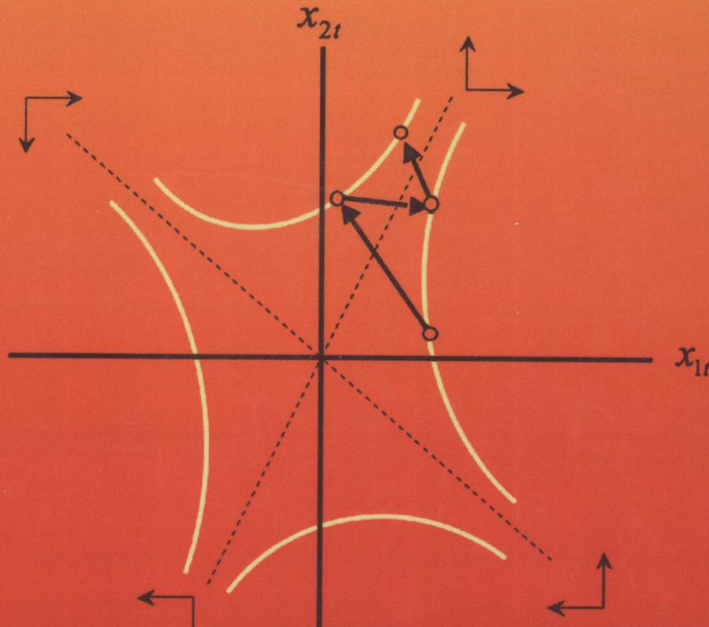
ODED GALOR

# Discrete Dynamical Systems

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# Preface

This book provides an introduction to discrete dynamical systems – a framework of analysis that is commonly used in the fields of biology, demography, ecology, economics, engineering, finance, and physics.

The book characterizes the fundamental factors that govern the quantitative and qualitative trajectories of a variety of deterministic, discrete dynamical systems, providing solution methods for systems that can be solved analytically and methods of qualitative analysis for those systems that do not permit or necessitate an explicit solution.

The analysis focuses initially on the characterization of the factors that govern the evolution of state variables in the elementary context of one-dimensional, first-order, linear, autonomous systems. The fundamental insights about the forces that affect the evolution of these elementary systems are subsequently generalized, and the determinants of the trajectories of multi-dimensional, nonlinear, higher-order, non-autonomous dynamical systems are established.[1]

Chapter 1 focuses on the analysis of the evolution of state variables in one-dimensional, first-order, autonomous systems. It introduces a method of solution for these systems, and it characterizes the trajectory of a state variable, in relation to a steady-state equilibrium of the system, examining the local and global (asymptotic) stability of this steady-state equilibrium. The first part of the chapter characterizes the factors that determine the existence, uniqueness and stability of a steady-state equilibrium in the elementary context of one-dimensional, first-order, linear autonomous systems. Although linear dynamical systems do not govern the evolution of the majority of the observed dynamic phenomena, they serve as an important benchmark in the analysis of the qualitative properties of the nonlinear systems in the

[1] For continuous dynamical systems see Arnold (1973), Hirsch and Smale (1974), and Hale (1980).

proximity of steady-state equilibria. The second part of the chapter examines the trajectories of nonlinear systems based on the characterization of the linearized system in the proximity of a steady-state equilibrium. The basic propositions established in Chapter 1 provide the conceptual foundations for the analysis of multi-dimensional, higher-order, non-autonomous, dynamical systems.

Chapter 2 analyzes the evolution of a vector of interdependent state variables in multi-dimensional, first-order dynamical systems. It develops a method of solution for these systems, based on the construction of a time-independent transformation that converts the dynamical system into a new one that is characterized by either independent state variables whose evolution can be determined based on the analysis of the one-dimensional case, or partially dependent state variables whose evolution are determined by the well established properties of the Jordan matrix. The analysis of linear multi-dimensional dynamical systems provides an important reference point in the analysis of multi-dimensional nonlinear systems in the proximity of their steady-state equilibrium. It provides the characterization of the linear approximation of multi-dimensional nonlinear systems around steady-state equilibria.

Chapter 3 characterizes the trajectory of a vector of state variables in multi-dimensional, first-order, linear dynamical systems. It examines the trajectories of these systems when the matrix of coefficients has real eigenvalues and the vector of state variables converges or diverges in a monotonic or oscillatory fashion towards or away from a steady-state equilibrium that is characterized by either a saddle point or a stable or unstable (improper) node. In addition, it examines the trajectories of these linear dynamical systems when the matrix of coefficients has complex eigenvalues and the system is therefore characterized by a spiral sink, a spiral source, or a periodic orbit.

