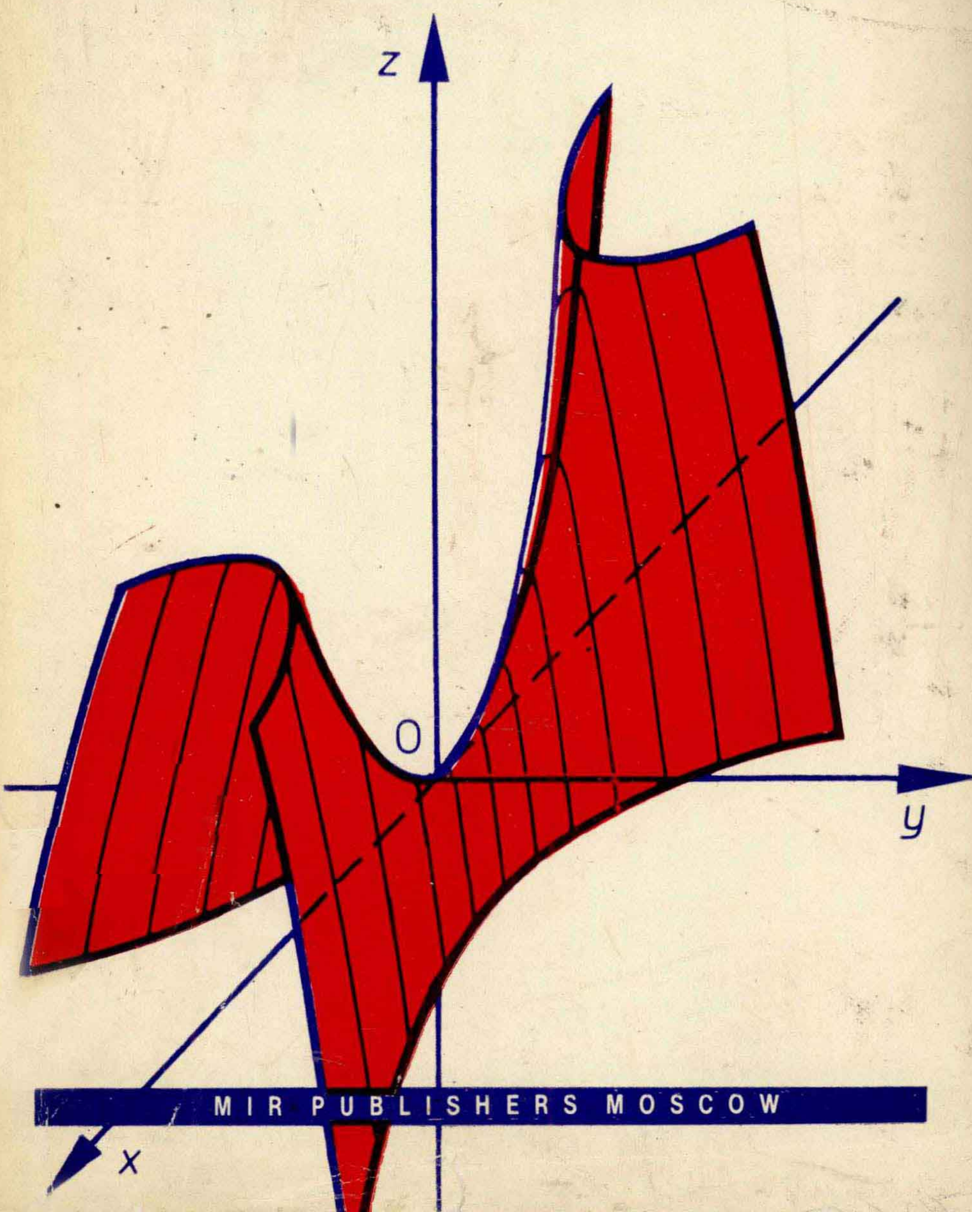
HIGHER MATHEMATICS

EDITED BY G.N. YAKOVLEV



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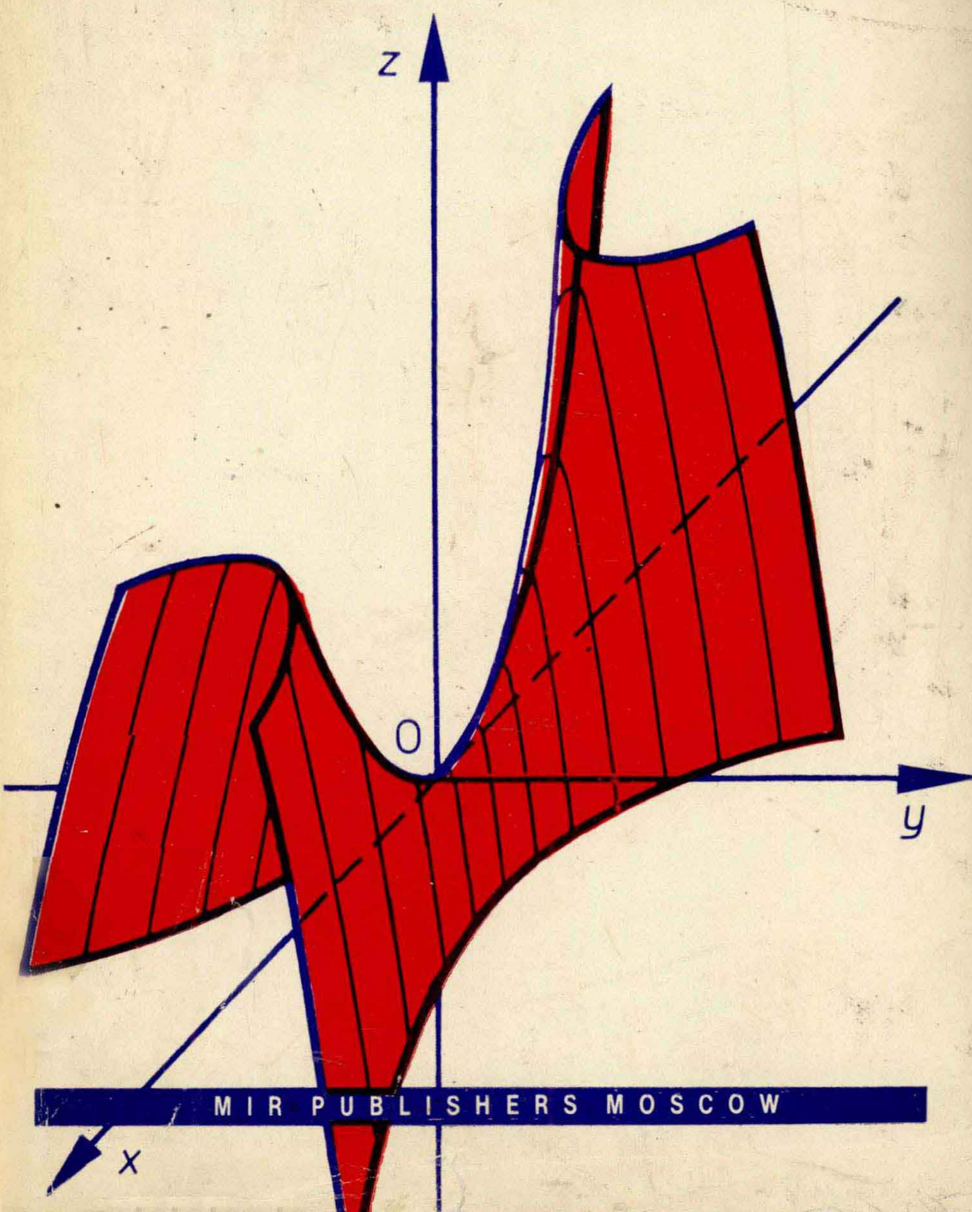

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HIGHER MATHEMATICS

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PREFACE TO THE ENGLISH EDITION

Mathematical methods are used in every field of science and engineering. This means that students, whatever their speciality, must have a solid theoretical grounding in mathematics and in solving practical problems. Mathematical analysis is the foundation of the mathematical education of scientists and engineers.

