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Groups, Rings and Algebras

A Conference in Honor of
Donald S. Passman
June 10–12, 2005
The University of Wisconsin-Madison
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With Best Wishes to
Donald S. Passman
for his 65^{th} birthday.

Preface

A conference in honor of Donald S. Passman, entitled *Groups, Rings and Algebras*, took place on June 10, 11 and 12, 2005 at the University of Wisconsin-Madison. The scientific purpose of the conference was a retroactive and proactive assessment of those areas of algebra related to his work. These include group rings, group theory, character theory, graded rings, enveloping algebras, group actions on algebras and rings, Hopf algebras and certain algebras arising from the study of noncommutative geometry. The principle speakers were Yuri Bahturin, Edward Formanek, Martin Isaacs, Martin Lorenz, Susan Montgomery, Lance Small, Toby Stafford, A. E. Zalesskiĭ and Efim Zelmanov. In addition, there were many contributed talks.

The social highlights of the meetings were a candle lit soirée at the Passman's on Friday evening and a banquet on the ninth floor of Van Vleck Hall on Saturday night.

Participants were invited to contribute papers. The submissions were refereed and those that were accepted are the contents of this volume. They are in final form and no version will be submitted for publication elsewhere.

Finally, it is our happy task to acknowledge those who made a difference and to thank them. First, we thank our financial sponsors the National Security Agency and the University of Wisconsin-Madison. In particular, we thank Michelle Wagner of the NSA and David Griffeath, Chair of the Mathematics Department. The excellent physical facilities in Van Vleck were provided by the Mathematics Department and the efforts of the staff of the department, especially Mark Castillo and Joan Wendt, were greatly appreciated.

W. Chin, J. M. Osterburg and D. Quinn

Biography of Donald S. Passman

Donald Steven Passman was born in New York City in 1940. He did his undergraduate work at the Polytechnic Institute of Brooklyn, receiving his B.S. degree in 1960, and his graduate studies at Harvard University, receiving his M.A. in 1961 and his Ph.D. in 1964. His thesis advisor was the famous algebraist Richard Brauer. He was an Assistant Professor at the University of California, Los Angeles (1964–1966) and at Yale University (1966–1969). In 1969, he was appointed an Associate Professor at the University of Wisconsin-Madison, and was promoted to the rank of full Professor in 1971. Since 1995, he has been the Richard Brauer Professor of Mathematics. Professor Passman has held visiting positions at U.C.L.A., the University of Warwick, and at IDA/CCR Princeton and LaJolla.

He is the author of six books, namely:

- **Permutation Groups**, Benjamin, New York, 1968.
- **Infinite Group Rings**, Marcel Dekker, New York, 1971.
- **The Algebraic Structure of Group Rings**, Wiley-interscience, New York, 1977. [Krieger, Malabar, 1985.]
- **Group rings, Crossed Products and Galois Theory**, CBMS Conference Notes, AMS, Providence, 1986.
- **Infinite Crossed Products**, Academic Press, Boston, 1989.
- **A Course in Ring Theory**, Wadsworth, Pacific Grove, 1991. [Chelsea-AMS, Providence, 2004.]

Professor Passman works in group theory, ring theory, group rings, Hopf algebras, and Lie algebras. He is the author of more than 160 research papers. His most significant papers would certainly include:

- *Nil ideals in group rings*, Michigan Math. J. **9** (1962), 374–384.
- *Group rings satisfying a polynomial identity*, J. Algebra **20** (1972), 103–117.
- *A new radical for group rings?*, J. Algebra **28** (1974), 556–572.
- *Infinite crossed products and group-graded rings*, Trans. AMS **284** (1984), 707–727.
- *The semiprimitivity problem for twisted group algebras of locally finite groups*, Proc. London Math. Soc. (3) **73** (1996), 323–357.
- *The Jacobson radical of group rings of locally finite groups*, Trans. AMS **349** (1997), 4696–4751.
- *Invariant ideals and polynomial forms*, Trans. AMS **354** (2002), 3379–3408.

One of his best puns is the title of:

- *It's essentially Maschke's theorem*, Rocky Mt. J. **13** (1983), 37–54.

Professor Passman has directed the dissertations of twelve doctoral students officially and several others unofficially. His many invited addresses include those at the 29th British Mathematical Colloquium (invited speaker, University of Edinburgh, 1977), the American Mathematical Society (invited speaker, Washington D.C., 1979), CBMS Conference (main lecturer, Mankato State University, 1985), and the Canadian Mathematical Society (plenary speaker, Windsor, Ontario, 1989). He has received numerous awards for his teaching and his writing. These include the Lester R. Ford Award (American Mathematical Society) in 1976 for his paper *What is a group ring?*, and the Deborah and Franklin Tepper Haimo Award for Distinguished University Teaching (Mathematical Association of America) in 2000.