Chapter 4 analyzes the trajectory of a vector of state variables in multi-dimensional, first-order, nonlinear systems. It utilizes the characterization of linear multi-dimensional systems to examine the trajectory of the nonlinear systems in light of the *Stable Manifold Theorem*. In particular, the analysis examines the properties of the local stable and unstable manifolds, and the corresponding global stable and unstable manifolds.

Chapter 5 characterizes the evolution of a vector of state variables in higher-order as well as non-autonomous systems. It establishes the solution method for these higher-order and non-autonomous systems and it analyzes the factors that determine the qualitative properties of these discrete dynamical systems in the linear and subsequently the nonlinear case. The analysis is based upon the transformation of higher-order and non-autonomous systems into a multi-dimensional first-order systems that can be examined based on the analysis in Chaps. 2–4. In particular, a one-dimensional second-order system is converted into a two-dimensional first-order system, a one-dimensional third-order system is transformed into a three-dimensional first-order system, a one-dimensional n$^{th}$-order system is converted into an n-dimensional first-order system, and an $n$- dimensional $m^{th}$-order system is transformed into an $n \times m$- dimensional first-order system. Similarly, the analysis of non-autonomous systems is based on their transformation into higher-dimension, time-independent (autonomous) systems that can be examined based on the analysis of multi-dimensional, first-order systems in Chaps. 2–4.

Chapter 6 provides a complete characterization of several representative examples of two-dimensional dynamical systems. These examples include a first-order linear system with real eigenvalues, a first-order linear system with complex eigenvalues that exhibits a periodic orbit, a first-order linear system with complex eigenvalues that exhibits a spiral sink, a first-order nonlinear system that is characterized by a oscillatory convergence, and a second-order one-dimensional system converted into a first-order, two-dimensional system characterized by a continuum of equilibria and oscillatory divergence.

The book is designed for advanced undergraduate and graduate students in the fields of demography, ecology, economics, engineering, evolutionary biology, finance, mathematics, and physics, who are familiar with differential calculus and linear algebra. Furthermore, it is a useful reference for researchers of applied disciplines in which discrete dynamical systems are commonly employed.

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# Contents

# 1

# One-Dimensional, First-Order Systems

This chapter analyzes the evolution of a state variable in one-dimensional, first-order, discrete dynamical systems. It introduces a method of solution for these systems, and it characterizes the trajectory of the state variable, in relation to its steady-state equilibrium, examining the local and global (asymptotic) stability of this steady-state equilibrium.

The first part of the chapter characterizes the factors determining the existence, uniqueness and stability of a steady-state equilibrium in the elementary context of one-dimensional, first-order, linear autonomous systems. Although linear dynamical systems do not necessarily govern the evolution of the majority of the observed dynamic phenomena, they serve as an important benchmark in the analysis of the qualitative properties of nonlinear systems, providing the characterization of the linear approximation of nonlinear systems in the proximity of steady-state equilibria. The second part of the chapter examines the trajectories of nonlinear systems based on the characterization of the linearized system in the proximity of a steady-state equilibrium.

The basic propositions derived in this chapter provide the conceptual foundations for the generalization of the analysis and the characterization of multi-dimensional, higher-order, non-autonomous, dynamical systems.

The qualitative analysis of these dynamical systems is based upon the examination of the factors that determine the actual trajectory of the state variable. However, as will become apparent, once the basic propositions that characterize the properties of these systems are derived, an explicit solution is no longer required in order to characterize the nature of these dynamical systems.

## 1.1 Linear Systems

Consider a one-dimensional, first-order, autonomous, linear difference equation that governs the evolution of a state variable, $y_t$, over time.

$$y_{t+1} = ay_t + b, \qquad t = 0, 1, 2, 3, \cdots, \tag{1.1}$$

where the value of the *state variable* at time $t$, $y_t$, is a real number, i.e., $y_t \in \Re$, the parameters $a$ and $b$ are constant real numbers, namely $a, b \in \Re$, and the initial value of the state variable at time 0, $y_0$, is given.[1]

The system is defined as a one-dimensional, first-order, autonomous, linear difference equation since it describes the evolution of a *one-dimensional* state variable, $y_{t+1}$, whose value depends in a *linear* and *time-independent* (autonomous) fashion on its value in the pervious period (first-order), $y_t$.

### 1.1.1 Characterization of the Solution

A solution to the difference equation $y_{t+1} = ay_t + b$ is a *trajectory* (or an *orbit*) of the state variable, $\{y_t\}_{t=0}^{\infty}$, that satisfies this law of motion at any point in time. It relates the value of the state variable at time $t$, $y_t$, to its initial value, $y_0$, and to the parameters $a$ and $b$.