This study aid was written in accordance with the new curriculum on higher mathematics and is intended for students of the industrial and education faculties of teacher training institutes. Attention is given to the applied and practical approach of the text and the application of computers.

The book is in two parts: Part 1. Analytic Geometry (Chapters 1-3) and Part 2. Mathematical Analysis (Chapters 4-12).

Chapter 1 provides all necessary theoretical elements of vector and linear algebra. Chapter 2 deals with analytic geometry in the plane, while Chapter 3 with analytic geometry in space. Chapter 4 treats real and complex numbers. Chapter 5 is dedicated to functions, sequences, and limits. Elements of differential calculus are discussed in Chapter 6 (functions of one variable) and Chapter 10 (functions of several variables). Chapters 7 and 8 deal with indefinite and definite integrals, respectively. Series are considered in Chapter 9. Chapter 11 provides a study of differential equations, and Chapter 12 is dedicated to the integral calculus of functions of several variables.

Each chapter contains a large number of examples and worked problems to make the subject matter more readily comprehensible and is amply supplied with exercises.

As a rule, worked examples in each section are arranged in order of increasing difficulty. The beginning and end of the proof of a statement are marked by the symbols □ and ■, respectively. The solution of each example is preceded by the symbol △, the end of the solution being symbolized by ▲.

The Authors

Part I

ANALYTIC GEOMETRY

Chapter 1

ELEMENTS OF VECTOR AND LINEAR ALGEBRA

1.1. Vectors. Operations on Vectors

1. Scalar and Vector Quantities. It is known that some quantities are completely defined by their numerical value (with a certain unit of measurement chosen). Such quantities are called scalar (or numerical). Length, area, volume, mass, and temperature are examples of scalar quantities. Other quantities, as, for instance, displacement, force, acceleration, and velocity, are defined not only by their numerical value, but also by direction in space. These are called vector quantities, or simply vectors. Returning to scalar quantities, we may give the following definition: a scalar quantity (or scalar) is a quantity that does not possess direction.

Geometrically, a vector quantity (a vector) can be represented with the help of a directed line segment whose length in a given scale of measurement is equal to the numerical value of the vector quantity and whose direction coincides with the direction of this quantity. A directed line segment is represented by an arrow. Figure 1 shows a directed line segment with initial point (or origin) O and terminal point A. It is denoted as $\overrightarrow{OA}$. Any other directed line segment is denoted in a similar way, provided that its beginning and terminus are known. For example, $\overrightarrow{AB}$ denotes a directed line segment with initial point A and terminal point B. The length of the directed line segment $\overrightarrow{AB}$ is denoted by $|\overrightarrow{AB}|$.

Thus, a vector is a quantity which can be represented by a directed line segment. This directed line segment is called the vector representing a given vector quantity. Thus, in geometry, a vector is a directed line segment.

Definition 1. Two vectors are equal if they are of the same length, lie on parallel or coinciding lines, and are in the same direction.

For example, in Fig. 2, where $ABCD$ is a rhombus, the vectors $\overrightarrow{AD}$ and $\overrightarrow{BC}$ are equal since $|\overrightarrow{AD}| = |\overrightarrow{BC}|$, $AD \parallel BC$ and the segments $\overrightarrow{AD}$ and $\overrightarrow{BC}$ are in the same direction. In this case we write: $\overrightarrow{AD} = \overrightarrow{BC}$.

The vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$ in Fig. 2 are not equal to each other since the straight lines AB and AD are not parallel. In this case we write: $\overrightarrow{AB} \neq \overrightarrow{AD}$. Further, since $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are in opposite directions, we have $\overrightarrow{AB} \neq \overrightarrow{CD}$.