Professor Passman continues to teach and to do research. He lives with his wife Marjorie in Madison, Wisconsin. They have two married children and five grandchildren. They enjoy both families: Barbara, Thomas, Samuel and Rebecca Brownsword of Montclair, New Jersey; and Pamela, Jonathan, Abraham, Jordan and Eve Passman of Minnetonka, Minnesota.

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Group Gradings on Associative Superalgebras.

Y. A. Bahturin and I. P. Shestakov

ABSTRACT. In this paper we describe all group gradings by a finite abelian group G of any simple associative superalgebra over an algebraically closed field F. Some restrictions on the characteristic of F apply.

1. Introduction

Let T be an abelian group, F a field. An associative algebra A is called a T-superalgebra if A is equipped with a grading by T, that is, $A = \bigoplus_{t\in T} A^t$ where each A^t is a vector subspace of A and $A^tA^s \subset A^{ts}$. A subspace (subalgebra, ideal) B of A is called *graded* if $B = \bigoplus_{t\in T}(B \cap A^t)$. A superalgebra A is called *simple* if A has no proper nonzero graded ideals. Using other terminology, a simple superalgebra is a *graded simple* algebra.

Before we start our discussion we introduce two types of gradings by groups on the matrix algebras [**4**]. If $A = M_n(F)$ the any n-tuple $(g_1, \ldots, g_n)$ of elements of G defines an *elementary* G-grading of A if we define $A_g = \text{Span}\,\{E_{ij} \mid g_i^{-1}g_j = g\}$. Here E_{ij} is usual matrix unit. Any grading obtained from this by an automorphism of A is also called elementary.

A grading of $A = M_n(F)$ by $G \cong \mathbb{Z}_n \times \mathbb{Z}_n$ is called an ε-grading, where ε is a primitive n^{th} root of 1, if $A_g = \text{Span}\,\{X_g\}$, for any $g \in G$. If a, b are the generators of G and $g = a^ib^j$ then $X_g = X_a^iX_b^j$. Finally,

$$(1)\quad X_a = \begin{bmatrix} \varepsilon^{n-1} & 0 & \cdots & 0 \\ 0 & \varepsilon^{n-2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \quad X_b = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

The mapping $\alpha : G \times G \longrightarrow F^*$ given by $\alpha(a^ib^j, a^kb^l) = \varepsilon^{-jk}$ is a multiplicative bicharacter on G and $X_gX_h = \alpha(g, h)X_{gh}$. The ratio $\beta(g, h) = \alpha(g, h)/\alpha(h, g)$ is

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a skew-symmetric bicharacter and $X_g X_h = \beta(g,h) X_h X_g$. If $T = T_1 \times \cdots \times T_k$, where $T_i \cong \mathbb{Z}_{n_i} \times \mathbb{Z}_{n_i}$, then on $A = M_{n_1}(F) \otimes \cdots \otimes M_{n_k}(F)$ we have a grading by T where the homogeneous elements are $X_t = X_{t_1} \otimes \cdots \otimes X_{t_k}$ for $t_1 \in T_1, \ldots, t_k \in T_k$ and $t = t_1 \ldots t_k$. We can define bicharacters α and β as product of bicharacters on T_i defined above and we will still have the same relations for the homogeneous generators of A as for individual M_{n_i}.

This grading is an example of so called *fine* gradings, that is, gradings of an algebra A by a group G such that $\dim A \leq 1$ for all $g \in G$. The support of any fine grading of $A = M_n(F)$ is always a subgroup and it was shown in [**4**] and [**8**] that any fine grading of $A = M_n(F)$ by a finite abelian group G over an algebraically closed field is equivalent to the grading defined just above, where $T = \operatorname{Supp} A$. Another result of [**4**], [**8**], and [**9**] is that any grading of $A = M_n(F)$ by a finite group G is equivalent to the tensor product $A = C \otimes D$ with an elementary grading on a matrix subalgebra C and a fine grading on another matrix subalgebra D, and $\operatorname{Supp} C \cap \operatorname{Supp} D = \{e\}$.

Notice that $A = M_n(F)$ with fine grading by a group T can be viewed as a twisted group algebra $F^\alpha[T]$ in the sense defined just below if we identify X_t with t, for every $t \in T$.

