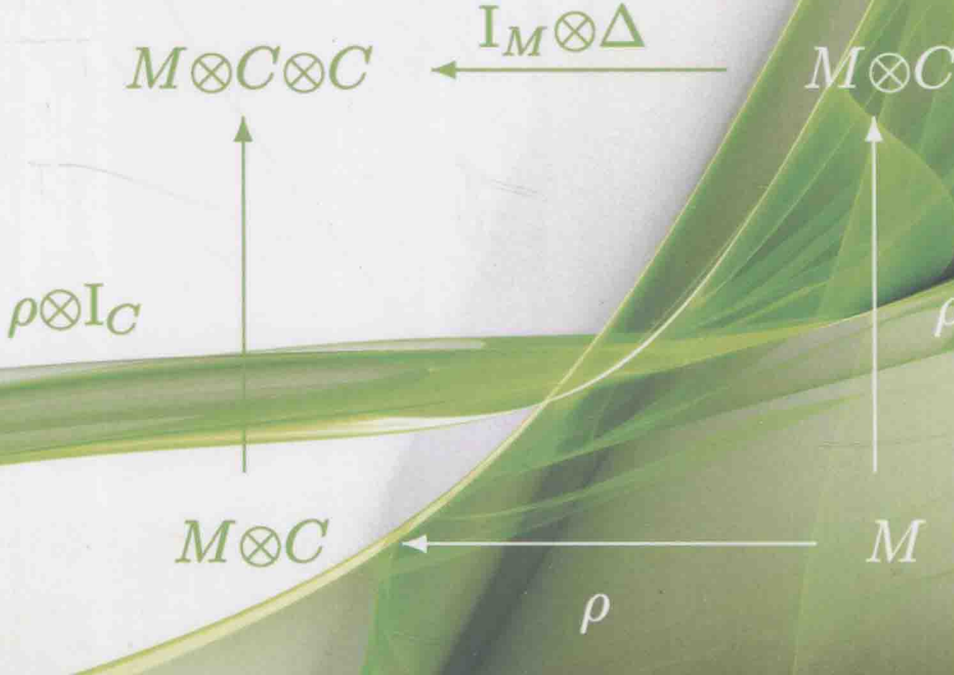
Series on Knots and Everything — Vol. 49

Hopf Algebras

David E Radford



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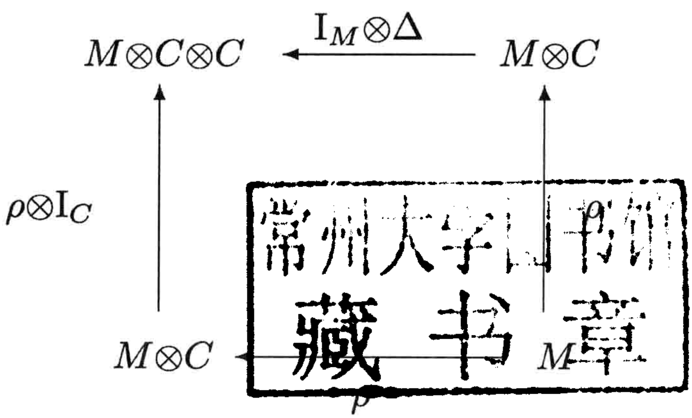

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Preface

The subject of Hopf algebras originated in an algebraic topology paper by Heinz Hopf [75] in 1941. Not long after a notion of Hopf algebra was formulated for the category of vector spaces over a field k. Hopf algebras of this type, which we refer to as Hopf algebras over a field, were seen to arise in a number of settings: as underlying algebras of affine groups in the theory of algebraic groups, as formal groups in number theory, and as universal enveloping algebras in Lie theory. This book is about Hopf algebras over a field k.