A vector can also be denoted by a single letter as in Fig. 4. In printing this letter is given in boldface type **a**, in writing it is given with

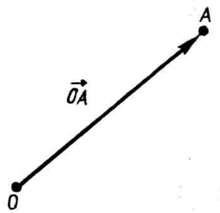

Fig. 1

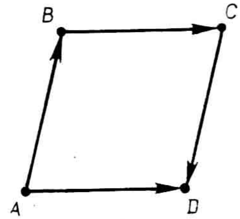

Fig. 2

a bar $\bar{a}$. If it is known that the vector **a** begins at the point A and terminates at the point B, then we write: $\mathbf{a} = \overrightarrow{AB}$. The length of the vector **a** is denoted by two vertical lines: $|\mathbf{a}|$. Sometimes, the length of a vector is called the absolute value.

A vector whose length is equal to zero is called the null vector. The null vector is symbolized by **0**, the number zero. The origin A of a null vector coincides with its terminus B, that is, $\mathbf{0} = \overrightarrow{AA} = \overrightarrow{BB}$.

Every nonzero vector is defined by length and direction. The null vector has no direction.

Definition 2. Two nonzero vectors whose directions coincide or are opposite are termed collinear. The null vector is considered collinear with any vector.

If two vectors **a** and **b** are collinear, then we write: $\mathbf{a} \parallel \mathbf{b}$. It is obvious that collinear vectors are those lying on parallel or coinciding straight lines. In Fig. 2, $\overrightarrow{AD} \parallel \overrightarrow{BC}$, $\overrightarrow{AB} \parallel \overrightarrow{CD}$, but $\overrightarrow{AB} \nparallel \overrightarrow{BC}$.

Definition 3. Two collinear vectors having the same absolute value (length) and opposite directions are called opposite vectors.

A vector which is in the direction opposite to a vector **a** is denoted by $-\mathbf{a}$. It is obvious that the vector $\overrightarrow{BA}$ is opposite to the vector $\overrightarrow{AB}$, and vice versa, that is, $\overrightarrow{BA} = -\overrightarrow{AB}$ and $\overrightarrow{AB} = -\overrightarrow{BA}$. In Fig. 2, $\overrightarrow{AB} = -\overrightarrow{CD}$.

Definition 4. Vectors in space are called coplanar if they are parallel to the same plane.

It is obvious that any two vectors in space are coplanar. In Fig. 3, where $ABCDEFGH$ is a parallelepiped, the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{AD}$, $\overrightarrow{FE}$, $\overrightarrow{FG}$, $\overrightarrow{HG}$, and $\overrightarrow{HE}$ are coplanar.

2. The Angle Between Two Vectors. The Projection of a Vector on an Axis. Suppose we are given two vectors **a** and **b**. We lay off these vectors from a certain point O, that is, we construct vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ such that $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$ (Fig. 4). Then the size of the interior angle AOB of the triangle AOB is defined as the angle between the vectors **a** and **b** and is denoted by $\widehat{(\mathbf{a}, \mathbf{b})}$ or by letters φ, ψ, etc. By definition, the angle between two vectors lies in the range

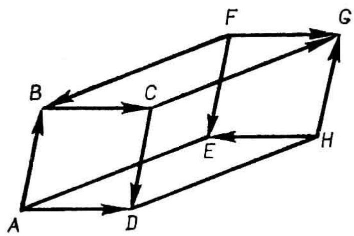

Fig. 3

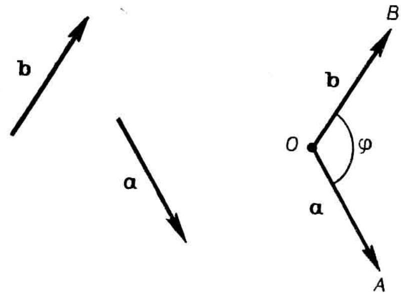

Fig. 4

from 0° to 180° (or from 0 to π radians). The angle between collinear vectors is equal to zero if they are in the same direction, and to 180° if they are in opposite directions.

