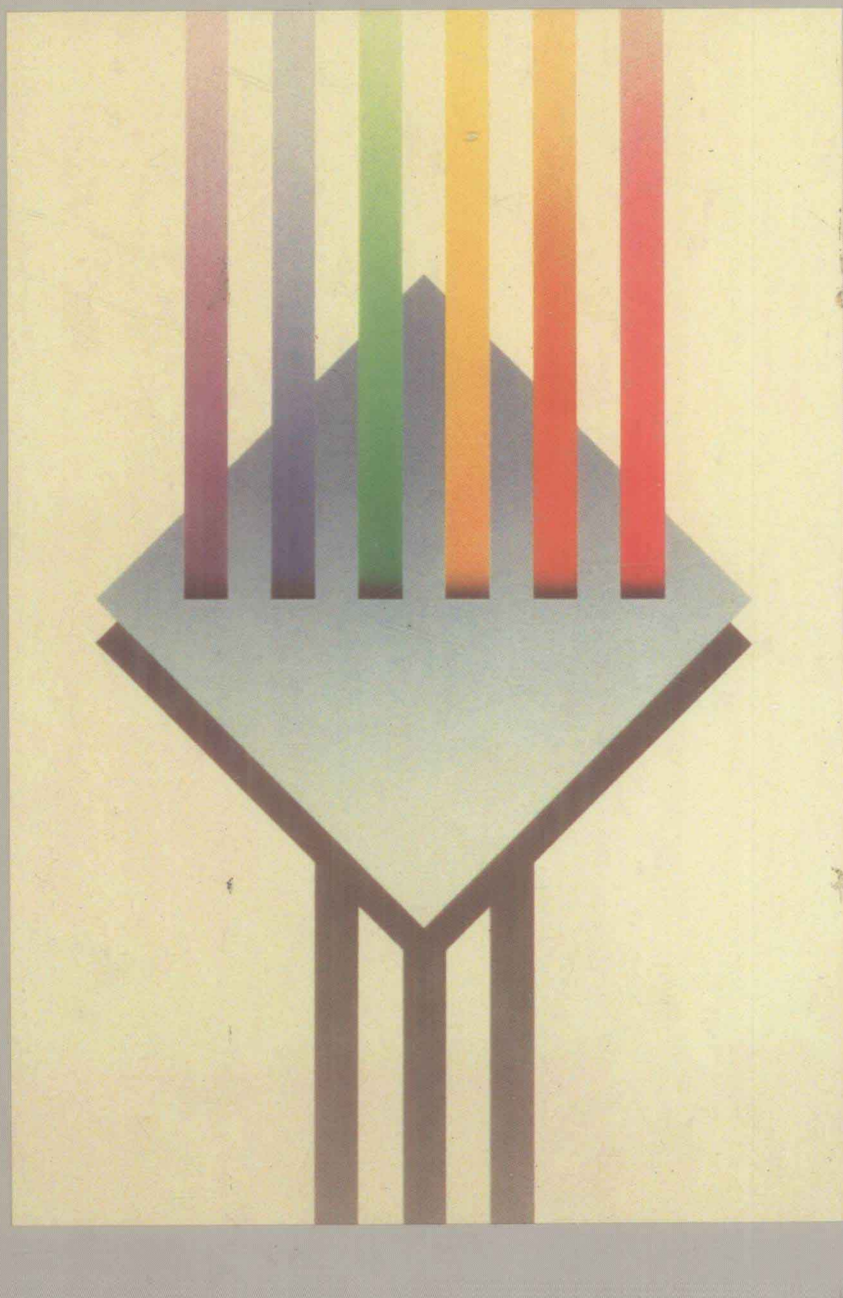
TRIGONOMETRY

GANTNER ■ GANTNER



TRIGONOMETRY

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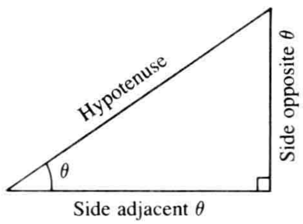
TRIGONOMETRIC FUNCTIONS OF ACUTE ANGLES