A general theory for them was beginning to emerge by the time of Sweelder's book [201] in 1969. Against the background of its development, applications to other areas were being discovered, for example to combinatorics and computer science [79, 185]. Other areas of mathematics were being treated in the context of Hopf algebras, for example Galois theory [30]. Ideas from ring theory were being applied to aspects of Hopf algebra theory [217]. Hopf algebra techniques were being used to provide purely algebraic proofs of results on algebraic groups [199, 202, 212, 213]. Affine group structures were being studied as algebraic objects related to Hopf algebras [215, 216].

The paper by Drinfel'd [44] in 1986 on quantum groups opened the floodgates for applications of Hopf algebras to physics, invariant theory for knots and links, and representations closely connected to Lie theory. The Hopf algebras involved are referred to as quantum groups, a term still lacking formal definition. Quantum group theory includes much more than Hopf algebras. A great number of mathematicians have contributed to the subject. Some contributions and references are given in chapter notes.

The subject of Hopf algebras continues to grow in many directions. More recent considerations have led to generalizations of the notion of Hopf algebra. We do not pursue the threads of evolution here; a quick foray into the literature will begin to reveal them.

This book is intended to be a graduate text and to be used by researchers in Hopf algebras and related areas. It does not replace standard texts, such as the book by Abe [1], by Dăscălescu, Năstăsescu, and Raianu [35], by Montgomery [133], or by Sweedler [201]. Each has some material not included in this one. This book reflects the deep influence of quantum groups on the subject and recent developments in the theory of pointed Hopf algebras.

Prerequisites are few. The reader is expected to be familiar with linear algebra, elementary abstract algebra including the tensor product and basic representation theory of algebras, and also with rudimentary knowledge of the language of category theory. What follows is an overview of the book chapter by chapter.

In Chapter 1 we set basic notation conventions. We cover linear algebra needed for the book in two brief sections, in discussion and through accompanying exercises.

Two notions from linear algebra play a basic role in this text, the rank of tensors and the concept of closed subspaces of a dual vector space. Most structures in Hopf algebras have an underlying vector space and give rise to important structures on the dual space. For any vector space V over k there is an inclusion reversing bijective correspondence between the subspaces of V and the closed subspaces of its linear dual V^*. The connection between structures on V and those on V^* via the bijection will be examined over and over again.

Most sections in this text come with a generous supply of exercises. Some exercises develop new ideas, new directions, or further results. Exercises for completing proofs tend to be accompanied by generous hints. Categorical aspects of material are usually developed in the exercises.

Chapter 2 is the first of a three-chapter treatment of coalgebras and their representations. Hopf algebras have an algebra and a coalgebra structure. Why the coalgebra structure is important. It affords the Hopf algebra with locally finite properties which algebras generally do not possess. It provides a way to form the tensor product of representations of the Hopf algebra's underlying algebra structure.

Algebras and coalgebras are dual structures in a categorical sense. We think of an algebra A over k formally as a triple (A, m, η), where

$$m : A\otimes A \longrightarrow A \qquad \text{and} \qquad \eta : k \longrightarrow A$$

are linear maps, m describes multiplication, η describes the unity of A via

$\eta(1_k) = 1_A$, such that certain commutative diagrams are satisfied which express that multiplication is associative and 1_A is the multiplicative neutral element of A. Thus a coalgebra C over k is formally a triple (C, Δ, ϵ), where

$$\Delta : C \longrightarrow C \otimes C \qquad \text{and} \qquad \epsilon : C \longrightarrow k$$

are linear maps, such that the commutative diagrams for A hold for C with arrows reversed, where C replaces A and ϵ replaces η. Δ is referred as the comultiplication map and ϵ to as the counit map.

Suppose A is an algebra over the field k. Then the dual vector space A^* contains a unique maximal coalgebra A^o derived from the algebra structure on A, called the dual coalgebra. When A is finite-dimensional $A^o = A^*$. Let C be a coalgebra over k. Then C^* has an algebra structure derived from the coalgebra structure of C, called the dual algebra.