If the angle between vectors is 90°, then these vectors are called perpendicular or orthogonal. If the vectors **a** and **b** are perpendicular, then we write: $\mathbf{a} \perp \mathbf{b}$.

Recall that an axis is defined as a straight line with a reference point, positive direction, and a length unit (scale) chosen. It is obvious that positive direction and unit of length on the axis can be specified with the help of a unit vector, that is, the vector whose length is equal to the unit of length. This vector is termed the unit vector of the axis. Figure 5 represents the axis l, where O is the reference point, and **e** the unit vector. The angle between the vector **a** and the axis l is defined as the angle between the vectors **a** and **e**.

Definition. The orthogonal projection (or simply projection) of a vector on an axis is a number equal to the product of the length of this vector by the cosine of the angle between the vector and axis.

The projection of the vector **a** on the axis l is denoted by the symbol $\text{pr}_l \mathbf{a}$. Thus, by definition

$$\text{pr}_l\, \mathbf{a} = |\, \mathbf{a}\, | \cos \varphi, \qquad (1)$$

where φ is the angle between the vector **a** and the axis l. If the angle between the vectors **a** and **e** is acute (Fig. 5, *a*), then the projec-

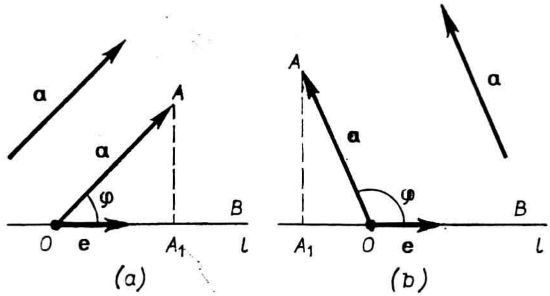

Fig. 5

tion of the vector **a** on the axis l is equal to the length of the line segment OA_1, where the point A_1 is the projection of the point A on the straight line l. Indeed,

$$\text{pr}_l\, \mathbf{a} = |\, \mathbf{a}\, | \cos \varphi = |\, \overrightarrow{OA}\, | \cos \angle AOA_1 = OA_1.$$

If the angle between the vectors **e** and **a** is obtuse (as in Fig. 5, *b*), then the projection of the vector **a** on the axis l is equal to the length of the line segment OA_1 taken with the minus sign. Indeed,

$$\text{pr}_l\, \mathbf{a} = |\, \mathbf{a}\, | \cos \varphi = |\, \mathbf{a}\, | \cos \angle BOA = -|\, \overrightarrow{OA}\, | \cos \angle A_1OA$$
$$= -OA_1.$$

If the vector **a** is perpendicular to the axis l, that is, $\varphi = 90°$, then $\text{pr}_l\, \mathbf{a} = |\, \mathbf{a}\, | \cos 90° = 0$.

Example. Compute the projection of the vector **a** on the axis l if $|\, \mathbf{a}\, | = 6$, and the angle φ between the vector and the axis is equal to $\frac{2\pi}{3}$.

△ We compute by the formula (1): $\text{pr}_l\, \mathbf{a} = |\mathbf{a}| \cos \varphi = 6 \cos \frac{2\pi}{3} = 6 \cos \left(\frac{\pi}{2} + \frac{\pi}{6}\right) = -6 \sin \frac{\pi}{6} = -3$. ▲

3. The Sum of Vectors. Let a body move from the point A to the point B, and then from the point B to the point C (Fig. 6). In mechanics the vector $\overrightarrow{AB}$ is called the displacement of the body from the point A to the point B. Similarly, the vector $\overrightarrow{BC}$ is the displacement of the body from the point B to the point C, while the vector $\overrightarrow{AC}$ is the displacement from the point A to the point C. The vector $\overrightarrow{AC}$ is then the sum of the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$. In this case we write

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}.$$

Let us give the general definition for the sum of two vectors.