The derivation of a solution may follow several methods. In particular, the intuitive method of iterations generates a pattern that can be easily generalized to a solution rule.

Given the value of the state variable at time 0, $y_0$, the dynamical system $y_{t+1} = ay_t + b$ implies that the value of the state variable at time 1, $y_1$, is

$$y_1 = ay_0 + b. \tag{1.2}$$

Given the value of the state variable at time 1, $y_1$, the value of the state variable at time 2, $y_2$, is uniquely determined.

$$y_2 = ay_1 + b = a(ay_0 + b) + b = a^2 y_0 + ab + b. \tag{1.3}$$

[1] Without loss of generality, the feasible domain of the time variable, $t$, is truncated to be the set of non-negative integers. Moreover, the initial condition is defined as the value of the state variable at time 0. In general, $t$ can be defined to be an element of any subset of the set of integers, and the initial value of the state variable, $y_0$, can be given at any point within this interval.

Similarly, the value of the state variable at time $3, 4, ..., t$, is

$$\begin{aligned} y_3 &= ay_2 + b = a(a^2 y_0 + ab + b) + b = a^3 y_0 + a^2 b + ab + b \\ &\vdots \qquad\qquad\qquad \vdots \\ y_t &= a^t y_0 + a^{t-1} b + a^{t-2} b + ... + ab + b. \end{aligned} \tag{1.4}$$

Hence, for $t = 1, 2, ...,$

$$y_t = a^t y_0 + b \sum_{i=0}^{t-1} a^i. \tag{1.5}$$

Since $\sum_{i=0}^{t-1} a^i$ is the sum of the geometric series, $\{1, a, a^2, a^3, ..a^{t-1}\}$, whose factor is $a$, it follows that

$$\sum_{i=0}^{t-1} a^i = \begin{cases} \frac{1-a^t}{1-a} & if \quad a \neq 1 \\ t & if \quad a = 1, \end{cases} \tag{1.6}$$

and therefore

$$y_t = \begin{cases} a^t y_0 + b\frac{1-a^t}{1-a} & if \quad a \neq 1 \\ y_0 \quad + bt & if \quad a = 1. \end{cases} \tag{1.7}$$

Alternatively,

$$y_t = \begin{cases} [y_0 - \frac{b}{1-a}]a^t + \frac{b}{1-a} & if \quad a \neq 1 \\ y_0 + bt & if \quad a = 1. \end{cases} \tag{1.8}$$

Thus, as long as an initial condition of the state variable is given, the entire trajectory of the state variable is uniquely determined.

The trajectory derived in (1.8) reveals the qualitative role of the parameter $a$, and to a lesser extent $b$, in the evolution of the state variable over time. These parameters determine whether the dynamical system evolves monotonically or in oscillations, and whether the state variable converges in the long run to a steady-state equilibrium, diverges asymptotically to plus or minus infinity, or displays a two-period cycle. Hence, a qualitative examination of a dynamical system requires the analysis of the asymptotic behavior of the system as time approaches infinity.

### 1.1.2 Existence of Steady-State Equilibria

Steady-state equilibria provide an essential reference point for a qualitative analysis of the behavior of dynamical systems. A *steady-state equilibrium* (alternatively defined as a *stationary equilibrium*, a *rest point*, an *equilibrium point*, or a *fixed point*) is a value of the state variable $y_t$ that is invariant under the law of motion dictated by the dynamical system.

**Definition 1.1.** *(A Steady-State Equilibrium)*
*A steady-state equilibrium of the difference equation* $y_{t+1} = ay_t + b$ *is* $\overline{y} \in \Re$ *such that*

$$\overline{y} = a\overline{y} + b.$$

Thus, if the state variable is at a steady-state equilibrium, it will remain there in the absence of any perturbations of the dynamical system due to either changes in the parameters $a$ and $b$ or direct perturbations in the value of the state variable itself. Namely, if $y_t = \bar{y}$ then $y_s = \bar{y}$ for all $s > t$.

As follows from Definition 1.1, as long as $a \neq 1$, there exists a unique steady-state equilibrium $\bar{y} = b/(1-a)$ for the difference equation $y_{t+1} = ay_t + b$. However, given the linear structure of the dynamical system, if $a = 1$ and $b = 0$ then in every time $t$, $y_{t+1} = y_t$ and the state variable does not deviate from its initial condition. In particular, $y_t = y_{t-1} = y_{t-2} = ... = y_0$ and the system is in a steady-state equilibrium where $\bar{y} = y_0$. In contrast, if $a = 1$ and $b \neq 0$, a steady-state equilibrium does not exist, and the state variable increases indefinitely if $b > 0$, or decreases indefinitely if $b < 0$.