A finite-dimensional subspace of C generates a finite-dimensional subcoalgebra of C. Therefore the theory of finite-dimensional algebras over k can be used to study coalgebras over k. And simple coalgebras are finite-dimensional.

The bijective correspondence between the subspaces of C and the closed subspaces of C^* is an important means for relating the coalgebra structures of C and the algebra structures of C^*. A significant operation in C is the wedge product which corresponds to multiplication of subspaces in C^*.

There is a very important finite-dimensional coalgebra which makes an appearance over and over again in the theory of Hopf algebras. This is $\mathrm{C}_n(k) = \mathrm{M}_n(k)^*$, where $\mathrm{M}_n(k)$ is the algebra of $n \times n$ matrices over the field k. Just as every finite-dimensional algebra over k is a subalgebra of $\mathrm{M}_n(k)$ for some $n \geq 1$, every finite-dimensional coalgebra over k is a quotient of $\mathrm{C}_n(k)$ for some $n \geq 1$.

Chapter 2 ends with the first of many dual constructions, here with the construction of the cofree coalgebra on a vector space. This coalgebra is the categorical dual of the free algebra on a vector space. The cofree coalgebra and free algebra play important roles in the structure theory of Hopf algebras.

Representation theory of coalgebras is a very extensive subject as is the representation theory of algebras. In Chapter 3 we focus on aspects of it needed for the sequel. Objects of study are comodules of C. These can be thought of as the rational C^*-modules, that is locally finite C^*-modules whose elements are annihilated by a *closed* cofinite ideal of C^*. Here the notion of closed subspace enters the theory.

Injective rational modules form an important class which we treat in detail. Results on them are applied to the study of indecomposable coalgebras.

Simple rational modules, or equivalently simple comodules, are finite-dimensional and are closely related to simple subcoalgebras. We begin to study the coradical of C, the sum of all simple subcoalgebras of C, in this chapter.

One of the most important structures associated with the coradical is the coradical filtration of C. Chapter 4 is devoted to its study. The terms of the coradical filtration of C correspond to the closures of powers of the Jacobson radical of C^*. Its terms can be expressed by the wedge product. Certain families of idempotents of C^* provide useful decompositions of the terms of the coradical filtration.

Several coalgebras are associated with C which have a simpler coradical and whose coradical filtrations are related to that of C in a good way. We show how they can be used to study C.

Chapter 5 introduces bialgebras and Chapter 7 Hopf algebras. Perhaps the relationship between bialgebras and Hopf algebras can be thought of as the relationship between monoids and groups. Group theory is very rich. For monoids assumptions are made at times to compensate for the lack of inverses.

Bialgebras are vector spaces with compatible algebra and coalgebra structures. A bialgebra H over k is formally a tuple $(H, m, \eta, \Delta, \epsilon)$, where (H, m, η) is an algebra over k and (H, Δ, ϵ) is a coalgebra over k, such that Δ and ϵ are algebra maps. Since algebra and coalgebra are dual concepts, the notion of bialgebra is self-dual. Bialgebras have the important property that the tensor product of representations of their underlying algebra structures can be formed. Hopf algebras are bialgebras with an antipode, an endomorphism analogous to the map of groups which takes elements to their inverses. For bialgebras assumptions are made at times to compensate for the lack of an antipode.

Two examples of Hopf algebras. The first is the group algebra $k[G]$ of a group G over k. Here

$$\Delta(g) = g \otimes g \qquad \text{and} \qquad \epsilon(g) = 1$$

for all $g \in G$. An element g of any bialgebra H over k which satisfies the two preceding equations is called grouplike and the set of grouplike elements of H is denoted $\mathrm{G}(H)$. The antipode S of $k[G]$ is determined by $S(g) = g^{-1}$ for all $g \in G$.