Definition. Let two vectors $\mathbf{a}$ and $\mathbf{b}$ be given. We lay off the vector $\mathbf{a}$ from some point A: $\mathbf{a} = \overrightarrow{AB}$, and then the vector $\mathbf{b}$ from the

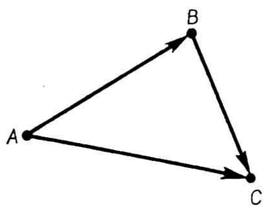

Fig. 6

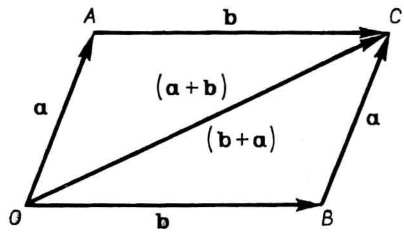

Fig. 7

point B: $\mathbf{b} = \overrightarrow{BC}$. Then the vector $\mathbf{c} = \overrightarrow{AC}$ is said to be the sum of the vectors $\mathbf{a}$ and $\mathbf{b}$. In this case we write:

$$\mathbf{c} = \mathbf{a} + \mathbf{b}.$$

The operation of finding the sum of vectors is called the addition of vectors. The above formulated rule for adding vectors is referred to as the triangle rule.

The operation of addition of vectors possesses the properties of commutativity and associativity, that is, for any vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ the following equalities hold true:

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \text{ (commutativity)},$$

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) \text{ (associativity)}.$$

□ Let us first prove the commutative property. We lay off the vectors $\mathbf{a}$ and $\mathbf{b}$ from some point O: $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$. Consider the case when the vectors $\mathbf{a}$ and $\mathbf{b}$ are noncollinear, that is, when the points O, A and B do not lie on the same straight line. On the line segments OA and OB we complete a parallelogram $OBCA$ (Fig. 7), then

$$\mathbf{a} + \mathbf{b} = \overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC},$$

$$\mathbf{b} + \mathbf{a} = \overrightarrow{OB} + \overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OC},$$

and, consequently, $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.

For the case when the vectors **a** and **b** are collinear, the proof is left to the student.

Let us now prove the associative property. We lay off the vector $\overrightarrow{OA} = \mathbf{a}$ from some point O, the vector $\overrightarrow{AB} = \mathbf{b}$ from the point A, finally the vector $\overrightarrow{BC} = \mathbf{c}$ from the point B (Fig. 8). Then

$$(\mathbf{a}+\mathbf{b})+\mathbf{c} = (\overrightarrow{OA}+\overrightarrow{AB})+\overrightarrow{BC} = \overrightarrow{OB}+\overrightarrow{BC} = \overrightarrow{OC},$$

$$\mathbf{a}+(\mathbf{b}+\mathbf{c}) = \overrightarrow{OA}+(\overrightarrow{AB}+\overrightarrow{BC}) = \overrightarrow{OA}+\overrightarrow{AC} = \overrightarrow{OC},$$

and consequently, $(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c})$. ■

Since the addition of vectors is associative, the sum of three and more vectors is written without parentheses. For instance, instead

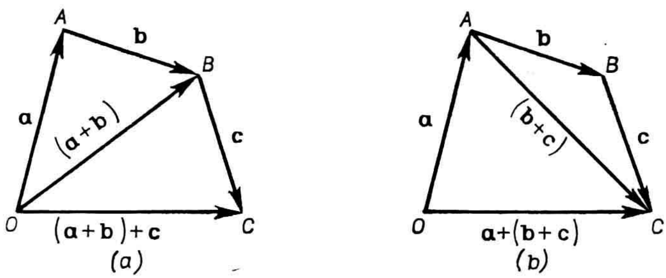

Fig. 8

of $(\mathbf{a}+\mathbf{b})+\mathbf{c}$ or $\mathbf{a}+(\mathbf{b}+\mathbf{c})$, we write: $\mathbf{a}+\mathbf{b}+\mathbf{c}$. If it is required to find the sum of three or more vectors, then we apply the so-called polygon rule (for the three-dimensional case, it would be more exact to call this rule the polygonal-line rule), which is the consequence of the triangle rule for adding two vectors. It consists in the following. Let us be given vectors **a**, **b**, **c**, **d**, etc. We choose some point O and construct the line segment $\overrightarrow{OA} = \mathbf{a}$, then the line segment $\overrightarrow{AB} = \mathbf{b}$, and so forth until all the vectors to be added are exhausted. The directed line segment emanating from the first vector and terminating at the terminus of the last vector thus closing the polygonal line obtained in the process of the described construction will be equal to the sum of the given vectors.

