RELATIVITY:

THE SPECIAL THEORY

BY

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Von Stund an sollen Raum für sich und Zeit für sich völlig zu Schatten herabsinken und nur noch eine Art Union der beiden soll Selbstständigkeit bewahren.

H. Minkowski, Raum und Zeit
[Physikalische Zeitschrift 10 (1909) 104]

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NOTATION

Latin suffixes take the values 1, 2, 3, 4 and Greek suffixes the range 1, 2, 3, with summation over the appropriate range of values in the case of a repeated suffix. Any exceptions to this rule are explicitly noted.

Real coordinates are written with superscripts, x^r. Coordinates with an imaginary time are written with subscripts, x_r, with $x_4 = ict$.

Partial derivatives are often indicated by a comma ($f_{,r} = \partial f/\partial x^r$). Covariant derivatives are indicated by a vertical stroke, $f_{r|s}$.

The signs of the components of the fundamental tensor g_{mn} are chosen so that (for real coordinates) the diagonal form is $(1, 1, 1, -1)$ and *not* $(-1, -1, -1, 1)$.

SYNGE: RELATIVITY

ERRATA

p. 65, Fig. 3: for "nul" read "null".

p. 102: for the three lines following eq. (158), substitute "all such transformations forming a six-parameter group of projective transformations of Euclidean 3-space into itself, equivalent to the six-parameter group of Lorentz transformations Λ of space-time".

p. 231, line 2: read "in the form".

p. 330, line 6 from end: for SYGNE read SYNGE.

PREFACE

This book originated in the notes of lectures given over a number of years in graduate courses at the University of Toronto and elsewhere. The basic idea is to present the essentials of relativity from the Minkowskian point of view, that is, in terms of the geometry of space-time. This geometrical approach is used to some extent in all expositions of relativity, but I have emphasised it more than is customary, because it is to me (and I think to many others) the key which unlocks many mysteries. My ambition has been to make space-time a real workshop for physicists, and not a museum visited occasionally with a feeling of awe.

As originally planned, the book was to cover both the special and general theories of relativity. But as it was being written, the charm of the special theory so worked on me that I found it impossible to confine it to the required limits and, in the end, the general theory had to be omitted. This is not wholly regrettable, because the special theory is by far the more firmly embedded in modern physics and should not be overshadowed by the general theory, as tends to be the case. However, I have left in Chapter I a foundation strong enough to support both the special and general theories.

To understand a subject, one must tear it apart and reconstruct it in a form intellectually satisfying to oneself, and that (in view of the differences between individual minds) is likely to be different from the original form. This new synthesis is of course not an individual effort; it is the result of much reading and of countless informal discussions, but for it one must in the end take individual responsibility. Therefore I apologise, if apology is necessary, for departing from certain traditional approaches which seemed to me unclear, and for insisting that the time has come in relativity to abandon an historical order and to present the subject as a completed whole, completed, that is, in its essentials. In this age of specialisation, history is best left to historians.

I have tried to include most of the famous relativistic results and to develope those relativistic formulae which may be regarded as basic, but, having done so, I have felt at liberty to go a little further along unfamiliar paths where the conclusions may, or may not, be physically significant. In every case I hope that I have made clear the assumptions from which the conclusions spring. In this category I may mention the reconstruction in relativistic form of the discrete system of Newtonian mechanics with action and reaction between its particles (Chapter VII), leading to a statistical concept of the energy tensor (Chapter VIII); the elastic collisions of particles (point-particles, that is) with conservation of angular momentum, leading to a surprising multiple determinacy and some rather intricate algebra (Chapter VII); and the models of uncharged "particles" of finite energy constructed out of singularity-free solutions of Maxwell's equations in vacuo, and model photons constructed according to the same plan (Chapter IX).