Going back to T-graded simple algebras, we mention that a complete description of finite-dimensional T-graded simple associative algebras over an algebraically closed field F of characteristic 0 or p such that $(p, |T|) = 1$ is given in [**3**], with some essential information true even in the case where T is an arbitrary, not necessarily abelian, group T. It is proven that any such algebra A has the form of $A = F^\sigma[H] \otimes M_k(F)$. Here H is a finite subgroup of T, σ a 2-cocycle on H with values in F^*, $F^\sigma[H]$ is a twisted group algebra of H by σ, with a natural H-grading. Also, $M_k(F)$ is the matrix algebras of order k over F endowed with an elementary grading defined by a k-tuple $(t_1, \ldots, t_k)$ of elements of G. As a graded vector space, A is spanned by the homogeneous elements of the form $h \otimes E_{ij}$ where E_{ij} is a matrix unit in $M_k(F)$ while $h \in H$ is an element of the natural basis of $F^\sigma[H]$) and the grading of $h \otimes E_{ij}$ is $t_i^{-1} h t_j$. Additionally, $H \cap \operatorname{Supp} M_k(F) = \{e\}$. In the case where T is abelian, the graded simple algebras, in a different language, have been determined in [**4**].

The best known case is that of $T = \mathbb{Z}_2$, which we will call *ordinary* associative superalgebras, and in this case the possible forms of A have been determined in [**11**]. All simple ordinary superalgebras are either simple in the non-graded sense or non-simple. In the former case any simple ordinary superalgebra has the form of $M_{k,l}$, that is, the matrix algebra $M_n(F)$, $n = k + l$, with an elementary grading given by the tuple $(\underbrace{0, \ldots, 0}_{k}, \underbrace{1, \ldots, 1}_{l})$ (see [**4**]). If A is not simple then, from the above, $A = F^\sigma[H] \otimes M_k(F)$ where H is nontrivial. The condition on the support tells us that then $M_k(F)$ remains ungraded. Also, $H = T$, and any 2-cocycle on $\mathbb{Z}_2$ is cohomologous to the identity cocycle, so that $F^\sigma[H] \cong F[T]$. If t is the generator for T, then $t^2 = e$ and A takes the well-known form of $A = (e \otimes M_k(F)) \oplus (t \otimes M_k(F))$ or, for shortness, $A = B \oplus tB$, where $B \cong M_k(F)$ is an ungraded matrix algebra, $t^2 = 1$ and the grading of every element in the component B is e while that in the component tB is t.

In this paper our goal is to describe finite-dimensional G-graded simple $\mathbb{Z}_2$-superalgebras R for a finite abelian group G, over any algebraically closed field F.

First, in Theorems 3 and 4 we settle the case where R is a simple associative algebra and then, in Theorem 6 we completely describe the gradings on superalgebras of the form $R = B \oplus tB$. Some restrictions on the characteristic of F are necessary and they can be found in the statements of respective theorems.

It is traditional to denote the components of $\mathbb{Z}_2$-graded superalgebra by $R_{\bar{0}}$ and $R_{\bar{1}}$, and we will follow this pattern in what follows.

2. Graded superalgebra structures on simple associative algebras. Fine gradings

If A is simple as a non-graded algebra, then it is just the matrix algebra with a T-grading. Assuming that A is G-graded as a T-superalgebra means that A is a matrix algebra $A = M_n(F)$ equipped with a $G \times T$-grading. All such gradings have been described in [**9**] even when the grading group was non-commutative. This approach does not allow to single out in the grading obtained the original G or $\mathbb{Z}_2$-grading. Thus instead we proceed as follows.

Let G be a finite abelian group, $R = M_n(F)$ a matrix algebra over an arbitrary field F of characteristic different from 2. Suppose that the superalgebra structure on R is given by an automorphism τ of order 2. Whatever basis chosen, τ is given by $\tau \circ X = \Phi^{-1} X \Phi$, for an appropriate nondegenerate matrix Φ of order n.

Now we assume that R is given a G-grading compatible with this superalgebra structure. This means that for any $g \in G$ we have $\tau \circ R_g = R_g$. In other words, is $R = R_{\bar{0}} \oplus R_{\bar{1}}$ is the superalgebra decomposition of R then

$$R_g = (R_g \cap R_{\bar{0}}) \oplus (R_g \cap R_{\bar{1}}), \text{ for any } g \in G.$$

Since any grading of R as an associative superalgebra is also a grading of R as an associative algebra, we have by [**4**] that $R = R^{(0)} \otimes R^{(1)}$ where each tensor factor $R^{(0)}$, respectively, $R^{(1)}$ is a matrix algebra with an elementary, respectively, fine grading. This suggests that in our searches of possible gradings on a matrix algebra $R = M_n(F)$, compatible with a superalgebra structure, we should look first at the cases where R is with one of these two gradings. An important formula is this: $M_{k,l} \otimes M_{r,s} \cong M_{kr+ls,ks+lr}$. This allows us, given a superalgebra structure on $M_n(F)$, which is determined by a partition $n = p + q$, $p, q \geq 0$, to proceed as follows. First, we find all elementary G-gradings on some $M_{k,l}$ and fine G-gradings on $M_{r,s}$, whose supports intersect trivially. If $p = kr + ls$ and $q = ks + lr$ than the tensor product of the previously determined gradings will give us a G-grading on $M_{p,q}$. (We also have $n = p + q = (k + l)(r + s)$. One could also write $n = p + \alpha q = (k + \alpha l)(r + \alpha s)$, where $\alpha^2 = 1$.) As usual, we would have