The second is the universal enveloping algebra $U(L)$ of a Lie algebra L over k. Here

$$\Delta(\ell) = 1 \otimes \ell + \ell \otimes 1 \qquad \text{and} \qquad \epsilon(\ell) = 0$$

for all $\ell \in L$ which determines the coalgebra structure of $U(L)$ since L generates $U(L)$ as an algebra and Δ, ϵ are algebra maps. An element ℓ of any bialgebra H over k which satisfies the two preceding equations is called primitive and the set of primitive elements of H is denoted $\mathrm{P}(H)$. The antipode of $U(L)$ is an algebra anti-endomorphism of $U(L)$. The antipode S is determined by $S(\ell) = -\ell$ for all $\ell \in L$.

Note that $1 \in \mathrm{G}(H)$ for any bialgebra H over k since Δ and ϵ are algebra maps. In particular $k1$ is a simple subcoalgebra of B. In terms of the coradical $k[G]$ and $U(L)$ are at the opposite ends of the spectrum. The coradical of $k[G]$ is all of $k[G]$. The coradical of $U(L)$ is the one-dimensional simple subcoalgebra $k1$ and is therefore as small as possible.

Chapter 5 emphasizes certain universal constructions for bialgebras. These have applications to Hopf algebras. They are based on the free algebra and the cofree coalgebra described earlier. The free algebra on a vector space over k is used to construct the free bialgebra on a coalgebra over k. The cofree coalgebra on a vector space over k is used to construct the dual counterpart to the free bialgebra, the cofree bialgebra on an algebra over k.

As algebras are quotients of free algebras on vector spaces, bialgebras are quotients of free bialgebras on coalgebras. We discuss a quotient construction in great detail since it is a model for construction of Hopf algebras from a typical presentation, a coalgebra which generates the Hopf algebra and relations among its elements.

Enter $\mathrm{C}_n(k)$ into the theory of bialgebras. Every finite-dimensional coalgebra over k is a quotient of $\mathrm{C}_n(k)$ for some $n \geq 1$. A finitely generated bialgebra over k is generated by a finite-dimensional subcoalgebra and therefore is the quotient of the free bialgebra on $\mathrm{C}_n(k)$ for some $n \geq 1$.

We have observed the notion of bialgebra is self dual. As there is a dual coalgebra and a dual algebra, there is dual bialgebra. Suppose A is a bialgebra over k. We have noted the algebra structure of A gives rise to a coalgebra A^o and the coalgebra structure of A accounts for an algebra structure on A^*. The subspace A^o of A^* is a subalgebra. The coalgebra structure of A^o together with this subalgebra structure makes A^o a bialgebra, called the dual bialgebra.

Chapter 6 is a short technical chapter which expands a bit on results used to analyze the antipode. Chapter 7 introduces Hopf algebras with an extensive discussion of the antipode and proceeds right away to the construction of two families of finite-dimensional Hopf algebras. Involved is the relation $xa = qax$, where q is a non-zero scalar. This relation arises in many well-known families of Hopf algebras. As a result Gaussian integers and q-binomial symbols play a role in the theory of Hopf algebras. We realize Hopf algebras in our two families as quotients of free bialgebras. A construction detail we discuss in depth is how to find a basis for the quotient. Generally there are various methods which might work. We demonstrate application of the Diamond Lemma for the first of many times.

The free and cofree bialgebra constructions of Chapter 5 are modified for Hopf algebras and variants of the cofree construction are introduced, notably the shuffle algebra. The shuffle algebra is used to study Hopf algebras whose coradical is one-dimensional.

In some cases an algebra multiplication can be "twisted" to form another algebra. There are notions of twisting for Hopf algebras, alterations of the algebra or the coalgebra structure, which give rise to other Hopf algebras. Twistings are important in the classification of certain families of Hopf algebras. Hopf algebras also give rise to others via the dual. The dual bialgebra H^o of a Hopf algebra H over k is a Hopf algebra. In particular H^* is a Hopf algebra over k when H is a finite-dimensional.

Chapter 8 describes the complete theory of Hopf modules; there is but one main result. Its power lies in numerous important applications. Let H be a Hopf algebra over k. A left H-Hopf module is a vector space with a left H-module and a left H-comodule structure which are compatible in a certain way. There are other compatibility conditions. These are discussed in Chapter 11.