Hence, following Definition 1.1,

$$\overline{y} = \begin{cases} \frac{b}{1-a} & if \quad a \neq 1 \\ \\ y_0 & if \quad a = 1 \text{ and } b = 0. \end{cases} \tag{1.9}$$

Thus, the necessary and sufficient conditions for the existence of a steady-state equilibrium are given by the values of the parameters $a$ and $b$, as stated in (1.9), that permit the system to have a steady-state equilibrium.

**Proposition 1.2.** *(Existence of Steady-State Equilibrium)*
*A steady-state equilibrium of the difference equation* $y_{t+1} = ay_t + b$ *exists if and only if*

$$\{a \neq 1\} \text{ or } \{a = 1 \text{ and } b = 0\}.$$

Hence, given the steady-state level of the state variable, $y_t$, as derived in (1.9), the solution to the difference equation $y_{t+1} = ay_t + b$ can be expressed in terms of the deviations of the initial value of the state variable, $y_0$, from its steady-state value, $\bar{y}$. Namely, substituting the value of $\bar{y}$ into the solution given by (1.8), it follows that

$$y_t = \begin{cases} (y_0 - \bar{y})a^t + \bar{y} & if \quad a \neq 1 \\ y_0 + bt & if \quad a = 1. \end{cases} \tag{1.10}$$

### 1.1.3 Uniqueness of Steady-State Equilibria

A steady-state equilibrium of the linear dynamical system, $y_{t+1} = ay_t + b$, is not necessarily unique. As depicted in Figs. 1.1, 1.3, 1.7, 1.9 and 1.10 for $a \neq 1$, the steady-state equilibrium is unique. However, as depicted in Fig. 1.5, for $a = 1$ and $b = 0$, a continuum of steady-state equilibria exists, reflecting the entire set of feasible initial conditions.

Necessary and sufficient conditions for the uniqueness of a steady-state equilibrium are given by the values of the parameters $a$ and $b$, as stated in (1.9), that permits the system to have a distinct steady-state equilibrium.

**Proposition 1.3.** *(Uniqueness of Steady-State Equilibrium)*
*A steady-state equilibrium of the difference equation* $y_{t+1} = ay_t + b$ *is unique if and only if*

$$a \neq 1.$$

### 1.1.4 Stability of Steady-State Equilibria

The stability analysis of the system's steady-state equilibria determines whether a steady-state equilibrium is attractive or repulsive for all or at least some set of initial conditions. It facilitates the study of the local, and often the global, properties of a dynamical system, and it permits the analysis of the implications of small, and sometimes large, perturbations that occur once the system is in the vicinity of a steady-state equilibrium.

A steady-state equilibrium is *globally* (asymptotically) stable if the system converges to this steady-state equilibrium regardless of the level of the initial condition, whereas a steady-state equilibrium is *locally* (asymptotically) stable if there exists an $\epsilon$- neighborhood of the steady-state equilibrium such that from every initial condition within this neighborhood the system converges to this steady-state equilibrium. Formally the definitions of local and global stability are as follows:[2]

**Definition 1.4.** *(Local and Global Stability of a Steady-State Equilibrium)*
*A steady-state equilibrium, $\overline{y}$, of the difference equation $y_{t+1} = ay_t + b$ is*

- *globally (asymptotically) stable if*

$$\lim_{t\to\infty} y_t = \overline{y} \qquad \forall y_0 \in \Re;$$

- *locally (asymptotically) stable if*

$$\lim_{t\to\infty} y_t = \overline{y} \quad \forall y_0 \text{ such that } |y_0 - \overline{y}| < \epsilon \text{ for some } \epsilon > 0.$$

Alternatively, if the state variable is in a steady-state equilibrium and upon a sufficiently small perturbation it converges asymptotically back to this steady-state equilibrium, then this equilibrium is *locally* stable. However, if regardless of the magnitude of the perturbation the

[2] The economic literature, to a large extent, refers to the stability concepts in Definition 4.2 as global stability and local stability, respectively, whereas the mathematical literature refers to them as global asymptotic stability and local asymptotic stability, respectively. The concept of stability in the mathematical literature is reserved to situations in which trajectories that are initiated from an $\epsilon$-neighborhood of a fixed point remain sufficiently close to this fixed point thereafter.

state variable converges asymptotically to this steady-state equilibrium, then the equilibrium is *globally* stable.