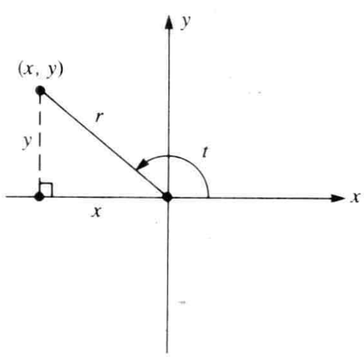


$$\sin \alpha = \frac{\text{side opposite } \theta}{\text{hypotenuse}}, \qquad \csc \alpha = \frac{\text{hypotenuse}}{\text{side opposite } \theta},$$

$$\cos \alpha = \frac{\text{side adjacent } \theta}{\text{hypotenuse}}, \qquad \sec \alpha = \frac{\text{hypotenuse}}{\text{side adjacent } \theta},$$

$$\tan \alpha = \frac{\text{side opposite } \theta}{\text{side adjacent } \theta}, \qquad \cot \alpha = \frac{\text{side adjacent } \theta}{\text{side opposite } \theta}.$$

TRIGONOMETRIC FUNCTIONS OF ARBITRARY AI

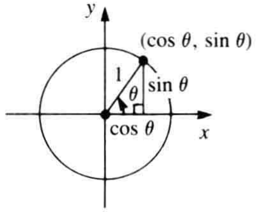


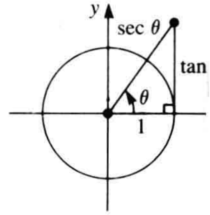
$$\sin t = \frac{y}{r},$$

$$\cos t = \frac{x}{r},$$

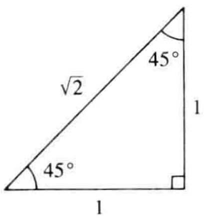
$$\tan t = \frac{y}{x}$$

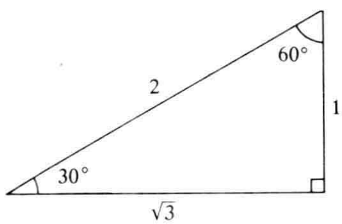
UNIT CIRCLE INTERPRETATION OF THE TRIGON





SPECIAL RIGHT TRIANGLES





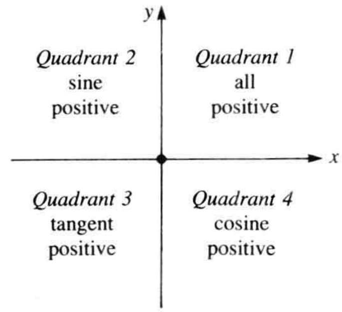
CONVERSION FORMULAS

$1° = \frac{\pi}{180}$ radians, $\quad$ 1 radian $= \left(\frac{180}{\pi}\right)^{\circ}$

SPECIAL VALUES OF TRIGONOMETRIC FUNCTIONS

θ (*Degrees*)	θ (*Radians*)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
0°	0	0	1	0	undefined	1	undefined
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	1	0	undefined	0	undefined	1

ALGEBRAIC SIGNS OF THE TRIGONOMETRIC FUNCTIONS



REFERENCE ANGLE CHART

Angle t in	Reference Angle *Degrees*	Reference Angle *Radians*
Quadrant 2	$180° - t$	$\pi - t$
Quadrant 3	$t - 180°$	$t - \pi$
Quadrant 4	$360° - t$	$2\pi - t$

LAW OF SINES

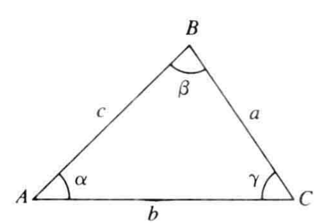
$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

LAW OF COSINES

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$



FUNDAMENTAL TRIGONOMETRIC RELATIONS

$$\tan t = \frac{\sin t}{\cos t}, \qquad \sec t = \frac{1}{\cos t},$$

$$\cot t = \frac{\cos t}{\sin t}, \qquad \csc t = \frac{1}{\sin t}.$$

PYTHAGOREAN IDENTITIES

$$\sin^2 t + \cos^2 t = 1,$$

$$\tan^2 t + 1 = \sec^2 t,$$

$$\cot^2 t + 1 = \csc^2 t.$$

NEGATIVE ANGLE IDENTITIES

$$\sin(-t) = -\sin t, \qquad \cos(-t) = \cos t,$$

$$\csc(-t) = -\csc t, \qquad \sec(-t) = \sec t,$$

$$\tan(-t) = -\tan t, \qquad \cot(-t) = -\cot t.$$

PERIODIC IDENTITIES

$$\sin(t \pm k2\pi) = \sin t, \qquad \cos(t \pm k2\pi) = \cos t,$$

$$\csc(t \pm k2\pi) = \csc t, \qquad \sec(t \pm k2\pi) = \sec t,$$

$$\tan(t \pm k\pi) = \tan t, \qquad \cot(t \pm k\pi) = \cot t.$$

COMPLEMENTARY ANGLE IDENTITIES

$$\sin\left(\frac{\pi}{2} - t\right) = \cos t, \qquad \cos\left(\frac{\pi}{2} - t\right) = \sin t,$$

$$\csc\left(\frac{\pi}{2} - t\right) = \sec t, \qquad \sec\left(\frac{\pi}{2} - t\right) = \csc t,$$

$$\tan\left(\frac{\pi}{2} - t\right) = \cot t, \qquad \cot\left(\frac{\pi}{2} - t\right) = \tan t.$$

