$$

where φ_{000} and φ_{001} are arbitrary analytic functions of $z = x + iy$. The substitution of this expression for U_{00} in the right-hand member of the second equation of the system (10) with $i = 1, j = 0$ yields the equation

$$\nabla^4 U_{10} = -2\frac{\partial^2}{\partial x^2}\nabla^2 U_{00},$$

whose solution is

$$U_{10} = \mathcal{R}[\varphi_{100}(z) + \bar{z}\varphi_{101}(z) - \tfrac{1}{4}\bar{z}^2\varphi_{001}'(z)],$$

in which the functions φ are analytic. Setting $i = 2, j = 0$ in equation (10) gives the equation for U_{20}, which can be solved by quadratures to yield

$$U_{20} = \mathcal{R}[\varphi_{200}(z) + \bar{z}\varphi_{201}(z) + \tfrac{1}{8}\bar{z}^2\varphi_{001}'(z) - \tfrac{1}{4}\bar{z}^2\varphi_{101}'(z) + \tfrac{1}{32}\bar{z}^3\varphi_{001}''(z)].$$

Continuation of this procedure leads to the general solution of the system (10) in the form

$$U_{ij} = \mathcal{R} \sum_{m=0}^{i+j+1} \bar{z}^m \varphi_{ijm}(z), \tag{12}$$

in which the functions φ_{ij0} and φ_{ij1} are arbitrary and the remaining ones satisfy the relations:

$$\begin{aligned}
-16(m+2)(m+1)\varphi_{ijm+2}''
&= (m+1)(8\varphi_{i-1\,j\,m+1}''' + 8\varphi_{i\,j-1\,m+1}''' + 4\varphi_{i-2\,j\,m+1}''' + 4\varphi_{i\,j-2\,m+1}''') \\
&+ (m+2)(m+1)(16\varphi_{i-1\,j\,m+2}'' + 16\varphi_{i\,j-1\,m+2}'' + 8\varphi_{i-2\,j\,m+2}'' + 8\varphi_{i\,j-1\,m+2}'') \\
&+ (m+3)(m+2)(m+1)(8\varphi_{i-1\,j\,m+3}' + 8\varphi_{i\,j-1\,m+3}' + 4\varphi_{i-2\,j\,m+3}' + 4\varphi_{i\,j-1\,m+3}') \\
&+ 2(\varphi_{i-1\,j-2\,m}^{(iv)} + 2\varphi_{i-2\,j-1\,m+4}^{(iv)} + \varphi_{i-1\,j-1\,m}^{(iv)} + \varphi_{i-2\,j-2\,m}^{(iv)}) \\
&+ (m+1)(8\varphi_{i-1\,j-2\,m+1}''' + 8\varphi_{i-2\,j-1\,m+1}''' + 16\varphi_{i-1\,j-1\,m+1}''' + 4\varphi_{i-2\,j-2\,m+1}''') \\
&+ (m+2)(m+1)(12\varphi_{i-1\,j-2\,m+2}'' + 12\varphi_{i-2\,j-1\,m+2}'' + 24\varphi_{i-1\,j-1\,m+2}'' \\
&+ 6\varphi_{i-2\,j-2\,m+2}'') + (m+3)(m+2)(m+1) \\
&\qquad \cdot (8\varphi_{i-1\,j-2\,m+3}' + 8\varphi_{i-2\,j-1\,m+3}' + 16\varphi_{i-1\,j-1\,m+3}' + 4\varphi_{i-2\,j-2\,m+3}') \\
&+ (m+4)(m+3)(m+2)(m+1) \\
&\qquad \cdot (2\varphi_{i-1\,j-2\,m+4} + 2\varphi_{i-2\,j-1\,m+4} + 4\varphi_{i-1\,j-1\,m+4} + \varphi_{i-2\,j-2\,m+4}).
\end{aligned}$$

The functions φ_{ij0} and φ_{ij1} are determined (essentially uniquely) by the boundary conditions. We do not write out the corresponding system of functional equations for their determination,[9] because its structure is entirely similar to equations (6) and (7).

Despite the formidable appearance of solution as given by the expression (9), specific computations indicate that a satisfactory approximate solution is obtained, in a number of problems, if one retains in the expansion only linear or quadratic terms in the deviation parameters ϵ_i. The computational labor in such cases is not excessively heavy.