*Global stability* of a steady-state equilibrium necessitates the *global uniqueness* of the steady-state equilibrium. If there is more than one steady-state equilibrium, none of the equilibria can be globally stable since there exist at least two points in the relevant space from which there is no escape and convergence from each of these steady-state equilibria to the other steady-state equilibrium is therefore not feasible.

**Proposition 1.5.** *(Necessary Condition for Global Stability of Steady-State Equilibrium)*
*A steady-state equilibrium of the difference equation* $y_{t+1} = ay_t + b$ *is globally (asymptotically) stable only if the steady-state equilibrium is unique.*

*Local stability* of a steady-state equilibrium necessitates the *local uniqueness* of the steady-state equilibrium. Namely, the absence of any additional point in the neighborhood of the steady-state from which there is no escape. If the system is characterized by a continuum of equilibria none of these steady-state equilibria is locally stable. There exists no neighborhood of a steady-state equilibrium that does not contain additional steady-state equilibria, and hence there exist initial conditions within an $\varepsilon$- neighborhood of a steady-state equilibrium that do not lead to this steady-state equilibrium in the long run. Thus, local stability of a steady-state equilibrium requires the local uniqueness of this steady-state equilibrium.

If the system is linear there is either unique steady-state equilibrium or continuum of (unstable) steady-state equilibria. Local uniqueness of a steady-state equilibrium therefore implies global uniqueness, and local stability therefore necessarily implies global stability.

As follows from the definitions of local and global stability, the stability of a steady-state equilibrium can be obtained by the examination of the properties of the system as time approaches infinity.

As follows from the solution for the difference equation $y_{t+1} = ay_t + b$, given by (1.10),

$$\lim_{t\to\infty} y_t = \begin{cases} [y_0 - \overline{y}] \lim_{t\to\infty} a^t + \overline{y} & \text{if} \quad a \neq 1 \\ y_0 + b \lim_{t\to\infty} t & \text{if} \quad a = 1, \end{cases} \tag{1.11}$$

and therefore the limit of the absolute value of the state variable, $|y_t|$, is

$$\lim_{t\to\infty}|y_t| = \begin{cases} |\overline{y}| & if \ \{|a|<1\} \ or \\ & \quad \{|a|>1 \ \& \ y_0=\overline{y}\} \\ |y_0| & if \ \ a=1 \ \& \ b=0 \\ \left.\begin{array}{ll} |y_0| & for \ t=0,2,4,... \\ |b-y_0| & for \ t=1,3,5,... \end{array}\right\} & if \ \ a=-1 \\ \infty & otherwise. \end{cases} \tag{1.12}$$

Thus, as follows from the property of the absolute value of the state variable $y_t$, as time approaches infinity, the absolute value of the parameter $a$ and the value of $b$ determine the long run value of the state variable. Moreover, the absolute value of the parameter $a$ determines whether a steady-state equilibrium $\bar{y}$ is globally stable.

In particular, in the feasible range of the parameter $a$ and $b$, the dynamical system exhibits five qualitatively different trajectories, characterized by the existence of a unique and globally stable steady-state equilibrium, a unique unstable, steady-state equilibrium, continuum of steady-state equilibria, inexistence of steady-state equilibria, and two-period cycles.

### A. Unique Globally Stable Steady-State Equilibrium ($|a| < 1$)

If the coefficient $|a| < 1$, then the system is globally (asymptotically) stable converging to the steady-state equilibrium $\overline{y} = b/(1-a)$, regardless of the initial condition, $y_0$. In particular, if $0 < a < 1$, then as depicted in the phase diagram in Fig. 1.1, the evolution of the state variable is characterized by monotonic convergence towards the steady-state equilibrium $\bar{y}$ regardless of the initial level of the state variable, $y_0$.

The steady state locus $y_{t+1} = y_t$ intersects with the linear difference equation, $y_{t+1} = ay_t + b$, at the steady-state equilibrium $\bar{y}$. Given $y_0$, the value of $y_1 = ay_0 + b$ can be read from corresponding value along the line $y_{t+1} = ay_t + b$. This value of $y_1$ can be mapped back to the $y_t$ axis via the $45^0$ line. Similarly, given $y_1$, the value of $y_2 = ay_1 + b$ can be read from the corresponding value along the line $y_{t+1} = ay_t + b$ and mapped back to the $y_t$ axis via the $45^0$ line. Hence, as depicted in Fig. 1.1, the state variable evolves along the depicted arrows of motion and converges monotonically to the steady-state equilibrium $\bar{y}$.