I would like to express my debt to the late Dr. L. Silberstein whose lectures at the University of Toronto over thirty years ago started my interest in relativity, to colleagues past and present, in particular Professors L. Infeld, C. Lanczos and E. Schrödinger, for many friendly discussions, to Professor N. L. Balazs who read a considerable portion of the manuscript and was fruitful in suggestions, and to Dr. G. H. F. Gardner who supplied an essential and rather subtle mathematical argument (see p. 239). I am most grateful to Dr. F. A. E. Pirani for his painstaking collaboration in proof-reading and checking formulae, and for suggestions which have unkinked the exposition at many points.

Dublin, June 1955 J. L. S.

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CHAPTER I

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§ 1. CONCEPTS

A scientific theory may be divided into three parts: (*a*) foundations, (*b*) accepted dogma, (*c*) excursions. The foundations are axioms, principles or laws (e.g. Newton's laws of motion or the first and second laws of thermodynamics). The accepted dogma consists of deductions from the foundations confirmed by observation and experiment, linking reason with nature in a satisfying way (e.g. Newtonian mechanics as it stood before relativity was thought of). The excursions wander out of the domain of accepted dogma, sometimes arousing in cautious minds a feeling that they have more imagination than solid fact in them (e.g. Maxwell's electromagnetic theory of light at the time when he put it forward — all excursions are not so successful!). A scientific theory is a living thing which grows and changes; fruitful excursions extend the body of accepted dogma and critical scrutiny of the foundations clarifies and sometimes modifies them.

If someone, otherwise well equipped in knowledge of mathematics and physics, wants to understand the theory of relativity, by what door is he to enter in? As the result of what process is he to find himself comfortably at home in the accepted dogma of relativity, capable of appreciating the foundations critically and able to discuss the excursions of others and to make his own?

Such questions are of course not pertinent to relativity only — they apply to any branch of theoretical physics, or indeed to any branch of science. But they are particularly difficult to answer satisfactorily in the cases of relativity and quantum mechanics, not merely because these subjects are comparatively young, but because they both uproot concepts usually accepted without question.

The ancient Greeks had their answer. The door by which they entered a subject was a set of axioms; granted these, all one had to do was to follow the processes of logical thought. To the works of Euclid and Archimedes, written in this axiomatic spirit, modern science owes its being. But it is not as simple as it looks. Axioms, we now realise, are not the self-evident truths they were long supposed to be, but rather the rules of a game, the pieces of which are elements or concepts which remain and must ever remain undefined because there is nothing in terms of which to define them. This was brought to light by the researches of Hilbert into the foundations of geometry at the end of the nineteenth century, and the knowledge that any theory with a claim to logical structure must start with undefined elements and unproved propositions is slowly permeating through science.

In fact, although it seems most natural to start with the foundations and build on them, we now realise that the foundations of a theory are actually the most elusive and confusing part of it. Anyone who tries to put a physical theory on an impeccable axiomatic basis soon realises that he has undertaken a major task, absorbing all his energy and leaving none for the body of the theory in which his real interest lies. The axiomatisation of physics is of great interest, but it is a job for the specialist in axiomatics, and the fruits of his labour are likely to be enjoyed rather by fellow specialists than by theoretical physicists at large. In brief, axiomatics do not provide the door we are seeking.

In modern works on theoretical physics axiomatisation has been largely abandoned. Instead, the entry to a new subject is by what may be called the "cuckoo-process". The eggs are laid, not on the bare ground to be hatched in the clear light of Greek logic, but in the nest of another bird, where they are warmed by the body of a foster mother, which, in the case of relativity, is the Newtonian physics of the nineteenth century. The student is first thoroughly indoctrinated with Newtonian physics, and he accepts its concepts as true to physical reality. Then, step by step, the concepts are modified, until finally he bites off the head of his foster mother and flies from the nest a full-fledged relativitist.

This cuckoo-process follows the true order of historical development in science and it has the advantage that at every stage of the transformation the learner has the comforting support of familiar surroundings. As each support falls away, it is replaced by another, constructed to the new pattern. But it is confusing. The concepts of Newtonian physics interlock with one another (e.g. force, acceleration, inertial mass and gravitational mass), and until one has finally reviewed all Newtonian concepts, there is always present a suspicion that the same word is being used with meanings which differ with the context.