An example of a left H-Hopf module is H with multiplication and comultiplication as module and comodule structures. Direct sums of Hopf modules are Hopf modules. Every non-zero left H-Hopf module M is isomorphic to the direct sum of copies of H. In particular M is a free left module over the algebra H. There is a subspace of $M_{co\,inv}$ of M described in terms of the comodule structure and any linear basis for M is a module basis.

When M is finite-dimensional the dual vector space M^* has the structure of a co-Hopf module. The Hopf module and co-Hopf module structures go a long way in accounting for the deeper properties of finite-dimensional Hopf algebras.

In Chapter 9 we consider when H is a free module over its Hopf subalgebras. Various conditions are given, some involving the coradical of H. The most important result is that finite-dimensional Hopf algebras are free over their Hopf subalgebras. This is established by using a modified notion of Hopf module. This result has very important implications for the structure of finite-dimensional Hopf algebras. There are examples of commutative Hopf algebras over the field of complex numbers which are not free over one of their Hopf subalgebras.

Chapter 10 is the most important chapter in the book on the structure of a finite-dimensional Hopf algebra H over k. The story is told in terms of non-zero integrals. A non-zero integral for H is a generator or a certain one-dimensional ideal of H. Existence and uniqueness (up to scalar multiple) of non-zero integrals follow as an application of the structure theorem for Hopf modules. This theorem also implies that only finite-dimensional Hopf algebras can possess a non-zero finite-dimensional ideal.

Semisimplicity of H or H^* as an algebra is determined by integrals. The trace function for endomorphisms of H can be computed in terms of an integral for H and an integral for H^*. Important formulas which express the trace of the square of the antipode in terms of integrals for H and H^* follow as a result.

There is a distinction between left and right integral for H. Suppose H is finite-dimensional. The relationship between the two accounts for a certain Hopf algebra automorphism of H. The composition of this automorphism with the transpose of its counterpart for H^* is the fourth power of the antipode of H. As a consequence the antipode of H has finite order as an endomorphism.

Generally one-dimensional ideals of H and H^* are important. Their generators are referred to as non-zero generalized integrals. Suppose H is not finite-dimensional. Then left (and right) integral can be defined for the algebra H^*. A non-zero left or right integral for H^* may not exist, but if so it is unique.

Chapter 11 is devoted to actions by a bialgebra or Hopf algebra H over k; especially actions on algebras or coalgebras. We discuss these actions since they account for useful constructions on tensor products where H as a factor.

Let ${}_H\mathcal{M}$ be the category whose objects are left H-modules and morphisms are module maps under function composition. Since H is a bialgebra the tensor product of two objects of ${}_H\mathcal{M}$ is defined and k has the structure of an object of ${}_H\mathcal{M}$. A left H-module algebra A is an algebra in ${}_H\mathcal{M}$; that is a k-algebra which is also a left H-module such that the algebra structure maps of A are module maps. For such an algebra A the smash product

$A\sharp H$ can be formed. The smash product is a k-algebra and is $A\otimes H$ as a vector space. A left H-module coalgebra is a k-coalgebra which is also a coalgebra in ${}_H\mathcal{M}$.

Now let ${}^H\mathcal{M}$ be the category whose objects are left H-comodules and whose morphisms are comodule maps under function composition. Since H is a bialgebra over k the tensor product of objects of ${}^H\mathcal{M}$ is defined and k has the structure of an object of ${}^H\mathcal{M}$. The notions of left H-comodule algebra and left H-comodule coalgebra are defined as above. If C is a left H-comodule coalgebra then the smash coproduct $C\natural H$ is defined. It is a k-coalgebra and as a vector space is $C\otimes H$. Smash products and smash coproducts are dual structures.

A natural question to ask at this point is when a smash product and smash coproduct structure form a bialgebra. The question brings us to a rather complicated setting which turns out to be important for several reasons.