Example 1. Given a parallelepiped $ABCDEFGH$ (see Fig. 3). Find the sum of the vectors $\overrightarrow{AB}$, $\overrightarrow{FG}$, $\overrightarrow{CG}$, $\overrightarrow{FE}$, $\overrightarrow{FB}$.

△ From the properties of the edges of the parallelepiped it follows that $\overrightarrow{FG}=\overrightarrow{BC}$, $\overrightarrow{FE}=\overrightarrow{GH}$, $\overrightarrow{FB}=\overrightarrow{HD}$. Therefore $\overrightarrow{AB}+\overrightarrow{FG}+$

$\overrightarrow{CG}+\overrightarrow{FE}+\overrightarrow{FB}=\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CG}+\overrightarrow{GH}+\overrightarrow{HD}=\overrightarrow{AD}$, where we have applied the polygonal-line rule. ▲

Example 2. Given a triangular pyramid $ABCD$ (Fig. 9). Find the sum $\overrightarrow{AB}+\overrightarrow{CD}+\overrightarrow{AC}+\overrightarrow{BC}+\overrightarrow{DA}$.

△ Applying the commutative and associative properties of the addition of vectors, we get $\overrightarrow{AB}+\overrightarrow{CD}+\overrightarrow{AC}+\overrightarrow{BC}+\overrightarrow{DA}=\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DA}+\overrightarrow{AC}=\mathbf{0}+\overrightarrow{AC}+\overrightarrow{AC}$. ▲

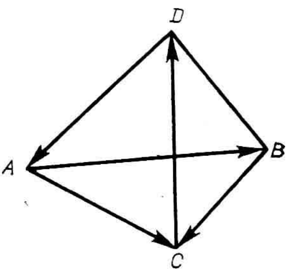

Fig. 9

It is obvious that for any vectors **b** and **c** the following equality is valid:

$$\text{pr}_l\,(\mathbf{b}+\mathbf{c}) = \text{pr}_l\,\mathbf{b} + \text{pr}_l\,\mathbf{c}. \qquad (1)$$

Indeed, if the projections of the vectors **b** and **c** on some axis l are of the same sign (Fig. 10, a), then the following equalities are valid:

$$\text{pr}_l\,(\mathbf{b}+\mathbf{c}) = \text{pr}_l\,\overrightarrow{AC} = A_1C_1$$
$$= A_1B_1 + B_1C_1 = \text{pr}_l\,\mathbf{b} + \text{pr}_l\,\mathbf{c},$$

and if the projections are of opposite sign (Fig. 10, b), then we have:

$$\text{pr}_l\,(\mathbf{b}+\mathbf{c})\ \text{pr}_l\,\overrightarrow{AC} = -A_1C_1 = A_1B_1 - B_1C_1 = \text{pr}_l\,\mathbf{b} + \text{pr}_l\,\mathbf{c}.$$

In both cases equality (1) is valid.

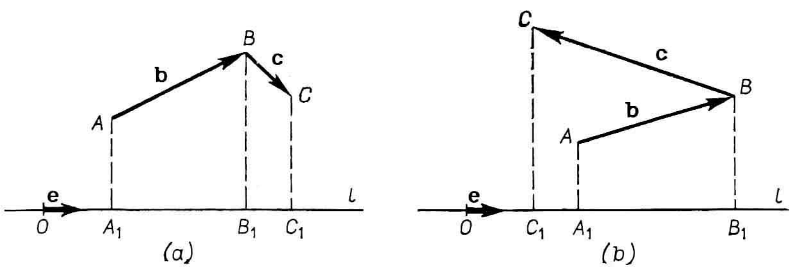

Fig. 10

4. The Difference Between Two Vectors. Recall that for any vector **a** its opposite vector is denoted by —**a**. Obviously, **a** + (—**a**) = **0**.