$$((M_{k,l})_{\bar{0}} \otimes (M_{r,s})_{\bar{0}}) \oplus ((M_{k,l})_{\bar{1}} \otimes (M_{r,s})_{\bar{1}}) = (M_{kr+ls,ks+lr})_{\bar{0}}$$

while

$$((M_{k,l})_{\bar{0}} \otimes (M_{r,s})_{\bar{1}}) \oplus ((M_{k,l})_{\bar{1}} \otimes (M_{r,s})_{\bar{0}}) = (M_{kr+ls,ks+lr})_{\bar{1}}.$$

Let us start with the case where the grading on $M_n(F)$ is fine and compatible with a $\mathbb{Z}_2$-grading $R = R_{\bar{0}} \oplus R_{\bar{1}}$. It follows from a very general lemma in [**4**] that the support H of this grading is a subgroup of the grading group G. Thus we have $R = \bigoplus_{h \in H} R_h$ and $R_h = \langle X_h \rangle$, for a nondegenerate matrix X_h. Let us also recall that the product in $R = M_n(F)$ with fine grading as above is defined by a *bicharacter* $\alpha : H \times H \to F^*$ as follows: $X_h X_k = \alpha(h,k) X_{hk}$, for any $h, k \in H$. The commutation relations in R take the form $X_h X_k = \beta(h,k) X_k X_h$ where $\beta(h,k) = \alpha(h,k)/\alpha(k,h)$ is a *skew-symmetric bicharacter* (see [**2**]).

It follows from the compatibility condition that there exists a subset $H_0 \subset H$ such that $R_{\bar{0}} = \bigoplus_{h\in H_0} R_h$ and $R_{\bar{1}} = \bigoplus_{h\notin H_0} R_h$. Since $R_{\bar{0}}R_{\bar{0}} \subset R_{\bar{0}}$ we have that $R_{hk} = R_h R_k \subset R_{\bar{0}}$ for any $h, k \in H_0$. Since we deal with finite groups, it follows that H_0 is a subgroup. If there is $g \in H \setminus H_0$ then using similar argument we easily derive that $H = H_0 \cup gH_0$ proving that H_0 is a subgroup of index 1 or 2. If $H = H_0$ then $R = R_{\bar{0}}$, that is, the superalgebra structure on R is trivial. If $R = R^{(0)} \otimes R^{(1)}$ as before, and the superalgebra structure on $R^{(1)}$ is trivial then $R_{\bar{0}} = (R^{(0)})_{\bar{0}} \otimes R^{(1)}$ while $R_{\bar{1}} = (R^{(0)})_{\bar{1}} \otimes R^{(1)}$. Thus, in the case where the G-grading is fine, everything is determined by a subgroup H_0 of index 2 in $H = \mathrm{Span}\,\{R\}$ and the superalgebra structure is given as follows:

$$R_{\bar{0}} = \mathrm{Span}\,\{X_h | h \in H_0\} \text{ and } R_{\bar{1}} = \mathrm{Span}\,\{X_h | h \notin H_0\}. \tag{2}$$

Obviously, this latter formula defines a $\mathbb{Z}_2$-grading on any algebra with fine grading. Since $\dim R_{\bar{0}} = \dim R_{\bar{1}}$ it follows that with such a grading we have $n = 2m$, for some m, and our superalgebra is isomorphic to $M_{m,m}$. Now if we denote the G-graded superalgebra just introduced by $P(H; H_0; \alpha)$ then the following theorem is true.

THEOREM 1. *Any G-graded associative superalgebra R which is a finite - dimensional simple algebra $M_n(F)$ whose G-grading is fine is isomorphic to a superalgebra $P(H; H_0; \alpha)$, for an appropriate subgroup $H \subset G$ of order n^2, a subgroup $H_0 \subset H$ of index at most 2 and a bicharacter α on H.*

Notice that in [**8**] the reader can find information about the cases where the matrix algebras admit fine grading and their description. The connections with the bicharacters can also be found in [**2**].

3. Graded superalgebra structures on simple associative algebras. Elementary gradings

The material of this section is to a large extent reminiscent of the respective parts of [**1**], where we deal with involution gradings of matrix algebras.

Following the argument in [**10**], we write the n-tuple, which defines our elementary grading, as $\theta = (g_1^{(k_1)}, \ldots, g_m^{(k_m)})$ where $g_i \neq g_j$ for $i \neq j$, $k_1, \ldots, k_m > 0$, $k_1 + \cdots + k_m = n$. The k-tuple $(k_1, \ldots, k_m)$ defines a partition of the matrices in R into blocks as indicated below. We also recall that τ is the "structural" automorphism of $R = M_n(F)$ as a $\mathbb{Z}_2$-superalgebra.