Let ${}^H_H\mathcal{M}$ be the category whose objects are left H-modules and left H-comodules and whose morphisms are module and comodule maps. Suppose A is an object of ${}^H_H\mathcal{M}$ which is both a left H-module algebra and a left H-comodule coalgebra. Then $A\otimes H$ is a bialgebra with algebra structure $A\sharp H$ and coalgebra structure $A\natural H$ if and only if A is a left H-module coalgebra, a left H-comodule algebra, and there is a certain compatibility between the module and comodule structures on A. We let ${}^H_H\mathcal{YD}$ be the full subcategory category of ${}^H_H\mathcal{M}$ whose objects satisfy this compatibility condition. ${}^H_H\mathcal{YD}$ is called a Yetter-Drinfel'd category and its objects are called left Yetter-Drinfel'd modules. Our conclusion: $A\otimes H$ is a k-bialgebra with the smash product and the smash coproduct structures if and only if A is a bialgebra in ${}^H_H\mathcal{YD}$. In this case we denote $A\otimes H$ with its bialgebra structure by $A\times H$ and call this bialgebra a biproduct.

We are glossing over the subtle points of how to form the tensor product of algebras and coalgebras in ${}^H_H\mathcal{YD}$. These important details are thoroughly examined in Chapter 11.

Now suppose that H is a Hopf algebra. Then ${}^H_H\mathcal{YD}$ is a braided monoidal category. These categories are a natural starting point for the construction of invariants of knots and links.

There is a way of characterizing biproducts of the form $B\times H$ when H is a Hopf algebra over k. Suppose A is a bialgebra over k. Then A is a biproduct of the form $A = B\times H$ if and only if there are bialgebra maps $A \underset{\pi}{\overset{\jmath}{\leftrightarrows}} H$ which satisfy $\pi\circ\jmath = \mathrm{I}_H$; loosely speaking if and only if there is a bialgebra projection from A onto H.

Chapter 12 is an in depth analysis of quasitriangular Hopf algebras, structures which produce invariants of knots and links, and sometimes of 3-manifolds. A quasitriangular Hopf algebra over k is a pair (H, R), where H is a Hopf algebra over k and $R \in H \otimes H$, which satisfies axioms guaranteeing that R is a solution to the quantum Yang–Baxter equation. Solutions to this equation provide a means of constructing invariants of knots and links. For invariants of 3-manifolds an extra bit of structure is needed, an element $v \in H$ called a ribbon element. The antipode of a quasitriangular Hopf algebra over k is bijective.

For the purposes of computing knot or link invariants we may assume (H, R) is minimal quasitriangular, meaning H is the smallest Hopf subalgebra K of H such that $R \in K \otimes K$. In this case H is finite-dimensional.

Finite-dimensional quasitriangular Hopf algebras abound. Suppose H is a finite-dimensional Hopf algebra over k. A quasitriangular Hopf algebra $(D(H), \mathcal{R})$ can be constructed, where $\mathrm{Dim}(D(H)) = \mathrm{Dim}(H)^2$, and H is a Hopf subalgebra of $D(H)$. The pair $(D(H), \mathcal{R})$ is called the Drinfel'd, or quantum, double.

Perhaps $D(H)$ is one of the most important examples of a finite-dimensional Hopf algebra. Chapter 13 is devoted to the study of $(D(H), \mathcal{R})$. It is minimal quasitriangular and factorizable. $D(H)$ is unimodular and the square of its antipode is an inner automorphism. Whether or not $(D(H), \mathcal{R})$ has a ribbon element is related to the formula for the fourth power of the antipode of H mentioned above.

