DIFFERENTIAL EQUATIONS AND APPLICATIONS

J. B. SCARBOROUGH

DIFFERENTIAL EQUATIONS AND APPLICATIONS

for Students of

Mathematics, Physics, and Engineering

BY

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PREFACE

Differential equations have long been recognized as one of the most powerful instruments for the investigation of natural phenomena. During and after a first course in them, however, the average student fails to realize their power and utility in bringing to light the hidden laws operating in the physical world. The writer thinks that a greater realization of the power and utility of differential equations would result if a course in them included more applications and in greater variety, devoted more time to the derivation of the equations, and put more emphasis on the interpretation of the solutions.

The chief aim of the present book is to show how differential equations arise, how to solve them in the most direct manner, and what the solutions mean or imply. An additional aim is to exhibit the power and utility of differential equations by showing the student what they can do. In the attempt to accomplish these aims, all differential equations in the applications are, with two or three exceptions, derived *ab initio* from fundamental principles or from known physical laws; they are derived and solved in *literal form*; and the solutions in most cases are interpreted and discussed as to their meaning and implications. This procedure shows the student how differential equations come about and what their solutions tell about the phenomenon in question.

Because of the large number and variety of applications, it has been thought best to separate the book into two parts, the first part dealing with the solution of differential equations and the second dealing with applications. Although the second part fills the greater number of pages, the first part contains an adequate treatment of all the methods and processes needed in the applications. When used as a textbook, the teacher may use such topics and problems from each part as best suits his purposes.

In Part I, I have attempted to give a clear and sufficient treatment of what I consider the best methods of solving the types of equations that arise in the applications. Special attention has been given to linear equations with constant coefficients because of their great importance in applied mathematics. Because I think the method of undetermined coefficients and the method of Laplace transforms are the best methods of solving the vast majority of such equations, I have taken special pains to explain the former method with unusual thoroughness and have explained the latter method sufficiently for the applications in this book. Many of the equations in Part II are solved by both methods for comparison. To facilitate the use of Laplace transforms I have appended short tables of direct and inverse transforms at the end of the book.

Certain types of differential equations with variable coefficients are of scarcely less importance than those with constant coefficients. I have therefore given a fairly extensive treatment of such equations in Chapter III.

Although the chapter on partial differential equations is not long, it treats in sufficient detail the types of equations that are of most importance in physics and engineering

In recent years the subject of Fourier's Series has become an important item in the educational equipment of engineers. That subject has therefore been given an adequate treatment in Chapter V.

The problems and topics treated in Part II have been selected because of their interest, instructive value, practical importance, or for all of these reasons. Some of them are above the elementary level, but they have all been brought within the easy understanding of students who have had the usual college course in physics and a first course in calculus. An effort has been made to give all the whys and wherefores pertaining to each problem or topic. The large number and wide variety of applications should give the instructor considerable latitude in the selection of problems to suit the needs and interests of his students.

In the applications of differential equations to natural phenomena, it is sometimes necessary to make legitimate approximations in the derivation of the equations or in their solution. When such approximations have been made, I have tried in all cases to justify them by showing their nature and magnitude, in order that the student may have confidence in the soundness of the results obtained.

The many solved problems enable the student to see how initial and boundary conditions are used to determine constants of integration under a variety of conditions. They also exhibit certain techniques that must sometimes be used in applied problems.

Numerical examples are included here and there to add interest, to illustrate the use of literal results, and to show the magnitudes of the quantities involved.

J. B. Scarborough

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Suggestions to Teachers

When this book is used as a textbook, some teachers may prefer to take the applications in Part II along with the text material of Part I. In that case the following suggestions are offered:

Chapters VI–X (except Arts. 68–69) can be studied after Chapter I is finished; Chapters XI (except Arts. 82–84) and XIII can be studied after Chapter II is finished; Chapters XVI, XVIII, XX, and XXI can also be studied after Chapter II is finished; Arts. 68–69, 82–84, and Chapters XII, XIV, XV, and XVII can be studied after Chapters III and IV are finished. Chapter XIX should be taken after Chapter V is finished.