Let us set

$$\varepsilon_1 = E_{11} + \cdots + E_{k_1k_1}, \ldots, \varepsilon_m = E_{k_{m-1}+1,k_{m-1}+1} + \cdots + E_{k_mk_m},$$

where the E_{ij}'s are the usual matrix units. Then $\varepsilon_1, \ldots, \varepsilon_m$ form a system of pairwise orthogonal idempotents of R with $1 = \varepsilon_1 + \cdots + \varepsilon_m$. Let us set $A_i = \varepsilon_i R \varepsilon_i$, which is the i-th diagonal block and let us write $A = A_1 \oplus \cdots \oplus A_m$, which is the identity component R_e of the grading we are dealing with. By our hypothesis, $\tau \circ A = A$. Let us write τ on R as $\tau(X) = \Phi^{-1} X \Phi$.

Since τ is an automorphism of A we have that $\tau(A_i) = A_{\sigma(i)}$ for a suitable permutation σ of $1, 2, \ldots, m$. Let ψ be the inner automorphism of R given by conjugation by the permutation matrix S which permutes the blocks A_i according to σ. Therefore the automorphism $\chi = \psi^{-1}\tau$ leaves every block A_i invariant $\chi(A_i) = A_i$, $i = 1, \ldots, m$.

Now the restriction of χ to A_i is an automorphism of this matrix algebra so that there exists a $k_i \times k_i$-matrix T_i such that $\chi(X) = T_i^{-1} X T_i$ for any $X \in A_i$. If we let $T = \mathrm{diag}(T_1, \dots, T_m)$, then the action of τ will coincide with conjugation by TS. Thus the conjugation by Φ and TS coincide on A.

If $\Psi = \Phi^{-1}TS$ and $\Psi = [\Psi_{ij}]$ then for any $X = \mathrm{diag}(X_1, \dots, X_m) \in A$ we have $X\Psi = \Psi X$ or $X_i \Psi_{ij} = \Psi_{ij} X_j$ for any $1 \le i, j \le m$. If $i = j$ then $X_i \Psi_{ii} = \Psi_{ii} X_i$ and it follows that $\Psi_{ii} = \lambda_i I$ is a scalar matrix, for some nonzero scalar λ_i. If $i \ne j$ then choosing $X_i = I_{k_i}$ and $X_j = 0$ we immediately obtain $\Psi_{ij} = 0$. So $\Psi = \mathrm{diag}(\lambda_1 I_{k_1}, \dots, \lambda_m I_{k_m})$ is a diagonal matrix. Since $\Phi = TS\Psi^{-1}$, it follows that $\Phi = [\Phi_{ij}]$ is a block matrix such that in each column of blocks and in each row of blocks we have exactly one nonzero square block T_i.

Now we should remember that $\tau^2 = 1$, that is, the conjugation by Φ^2 is trivial. In other words, $\Phi^2 = \mu I$, for an appropriate nonzero scalar μ. This means that the above permutation σ is the product of independent cycles of length 2. Therefore, permuting the whole blocks, we may assume that Φ has the form of

$$\Phi = \sqrt{\mu}\,\mathrm{diag}\left\{\begin{bmatrix} 0 & U_1 \\ U_1' & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & U_r \\ U_r' & 0 \end{bmatrix}, V_1, \dots, V_s, \right\}.$$

where $s + 2r = m$. Recalling $\Phi^2 = \mu I$ we obtain

$$I = \mathrm{diag}\left\{U_1U_1', U_1'U_1 \dots, U_rU_r', U_r'U_r, V_1^2, \dots, V_s^2\right\}.$$

It follows then that $U_1' = U_1^{-1}, \dots, U_r' = U_r^{-1}$, $V_1^2 = I, \dots, V_s^2 = I$. Thus the conjugation by Φ is the same as that by

$$\Phi' = \mathrm{diag}\left\{\begin{bmatrix} 0 & U_1 \\ U_1^{-1} & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & U_r \\ U_r^{-1} & 0 \end{bmatrix}, V_1, \dots, V_s, \right\}.$$

Now let us notice that the elementary grading preserves if we conjugate Φ' by

$$C = \mathrm{diag}\left\{C_1, C_1', C_r, C_r', D_1, \dots, D_s\right\},$$

with the same splitting into blocks. We will then have

$$\begin{aligned} C^{-1}\Phi' C &= \mathrm{diag}\left\{\begin{bmatrix} 0 & C_1^{-1}U_1C_1' \\ (C_1')^{-1}U_1^{-1}C_1 & 0 \end{bmatrix}, \dots, \right. \\ &\left. \begin{bmatrix} 0 & C_r^{-1}U_rC_r' \\ (C_r')^{-1}U_r^{-1}C_r & 0 \end{bmatrix}, D_1^{-1}V_1D_1, \dots, D_s^{-1}V_sD_s, \right\}. \end{aligned}$$