The plan adopted in this book is a compromise. Formal axiomatics are avoided, but a serious effort is made to state the assumptions with enough clarity to show that relativity is essentially a logical structure for which any interested logician might seek a system of axioms if he wanted to. The appeal to undefined elements, essential in an axiomatic treatment but repellent to most physicists as an unnecessary mystification, is avoided by taking over concepts from Newtonian physics and setting them in a new background. There is, of course, a danger in this, for even in Newtonian physics different people have slightly different ways of looking at concepts, and so the theory of relativity, created in the mind of the individual, may have a slightly subjective character depending on who the individual is. This is unavoidable; only by the give-and-take of scientific conversation can anyone be sure that his concepts are the same as the concepts of others, or, if they are not the same, find out how they differ.

It might seem that this is the cuckoo-process over again, but it is not. We shall from the first turn a cold and sceptical eye on Newtonian physics, never admitting a bunch of interlocking Newtonian concepts but only concepts one at a time, alone and disinfected.

Let us now get to work. We start with a tabula rasa, a clean sheet, a mind in a state of intellectual nudity.

Into this void we admit at once the whole body of *pure mathematics*, or at least those portions of pure mathematics which we may have occasion to use later. *Applied* mathematics on the other hand is excluded, for almost all applied mathematics deals with Newtonian physics, and the words used in it evoke Newtonian concepts and these we are prepared to admit only singly and under scrutiny.

This embargo on applied mathematics is serious, for it excludes the dynamics of particles and rigid bodies, celestial mechanics, Lagrangian and Hamiltonian methods, hydrodynamics, elasticity and electrodynamics. The trouble with these subjects is that they all involve the Newtonian concept of *time*, and that, as we shall see in due course, is one of those Newtonian concepts which we shall *not* take over into relativity. If the subjects listed above are to appear in the theory of relativity, they must appear in a revised form. However the student of relativity has not wasted his time in the study of Newtonian subjects, for in relativity we frequently seek contact with Newtonian physics in certain limiting cases, and such contacts are understandable only to one familiar with Newtonian theories.

The subject of *geometry* deserves special mention. Is it pure mathematics or applied mathematics? There is considerable confusion on this question, because the word is used to cover two entirely different things. In so far as it is axiomatic (logical deduction from axioms dealing with undefined elements), it is pure mathematics and as such admissible into our relativistic scheme, provided we do not subconsciously define the undefined elements physically and accept the axioms concerning them. In so far as geometry deals with the form of actual things and their measurement, it is definitely applied mathematics, and as such excluded from our relativistic scheme, at least until such time as we are ready to consider the possibility of admitting it.

There is another way of looking at this question of geometry. The positive integers, which correspond to the primitive physical operation of *counting*, form the basis of a considerable portion of mathematics. From them we derive negative numbers, irrational numbers and complex numbers, and hence the body of mathematical analysis. Relativity is not so revolutionary as to question the validity of counting, and in fact it accepts all mathematics based on counting. Thus the relationship of geometry to relativity is most satisfactorily established when geometry is regarded *analytically*, a "point" being nothing but a set of numbers (its coordinates) and a "line" a set of points. This is a most fruitful way to look at geometry, and we shall make extensive use of it. It is only when geometry purports to deal with "physical space" that we must view it with extreme caution. This does not mean that the geometry of physical space will not be discussed, but only that we reserve the right to discuss it at our own time and in our own terms.

For historical accounts of the theory of relativity, the reader may consult DUGAS [1950] and WHITTAKER [1953], or for more mathematical detail, PAULI [1920].[1] Out of the work of Lorentz and Poincaré the special theory of relativity emerged, EINSTEIN [1905] clearing up philosophical difficulties by destroying the concepts of ether and absolute simultaneity and MINKOWSKI [1909] giving the theory a clear mathematical form in terms of the geometry of space-time. After some tentative approaches, the general theory of relativity (the new theory

[1] For these and other references, see p. 435.

of gravitation) took final shape in 1915–1916 (EINSTEIN [1915, 1916])[1].