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PART I

SOLUTION OF DIFFERENTIAL EQUATIONS

CHAPTER I

EQUATIONS OF THE FIRST ORDER. SPECIAL EQUATIONS OF THE SECOND AND HIGHER ORDERS

1. Definitions and Preliminary Statements. A *differential equation* is an equation that involves derivatives or differentials. Thus the following are differential equations of several types:

(1) $$\frac{dy}{dx} + 3y \tan x = 0$$

(2) $$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = \sin t$$

(3) $$y\frac{dy}{dx} - 2x\left(\frac{dy}{dx}\right)^2 - xy\left(\frac{d^2y}{dx^2}\right)^2 = e^{2x}$$

(4) $$y^2\frac{\partial z}{\partial x} - ay\frac{\partial z}{\partial y} = bxz$$

(5) $$\frac{\partial^2 z}{\partial x^2} = k^2\frac{\partial^2 z}{\partial t^2}$$

An *ordinary* differential equation is a differential equation involving only one independent variable. Thus (1), (2), and (3) above are ordinary differential equations.

A *partial* differential equation is a differential equation that involves two or more independent variables, as (4) and (5) above.

The *order* of a differential equation is the order of the highest derivative in it. Thus, in the above list (1) is of the first order and all the others are of the second order.

The *degree* of a differential equation is the degree of the highest-order derivative appearing in it when the equation is free from radicals. In the above list (3) is of the second degree, while all the others are of the first degree.

A differential equation in which the dependent variable and all its derivatives are of the first degree and none multiplied together is called a *linear* equation. Stated otherwise, a differential equation is linear if it is linear in the dependent variable and all its derivatives.

A partial differential equation of the *first order* is defined to be linear if the partial derivatives are of the first degree and the dependent variable is of any degree and occurs in any manner (see Art. 32).

In the above list of equations all except (3) are linear.

To *solve* a differential equation is to integrate it in some manner and thereby obtain an equation which does not contain derivatives. The *solution* of a differential equation is thus any value of the function which satisfies the given equation.

Only a few simple differential equations can be integrated directly. The solutions of most equations must be found by indirect methods, each type having its own method of solution. In order to solve differential equations with the least amount of labor, the student must be able to recognize each of the several types and recall the proper method of solving it.

The number of differential equations which can be integrated in exact, finite form is very limited. When no exact, finite solution can be found, the solution must be expressed as a power series or else tabulated step by step by numerical processes.

The student will recall from calculus that, except in the case of definite integrals, it was necessary to add a constant of integration each time an integration was performed. The same is true in the integration of differential equations. Hence we may infer that the solution of a second-order ordinary differential equation, for example, will contain two constants of integration, since the equivalent of two integrations must be performed in solving the equation. It is proved in works dealing with the theory of differential equations that the *general* and *complete* solution of an ordinary differential equation of the nth order must contain n arbitrary constants of integration, and no more.

A *particular solution* of a differential equation is any solution which can be obtained from the general solution by assigning fixed values to some or all of the arbitrary constants in the latter.

A third kind of solution of a differential equation is the *singular solution*. It contains no arbitrary constants and cannot be derived from the general solution by assigning fixed values to the constants in that solution. Singular solutions will not be considered in this book.

Since every solution of a differential equation must satisfy the given equation, it is usually an easy matter to check the correctness of a solution by substituting it into the given differential equation. Such a check should be made in any case of importance, or whenever there is any doubt as to the correctness of a solution.

As an example of such a check, we will show that

$$\text{(a)} \qquad y = c_1x^3 + c_2 + c_3 \ln x$$

is a solution of the differential equation

$$\text{(b)} \qquad x^2 \frac{d^3y}{dx^3} - 2\frac{dy}{dx} = 0$$