If we choose appropriate $C_1, \dots, D_s$ then we can reduce Φ' to the form of

(3)

$$\Phi'' = \mathrm{diag}\left\{\begin{bmatrix} 0 & I_{k_2} \\ I_{k_1} & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & I_{k_{2r}} \\ I_{k_{2r-1}} & 0 \end{bmatrix}, \begin{bmatrix} I_{p_1} & 0 \\ 0 & -I_{q_1} \end{bmatrix}, \dots, \begin{bmatrix} I_{p_s} & 0 \\ 0 & -I_{q_s} \end{bmatrix}\right\},$$

where $k_{2r+1} = p_1 + q_1, \dots, k_{2r+s} = p_s + q_s$ and of course $k_1 = k_2, \dots, k_{2r-1} = k_{2r}$.

Let us call the blocks $\begin{bmatrix} 0 & I_k \\ I_k & 0 \end{bmatrix}$ the blocks of the first kind while $\begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$ the blocks of the second kind. We want to show that if τ is compatible with an elementary grading then the respective conjugating matrix Φ'' cannot have both blocks of the first and the second kind.

Indeed, consider any block matrix X all of whose blocks except $X_{1,2r+1}$ are zero. This matrix is homogeneous with respect to our grading and its degree is $g_1^{-1}g_{2r+1}$. If we apply τ then $Y = \Phi^{-1}X\Phi$ will be a matrix in the second row of blocks and still the $2r+1$ column of blocks. Thus the degree of Y will be $g_2^{-1}g_{2r+1}$, which implies $g_1 = g_2$, a contradiction.

Thus there are two cases: the first, where Φ'' has only blocks of the first kind and the second, when it has only blocks of the second kind.

Suppose we have only blocks of the first kind. Then $m = 2r$ and the defining tuple has the form $\theta = (g_1^{(k_1)}, g_2^{(k_1)} \ldots, g_{2r-1}^{(k_{2r-1})}, g_{2r}^{(k_{2r-1})})$ where $g_i \neq g_j$ for $i \neq j$, $k_1, \ldots, k_{2r-1} > 0$. However there are some relations between the components of θ. For example if in X as above, only the block X_{12} is nonzero then in $Y = \tau(X)$ only the block Y_{21} is nonzero. This implies $g_1^{-1}g_2 = g_2^{-1}g_1$, or $(g_1^{-1}g_2)^2 = e$. Similarly, $(g_3^{-1}g_4)^2 = \ldots = (g_{2r-1}^{-1}g_{2r})^2 = 1$. Also, if X is in row $2t-1$ and column $2s-1$ then $Y = \tau(X)$ is in the row $2t$ and column $2s$, which implies $g_{2t-1}^{-1}g_{2s-1} = g_{2t}^{-1}g_{2s}$. Quite similarly, $g_{2t-1}^{-1}g_{2s} = g_{2t}^{-1}g_{2s-1}$, $g_{2t}^{-1}g_{2s-1} = g_{2t-1}^{-1}g_{2s}$, and $g_{2t}^{-1}g_{2s} = g_{2t-1}^{-1}g_{2s-1}$. Obviously, we need only two of these latter relations $g_{2t-1}^{-1}g_{2s-1} = g_{2t}^{-1}g_{2s}$ and $g_{2t-1}^{-1}g_{2s} = g_{2t}^{-1}g_{2s-1}$, for all possible choices of s and t. A quick analysis shows that these conditions are equivalent to the following ones. There exists in H an element h of order 2 such that $g_{2t} = g_{2t-1}h$ for any $t = 1, \ldots, r$.

Notice, that in this case we obtain $n = 2l$ and the superalgebra grading on $R = M_n(F)$ is defined as follows. We should consider splitting matrices into the blocks of sizes $(2k_1, 2k_3, \ldots, 2k_{2r-1})$ by merging the first and the second rows (columns) of blocks,..., the $(2r-1)^{\text{st}}$ and the $2r^{\text{th}}$ rows (columns) of blocks. Then in the intersection of each doubled row and column there is a matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$. If all these submatrices of some matrix X are of the form $\begin{bmatrix} A & B \\ B & A \end{bmatrix}$ then $X \in R_{\bar{0}}$. If all submatrices have the form $\begin{bmatrix} A & -B \\ B & -A \end{bmatrix}$ then $X \in R_{\bar{1}}$. We will call the G-graded superalgebras as just described the G-graded superalgebras of the first kind and denote by $Q(\theta; h)$. It is easy to observe that in the case of such superalgebra R we have $\dim R_{\bar{0}} = \dim R_{\bar{1}}$, that is, as a (non-graded superalgebra) R is isomorphic to $M_{m,m}$ where $n = 2m$, for some integer m. Another important observation is that $R_{\bar{0}} \cap R_e \cong M_{k_1} \oplus M_{k_3} \oplus \cdots \oplus M_{k_{2r-1}}$.