In the present chapter historical order is abandoned and a basis laid down wide enough to support both the special and the general theories. This avoids a cuckoo-process by which the special theory is first developed and later eaten up by the general theory; if this delays a little the reader's contact with the details of relativity, it will, it is hoped, save him from headaches later on.

We shall now start to take over Newtonian concepts into relativity, beginning with the concepts of *event* and *particle*.

§ 2. EVENTS AND PARTICLES

The word *event* does not occur frequently in Newtonian physics, but this is accidental, because the concept is quite clear. Anything that happens is an *event*, but (just as in geometry we sharpen the concept of a point) we sharpen the concept of an event to mean an occurrence which takes up no room and has no duration. To emphasise this sharpening we may call it a *point-event*, but it is unnecessary to do so because we shall always use the word *event* in this sharpened sense.

To stimulate the imagination we may think of an event as an explosion or collision, but this dramatic or catastrophic association is not essential. *Any* occurrence, sharpened as aforesaid, is an event, and we can of course imagine possible events as well as those which we think of as actually occurring. We shall presently consider the totality of all possible events (space-time).

The concept of a *material particle* also is taken over from Newtonian physics. We are familiar with the way in which it appears in physical theory — a moving point with a number (mass) associated with it, and perhaps a second number (electric charge). This concept we accept in relativity, with one slight reservation; perhaps the mass is not constant. We exercise the same discrimination as in Newtonian physics regarding the circumstances under which a real piece of matter may be treated as a particle; it may on occasion be an electron, an atom, a billiard ball, a planet, a star, or even a nebula.

It is convenient to introduce also the *particle of light* or *photon*, the properties of which will be discussed later. For the moment we think of it as a moving point. There is nothing un-Newtonian in the idea of a small parcel of light.

[1] English translations of a number of fundamental papers are contained in LORENTZ [1923].

We have now taken over from Newtonian physics the concepts of *event* and *particle* (material particle and photon). They are linked together by the fact that *the history of a particle is a continuous sequence of events*.

§ 3. SPACE-TIME

In Newtonian physics an event may be identified by four numbers (x, y, z, t), where (x, y, z) are the rectangular Cartesian coordinates of the place where it occurs and t the time at which it occurs. But we do not have to use these numbers. If we define (X, Y, Z, T) as four functions of (x, y, z, t), then the values of (X, Y, Z, T) serve to identify an event. The essential thing is that an event needs *four* numbers to identify it, and for that reason we say that in Newtonian physics *the totality of all possible events form a four-dimensional continuum*.

This italicised statement we take over into relativity, without necessarily taking over the chain of thought leading up to it. We are not yet ready to discuss whether the concepts of rectangular Cartesian coordinates (x, y, z) and time t are acceptable in relativity. Actually, we shall accept them later with important reservations. For the present let us not think about them, but accept as a fundamental hypothesis of relativity the statement that *the totality of all possible events form a four-dimensional continuum*. This continuum we call *space-time*; we are not at all in a position to remove the hyphen and speak of *space* and *time* separately.

In regard to space-time, the emphasis is on the word *four*. It is four-dimensional, not three-dimensional or five-dimensional. In saying that it is four-dimensional, we mean that an event is identified by four numbers, say (x^1, x^2, x^3, x^4), which numbers we call the *coordinates* of the event, or *space-time coordinates* if we want to be emphatic.

It is well at this point to interject a cautionary remark which does not affect the general line of thought. In Newtonian physics the space-time of all possible events is covered once over by the coordinates (x, y, z, t), each of these four coordinates ranging from $-\infty$ to $+\infty$. There is in fact a one-to-one correspondence between events and tetrads of numbers (x, y, z, t). It is not asserted that the space-time of relativity is covered once over by the coordinates (x^1, x^2, x^3, x^4) ranging from $-\infty$ to $+\infty$. It is enough to think that a portion of space-time is covered once over by these coordinates with suitable ranges. To cover the whole of space-time it may be necessary to use