Definition. For any vectors **a** and **b** the sum **a** + (—**b**) is called their difference and is denoted as **a** — **b**.

Consequently, by definition, **a** — **b** = **a** + (—**b**).

Let us give the rule for finding the difference between two vectors. Given vectors **a** and **b** (Fig. 11). We lay off the vector **a** from some point O: $\mathbf{a}=\overrightarrow{OA}$, and the vector $(-\mathbf{b})$ from the point A: $-\mathbf{b}=\overrightarrow{AB}$. Then $\mathbf{a}-\mathbf{b}=\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}$.

5. The Product of a Vector by a Number. For vectors, besides the operations of addition and subtraction, the operation of multiplication of a vector by a number is also defined.

Definition. The product of a nonzero vector $\mathbf{a}$ by a number $\lambda \neq 0$ is a vector whose length is equal to $|\lambda|\,|\mathbf{a}|$, the direction coinciding

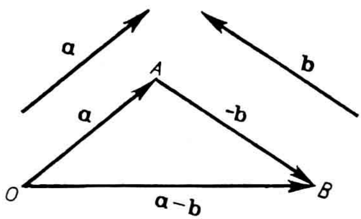

Fig. 11

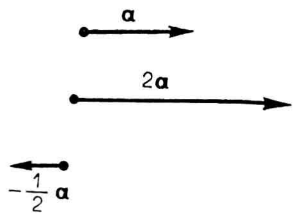

Fig. 12

with the direction of the vector $\mathbf{a}$ or being in the opposite sense, depending on whether the number is positive or negative. If $\lambda = 0$, the product is the null vector.

The product of the vector $\mathbf{a}$ by the number λ is denoted by $\lambda\mathbf{a}$ (the numerical factor is usually written first). The product of the null vector by any number λ, as well as the product of any vector by the number zero, is the null vector. By definition

$$|\lambda\mathbf{a}| = |\lambda|\,|\mathbf{a}|,$$

where $|\lambda|$ is the absolute value of λ. Figure 12 shows the vectors $\mathbf{a}$, $2\mathbf{a}$, and $-\frac{1}{2}\mathbf{a}$.

The operation of multiplication of a vector by a number possesses the associative and distributive properties, that is, for any vectors $\mathbf{a}$ and $\mathbf{b}$ and any numbers λ, μ the following equalities hold:

$$\lambda(\mu\mathbf{a}) = (\lambda\mu)\,\mathbf{a}, \tag{1}$$

$$\lambda\mathbf{a} + \mu\mathbf{a} = (\lambda+\mu)\,\mathbf{a}, \tag{2}$$

$$\lambda\mathbf{a} + \lambda\mathbf{b} = \lambda(\mathbf{a}+\mathbf{b}). \tag{3}$$

☐ Let us prove equality (1). If $\mathbf{a} = \mathbf{0}$ or $\lambda\mu = 0$, then the equality $\lambda(\mu\mathbf{a}) = (\lambda\mu)\,\mathbf{a} = \mathbf{0}$ is obvious. Let $\mathbf{a} \neq \mathbf{0}$, $\lambda\mu \neq 0$, and $\mathbf{a} = \overrightarrow{OA}$. Then the vectors $\lambda(\mu\overrightarrow{OA})$ and $(\lambda\mu)\,\overrightarrow{OA}$ lie on the straight line $\overrightarrow{OA}$, have the same length equal to $|\lambda|\,|\mu|\,|\overrightarrow{OA}|$, and are in the same direction (in the direction of $\overrightarrow{OA}$ if $\lambda\mu > 0$, and in the op-