The perturbational procedure, described above, naturally is not the only one that can be applied to anisotropic problems. Thus, one can introduce the perturbation parameters ϵ_i not in the roots μ_i of characteristic equations, but directly into the coefficients of equation (8) after it has been divided through by the numerically largest coefficient c_i. For example, if equation (8) is written in the form

$$2\frac{c_1}{c_3}\frac{\partial^4 U}{\partial x^4} + 2\frac{\partial^4 U}{\partial x^2\,\partial y^2} + 2\frac{c_5}{c_3}\frac{\partial^4 U}{\partial y^4} = 0,$$

[9] A discussion of these equations is too lengthy to be included in this paper.

and if we set

$$1 - \epsilon_x \equiv 2\frac{c_1}{c_3},$$

$$1 - \epsilon_y \equiv 2\frac{c_5}{c_3},$$

the equation assumes the form

$$(13) \qquad (1-\epsilon_x)\frac{\partial^4 U}{\partial x^4} + 2\frac{\partial^4 U}{\partial x^2\,\partial y^2} + (1-\epsilon_y)\frac{\partial^4 U}{\partial y^4} = 0,$$

where the ϵ_x and ϵ_y are new deviation parameters.[10]

If ϵ_x and ϵ_y are small compared with unity, it is reasonable to seek solution of equation (13) in the form

$$(14) \qquad U(x, y) = \sum_{i,j=0}^{\infty} U_{ij}(x, y)\epsilon_x^i\epsilon_y^j.$$

This time the procedure that has led to equations (10) leads to a simpler system of equations:

$$(15) \qquad \begin{aligned} &\nabla^4 U_{00} = 0, \\ &\nabla^4 U_{ij} = \frac{\partial^4}{\partial x^4} U_{i-1\,j} + \frac{\partial^4}{\partial y^4} U_{i\,j-1} \qquad (i, j = 1, 2, \cdots). \end{aligned}$$

Since the formal structure of these equations is typified by the system (11), we again seek the solution in the form (12) and obtain the following relations for the functions $\varphi_{ij\,m+2}$ $(m = 0, 1, 2, \cdots)$:

$$\begin{aligned} &16(m+2)!\varphi''_{i\,j\,m+2} \\ &\quad = [m!\varphi^{(iv)}_{i-1\,j\,m} + 4(m+1)!\varphi'''_{i-1\,j\,m+1} + 6(m+2)!\varphi''_{i-1\,j\,m+2} \\ &\qquad\qquad + 4(m+3)!\varphi'_{i-1\,j\,m+3} + (m+4)!\varphi_{i-1\,j\,m+4}] \\ &\quad + [m!\varphi^{(iv)}_{i\,j-1\,m} - 4(m+1)!\varphi'''_{i\,j-1\,m+1} + 6(m+2)!\varphi''_{i\,j-1\,m+2} \\ &\qquad\qquad - 4(m+3)!\varphi'_{i\,j-1\,m+3} + (m+4)!\varphi_{i\,j-1\,m+4}]. \end{aligned}$$

As in the former case, the functions φ_{ij0} and φ_{ij1} are subject to the determination from prescribed boundary conditions.[11]

Another variant of the perturbation scheme, leading to simple recursion formulas, makes use of equation (8) in the form[12]

$$(16) \qquad \frac{\partial^4 U}{\partial x^4} + 2\kappa\frac{\partial^4 U}{\partial x\,\partial \eta} + \frac{\partial^4 U}{\partial \eta^4} = 0,$$

---

[10] These parameters are obviously functions of ϵ_1 and ϵ_2 used in the preceding discussion, since the coefficients in equation (13) are symmetric functions of the roots μ_i of the characteristic equation.

[11] These computations were performed by Mr. Harold Luxenberg, Research Assistant, University of California (Los Angeles) who made use of them in the analysis of deflection of thin orthotropic elastic plates. In particular, this variant of the perturbational procedure leads readily to the known solution of the problem of deflection of a clamped elliptic plate subjected to a uniform normal load.

[12] This variant was considered by Mr. Julius Brandstatter, Research Assistant, University of California (Los Angeles), who is responsible for the calculations given in the remainder of this section.

which is obtained from equation (8) by making the transformation

$$\eta = \left(\frac{c_1}{c_5}\right)^{1/4} y.$$

The value of κ in terms of the original coefficients c_i is given by the expression $\kappa = c_3/(c_1c_5)^{1/2}$.

There are two cases to be considered, depending on whether $|\kappa|$ is less than or greater than 1.

If $|\kappa| < 1$, we set $\kappa = 1 - \epsilon$ and seek the solution in the form

$$U = \sum_{i=0}^{\infty} U_i(x, \eta)\epsilon^i. \tag{17}$$

The substitution of this assumed solution in equation (16) leads to the following relations to be satisfied by the functions $U_i(x, \eta)$:

$$\nabla^4 U_i = 2\frac{\partial^4 U_{i-1}}{\partial x^2\, \partial \eta^2}, \tag{18}$$

where

$$\nabla^4 \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \eta^2}\right).$$

Now, if we introduce a complex variable $z = x + i\eta$, we can rewrite the relations (18) in the form

$$\nabla^4 U_i = -2\left(\frac{\partial^4}{\partial z} - 2\frac{\partial^4}{\partial z\, \partial \bar{z}} + \frac{\partial^4}{\partial \bar{z}^4}\right) U_{i-1},$$

which are satisfied by

$$U_i = \mathcal{R} \sum_{m=0}^{2i+1} \varphi_{im}(z)\bar{z}, \tag{19}$$

in which the $\varphi_{im}(z)$ are analytic functions of z.