In the second case we will have

$$\Phi'' = \operatorname{diag}\{I_{p_1}, -I_{q_1}, \ldots, I_{p_m}, -I_{q_m}\}.$$

Since the conjugation by a diagonal matrix does not change the position of the blocks there are no restrictions on the elements of the tuple θ. As superalgebra, R is isomorphic to $M_{p_1+\cdots+p_m, q_1+\cdots+q_m}$. The structure of $R_{\bar{0}}$ and $R_{\bar{1}}$ is the following. Each row of blocks and each column of blocks is split into two by partitions $k_1 = p_1+q_1, \ldots, k_m = p_m+q_m$. Thus in each position ij of the original splitting into the blocks we will find a block matrix $X_{ij} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. A matrix X is in $R_{\bar{0}}$ if each X_{ij} has the form $X_{ij} = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$. A matrix X is in $R_{\bar{1}}$ if each X_{ij} has the form $X_{ij} = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$. If we denote by $\bar{p}$ the vector $\bar{p} = (p_1, p_2, \ldots, p_m)$ then we will denote the G-graded superalgebra just defined by $A(\theta; \bar{p})$. In the case $R = A(\theta; \bar{p})$ we have $R_{\bar{0}} \cap R_e \cong M_{p_1} \oplus M_{q_1} \oplus \cdots \oplus M_{p_m} \oplus M_{q_m}$.

Notice that, as G-graded superalgebras, none of the superalgebras $Q(\theta; h)$ is isomorphic to any of $A(\theta; \bar{p})$. Indeed, as noted at the beginning of this section, the identity component of the elementary grading defined by a tuple $(\theta = (g_1^{(k_1)}, \dots, g_m^{(k_m)})$ where $g_i \neq g_j$ for $i \neq j$, $k_1, \dots, k_m > 0$ is isomorphic to $M_{k_1} \oplus \dots \oplus M_{k_m}$, that is, has m simple components. So, if two superalgebras R_1 and R_2, one from the first family and the other from the second one, are isomorphic as G-graded superalgebras, then their defining tuples must have the same number components, say, m. It is immediate that $m = 2l$, for some l. But then the number of simple components in $(R_1)_{\bar{0}} \cap (R_1)_e$, as mentioned, above, is l whereas for the second algebra this number can be anything between $m = 2l$ and $2m = 4l$.

Now we can formulate the main result about the elementary gradings of associative superalgebras which are simple associative algebras.

THEOREM 2. *Let G be an arbitrary finite abelian group, F an algebraically closed field of characteristic different from 2, R a G-graded finite-dimensional associative superalgebra which is simple as an associative algebra, whose grading is elementary. Then as a G-graded superalgebra R is isomorphic to one of the algebras $Q(\theta; h)$ or $A(\theta; \bar{p})$.*

4. Graded superalgebra structures on simple associative algebras. General case

In this section we are very close to [**7**]. Suppose an algebra R is equipped with an automorphism φ and S is a φ-invariant subalgebra of R. We say that S is φ-simple if S has no φ-invariant ideals different from S and $\{0\}$.

LEMMA 1. *Let $R = C \otimes D = \bigoplus_{g \in G} R_g$ be a G-graded matrix algebra $M_n(F)$ with an elementary grading on C and fine grading on D. Let φ be a G-graded automorphism of R, whose restriction to R_e is of order 2. Let E denote the identity element of R. Then*

1. *$R_e = C_e \otimes E$ is a φ-invariant subalgebra of R.*
2. *There are φ-simple ideals $B_1, \dots, B_m$ of C_e such that $C_e = B_1 \oplus \dots \oplus B_m$.*
3. *The action of φ on R_e is a conjugation by an $n \times n$-matrix $S = S_1 \otimes E + \dots + S_m \otimes E$ where $S_i \in B_i C B_i$ and either $S_i = \begin{bmatrix} I_{p_i} & 0 \\ 0 & -I_{q_i} \end{bmatrix}$, $p_i + q_i = k_i$ if $B_i \cong M_{k_i}$ or $S_i = \begin{bmatrix} 0 & I_{k_i} \\ I_{k_i} & 0 \end{bmatrix}$, if $B_i \cong M_{k_i} \oplus M_{k_i}$.*
4. *The centralizer of R_e in R decomposes as $Z_1 \otimes D_1 \oplus \dots \oplus Z_m \otimes D_m$ where $D_1, \dots, D_m$ are φ-invariant subalgebras isomorphic to D and $Z_i = Z_i' \otimes E$ where Z_i' is the center of B_i.*
5. *If one of B_i is simple then all of them are simple and in this case both C and D are φ-invariant.*