Chapter 14 deals with the dual notions of quasitriangular algebra, bialgebra, and Hopf algebra; these are coquasitriangular coalgebra, bialgebra, and Hopf algebra respectively. Here R is replaced by a bilinear form and the quasitriangular axioms are replaced by axioms for the form. A proof that the antipode of a quasitriangular Hopf algebra is bijective can be modified to give a proof that the antipode of a coquasitriangular Hopf algebra is bijective. The free bialgebra on a coalgebra is the basis for the construction of the free coquasitriangular bialgebra on a coquasitriangular coalgebra over k.

Two classes of Hopf algebras have been studied extensively; pointed and semisimple. A Hopf algebra is pointed if its simple subcoalgebras are one-dimensional. This is equivalent to saying its coradical is a Hopf subalgebra isomorphic to a group algebra $k[G]$. Pointed Hopf algebras figure prominently in the theory of quantum groups.

Classification of pointed Hopf algebras is complete in the finite-dimensional case when k is algebraically closed of characteristic zero, G is abelian, and minor constraints are placed on orders of elements of G. For many important classes of pointed Hopf algebras G is abelian.

The purpose of Chapter 15 is to set the stage for the study of pointed Hopf algebras and their classification. We provide a detailed construction of the quantized enveloping algebras and their generalizations. One of the earliest results for Hopf algebras was the structure theorem for cocommutative pointed Hopf algebras when k has characteristic zero. Cocommutative means the dual algebra is commutative. Such a Hopf algebra is a smash product $U\sharp k[G]$, where U is the universal enveloping algebra of a Lie algebra. Generally a pointed Hopf algebra over any field k is a crossed product $\mathcal{U}\sharp_\sigma k[\mathcal{G}]$, where $\mathcal{U}$ is an indecomposable pointed Hopf algebra and $\mathcal{G}$ is a quotient of G.

Now we consider a recipe for classification. Since the coradical $k[G]$ of H is a Hopf subalgebra of H, there is an associated graded Hopf algebra $\mathrm{gr}(H)$ with coradical $k[G]$ and $\mathrm{gr}(H) = B \times k[G]$ is a biproduct, where B is a pointed irreducible Hopf algebra in the Yetter-Drinfel'd category ${}_{k[G]}^{k[G]}\mathcal{YD}$. By definition B is a Nichols algebra when it is generated as an algebra by its space of primitive elements $B(1)$.

The recipe for classification is to pass from H to $\mathrm{gr}(H)$, show that B is a Nichols algebra, find defining relations for B, and use them to determine defining relations for H. The analysis of Nichols algebras is well beyond the scope of this book. The space $B(1)$ is an object of ${}_H^H\mathcal{YD}$. We do show for every object V of ${}_H^H\mathcal{YD}$ there is a Nichols algebra $B(V)$ in ${}_H^H\mathcal{YD}$ with $B(V)(1) = V$ and show that Nichols algebras can be defined by a universal mapping property.

Suppose that k is algebraically closed of characteristic zero. Chapter 16 treats a few topics in the theory of finite-dimensional Hopf algebras over k. In the first section semisimple Hopf algebras are characterized in various ways and the fact that their antipodes are involutions is established.

For a given dimension there are only finitely many isomorphism classes of semisimple Hopf algebras. This is not the case for pointed Hopf algebras as shown in the second section. The third and last section of the chapter states some very useful results for classification of semisimple Hopf algebras over k. Just a few proofs are included.

The chapter notes contain a lengthy discussion of the current status of classification of finite-dimensional Hopf algebras, in particular of semisimple ones. In contrast to the pointed case, classification of semisimple Hopf algebras over k is still in beginning stages.

It has been my great pleasure to get to know many who have contributed to Hopf algebras and related areas. Their collective contributions are far to numerous to completely list here. I would like to recognize my coauthors: Nicholás Andruskiewitsch, Robert Grossman, Robert Heyneman, Louis Kauffman, Leonid Krop, Larry Lambe, Richard Larson, Kenneth Newman, Stephen Sawin, Hans-Jürgen Schneider, Earl Taft, Jacob Towber, Sara Westreich, and Robert Wilson. Work with them was always interesting and fruitful.

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David E. Radford

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