It is easy to verify that the functions φ_{im} must satisfy the conditions

$$\begin{aligned} 8(m+2)(m+1)\varphi''_{i\ m+2} = -\varphi^{(iv)}_{i-1\ m} &+ 2(m+2)(m+1)\varphi''_{i-1\ m+2} \\ &- (m+4)(m+3)(m+2)(m+1)\varphi_{i-1\ m+4}. \end{aligned}$$

If $|\kappa| > 1$, we set $\kappa = 1/(1 - \epsilon)$ and again seek the solution in the form (17). This time the U_i satisfy the differential equation

$$\nabla^4 U_i = \left(\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial \eta}\right) U_{i-1},$$

or

$$\nabla^4 U_i = 2\left(\frac{\partial^4}{\partial z^4} + 6\frac{\partial^4}{\partial z^2\, \partial \bar{z}^2} + \frac{\partial^4}{\partial \bar{z}^2}\right) U_{i-1},$$

where $z = x + i\eta$ and the functions $\varphi_{im}(z)$ in (19) are bound by the relations:

$$\begin{aligned} 8(m+2)(m+1)\varphi''_{i\ m+2} = \varphi^{(iv)}_{i-1\ m} &+ 6(m+2)(m+1)\varphi''_{i-1\ m+2} \\ &+ (m+4)(m+3)(m+2)(m+1)\varphi_{i-1\ m+4}. \end{aligned}$$

In both of the foregoing cases φ_{i0} and φ_{i1} are arbitrary functions determined by the boundary conditions.

We have already remarked that the first of the perturbation procedures described in this section can be applied to the general case of anisotropic media with only one plane of elastic symmetry which is typified by equation (1).

A special case of this procedure, applied to equation (16) is of interest.

The characteristic equation (4) associated with equation (16) is

$$\mu^4 + 2\kappa\mu^2 + 1 = 0.$$

If $\kappa > 1$, the roots of this equation are pure imaginary, and we consider the case for which the deviation parameters ϵ_i defined by

$$\mu_1 = i(1 + \epsilon_1), \qquad \mu_2 = i(1 + \epsilon_2), \qquad \mu_3 = \bar{\mu}_1, \qquad \mu_4 = \bar{\mu}_2$$

are less than unity. The ϵ_i are related to κ by the formula

$$2\kappa = 2(1 + \epsilon_1 + \epsilon_2) + \epsilon_1^2 + \epsilon_2^2.$$

We assume that U is represented in the double series

$$U = \sum_{i,j=0}^{\infty} U_{ij}(x, \eta)\epsilon_1^i\epsilon_2^j,$$

with $U_{ij} = U_{ji}$, and find that

$$\nabla^4 U_{ij} = -\frac{\partial^4}{\partial x^2\,\partial \eta^2}(2U_{i-1\,j} + 2U_{i\,j-1}U_{i-2\,j} + U_{i\,j-2}),$$

or

$$\nabla^4 U_{ij} = \left(\frac{\partial^4}{\partial z^4} - 2\frac{\partial^4}{\partial z^2\,\partial \bar{z}^2} + \frac{\partial^4}{\partial \bar{z}^4}\right)(2U_{i-1\,j} + 2U_{i\,j-1} + U_{i-2\,j} + U_{i\,j-2}),$$

where $z = x + i\eta$.

These equations have solutions of the form

$$U_{ij} = \mathcal{R}\sum_{m=0}^{2i+2j+1} \varphi_{ijm}(z)\bar{z}^m,$$

where the analytic functions φ_{ijm} satisfy the recursion relations:

$$\begin{aligned} &16(m+2)(m+1)\varphi''_{i\,j\,m+2} \\ &\quad = 2\varphi^{(iv)}_{i-1\,j\,m} + 2\varphi^{(iv)}_{i\,j-1\,m} + \varphi^{(iv)}_{i-2\,j\,m} + \varphi^{(iv)}_{i\,j-2\,m} \\ &\quad - 2(m+2)(m+1)(2\varphi''_{i-1\,j\,m+2} + 2\varphi''_{i\,j-1\,m+2} + \varphi''_{i-2\,j\,m+2} + \varphi''_{i\,j-2\,m+1}) \\ &\quad + (m+4)(m+3)(m+2)(m+1)(2\varphi_{i-1\,j\,m+4} + 2\varphi_{i\,j-1\,m+4} + \varphi_{i-2\,j\,m+4} \\ &\qquad\qquad + \varphi_{i\,j-2\,m+4}). \end{aligned}$$

The functions φ_{ij0} and φ_{ij1} remain arbitrary and are subject to the determination from the boundary conditions.

It should be emphasized that the perturbational procedures described in this section cannot be expected to yield satisfactory solutions where the media under consideration are highly anisotropic. However, computations now in progress indicate that useful approximate solutions can be obtained even for such anisotropic materials as wood.

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