Proof. Since $D_e = \mathrm{Span}\,\{E\}$ and $\mathrm{Supp}\,C \cap \mathrm{Supp}\,D = \{e\}$, it follows that $R_e = C_e \otimes E$. Since φ respects the G-grading, it follows that $\varphi * (C_e \otimes E) = C_e \otimes E$, that is, $C_e \otimes E$ is φ-invariant. We have already seen at the beginning of Section 3 that C_e decomposes as the sum of ideals $B_1, \dots, B_m$ each of which is φ-invariant, in other words, each B_i is φ-simple. Now it is well-known that with respect to an appropriate basis, the structural map φ of a simple superalgebra has one of the form indicated in Claim (3). Claim (4) is a simple exercise. As for (5), this is the same argument as in Section 3: if, say, B_1 is simple and B_2 is not then $B_2 = P \oplus Q$. The degree of any element in B_1CP will be $g_1^{-1}g_2$. Now φ maps B_1 into itself and changes places of P and Q. Therefore, B_1CP is mapped into $B_1RQ = B_1CQ \otimes D$. The degree of a homogeneous element in $B_1CQ \otimes D$ is $g_1^{-1}g_3t$ for an appropriate $t \in T$. Since φ should respect the degree, we must have $g_3^{-1}g_2 = t$. Here $g_3^{-1}g_2 \in \operatorname{Supp} C$ (this is the degree of any element in QCP) and so we must have $t = 1$ and $g_2 = g_3$, a contradiction. Now the same argument shows that if B_1 and B_2 are simple, we must have B_1CB_2 mapped into itself. Thus if at least one of B_i is simple then by the above all of them are simple and then $C = \sum_{ij} B_iCB_j$ is φ-invariant. In this case also D, which is the centralizer of C in R, has to be φ-invariant, and the proof of Claim (5) is complete. □

The above Lemma applies to graded superalgebras because they are defined by a G-grading and a G-graded automorphism φ of order 2. Claim (5) of Lemma 1 shows that the situation where the elementary and fine components of a G-grading are subsuperalgebras arise quite naturally. Thus this is an appropriate place to formulate the following result. The proof is a combination of what we have obtained in Sections 2 and 3.

THEOREM 3. *Let G be an arbitrary finite abelian group, F an algebraically closed field of characteristic different from 2, R a G-graded finite-dimensional associative superalgebra which is simple as an associative algebra. If fine and elementary components of R as a G-graded algebra, are subsuperalgebras then, as a G-graded superalgebra, R is isomorphic to a superalgebra of one of the following two classes:*

(1) $Q(\theta; h) \otimes P(H; H_0; \alpha)$;
(2) $A(\theta; \bar{p}) \otimes P(H; H_0; \alpha)$.

The parameters used are defined in Theorems 1 and 2. None of the superalgebras of one of these classes with a nontrivial tuple θ can be isomorphic to a superalgebra in the other class. The G-grading and $\mathbb{Z}_2$-grading on the tensor products are defined canonically.

The nonisomorphism claim follows in the same way as in the proof of Theorem 2 because the "identity" components R_e and $R_e \cap R_{\bar{0}}$ of the general R are the same as of its elementary factor.

Now we have to consider the case where the subalgebras C and D are not subsuperalgebras. Let us consider the decomposition $C_e = B_1 \oplus \ldots \oplus B_m$ found in Lemma 1. In this case, as we have seen before, all $B_1, \ldots, B_m$ in the decomposition of C_e are not simple as associative algebras. Let us write $(B_i)_e = P_i \oplus Q_i$. In this case also the defining tuple of the elementary grading on C has the form $\theta = \left[g_1^{(k_1)}, h_1^{(k_1)}, \ldots, g_m^{(k_m)}, h_m^{(k_m)}\right]$. Let e_i be the identity element of P_i and f_i the identity of Q_i. Then the identity of B_i will be $E_i = e_i + f_i$.

The action of φ on R is defined by the rule $\varphi * A = \Phi^{-1} A \Phi$ for some matrix Φ, $A \in R$. If $A \in C_e$ then $A = A_1 \otimes E + \ldots + A_m \otimes E$ where $A_i \in B_i$, $1 \leq i \leq m$. By Lemma 1 φ acts on A as $\varphi * A = S^{-1} AS$ where $S = S_1 \otimes E + \ldots + S_m \otimes E$ with $S_i \in B_iCB_i$ and $S_i = \begin{bmatrix} 0 & I_{k_i} \\ I_{k_i} & 0 \end{bmatrix}$, if B_i is identified with $M_{k_i} \oplus M_{k_i}$. Hence ΦS^{-1} commutes with any $A \in R_e$ and so is an element of the centralizer of R_e in R, which was determined in Claim (4) of Lemma 1. Thus we have

$$\Phi = S_1Y_1 \otimes Q_1 + \ldots + S_mY_m \otimes Q_m \tag{4}$$