SINE SUM AND DIFFERENCE IDENTITIES

$$\sin(s + t) = \sin s \cos t + \cos s \sin t,$$

$$\sin(s - t) = \sin s \cos t - \cos s \sin t.$$

COSINE SUM AND DIFFERENCE IDENTITIES

$$\cos(s + t) = \cos s \cos t - \sin s \sin t,$$

$$\cos(s - t) = \cos s \cos t + \sin s \sin t.$$

TANGENT SUM AND DIFFERENCE IDENTITIES

$$\tan(s + t) = \frac{\tan s + \tan t}{1 - \tan s \tan t},$$

$$\tan(s - t) = \frac{\tan s - \tan t}{1 + \tan s \tan t}.$$

DOUBLE ANGLE IDENTITIES

$$\sin 2t = 2 \sin t \cos t,$$

$$\cos 2t = \cos^2 t - \sin^2 t$$

$$= 2\cos^2 t - 1$$

$$= 1 - 2\sin^2 t,$$

$$\tan 2t = \frac{2 \tan t}{1 - \tan^2 t}.$$

POWER FORMULAS

$$\sin^2 t = \frac{1 - \cos 2t}{2},$$

$$\cos^2 t = \frac{1 + \cos 2t}{2}.$$

HALF-ANGLE IDENTITIES

$$\sin\frac{s}{2} = \pm\sqrt{\frac{1-\cos s}{2}},$$

$$\cos\frac{s}{2} = \pm\sqrt{\frac{1+\cos s}{2}},$$

$$\tan\frac{s}{2} = \frac{\sin s}{1+\cos s} = \frac{1-\cos s}{\sin s}.$$

PRODUCT TO SUM IDENTITIES

$$2\sin s\cos t = \sin(s+t) + \sin(s-t),$$

$$2\cos s\sin t = \sin(s+t) - \sin(s-t),$$

$$2\cos s\cos t = \cos(s-t) + \cos(s+t),$$

$$2\sin s\sin t = \cos(s-t) - \cos(s+t).$$

SUM TO PRODUCT IDENTITIES

$$\sin u + \sin v = 2\sin\frac{u+v}{2}\cos\frac{u-v}{2},$$

$$\sin u - \sin v = 2\cos\frac{u+v}{2}\sin\frac{u-v}{2},$$

$$\cos v + \cos u = 2\cos\frac{u+v}{2}\cos\frac{u-v}{2},$$

$$\cos v - \cos u = 2\sin\frac{u+v}{2}\sin\frac{u-v}{2}.$$

INVERSE SINE FUNCTION

$y = \sin^{-1}x = \arcsin x$, where $-1 \le x \le 1$,

means

$x = \sin y$, where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

INVERSE COSINE FUNCTION

$y = \cos^{-1}x = \arccos x$, where $-1 \le x \le 1$,

means

$x = \cos y$, where $0 \le y \le \pi$.

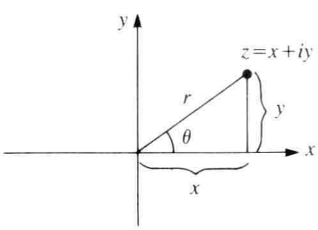
INVERSE TANGENT FUNCTION

$y = \tan^{-1}x = \arctan x$, where x is any real number,

means

$x = \tan y$, where $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

TRIGONOMETRIC FORM OF A COMPLEX NUMBER



$$z = x + iy = r(\cos\theta + i\sin\theta)$$

DeMOIVRE'S THEOREM

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

TRIGONOMETRY

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To Peggy and Cindi
and to our mother, Hilda

PREFACE

Trigonometry is designed for use in a one-semester or one-quarter course in trigonometry, with a prerequisite of intermediate algebra. A background in geometry is desirable, but many students take trigonometry with little or no formal geometry. For this reason, the geometric concepts used in the development of trigonometry are reviewed in Chapter 1.

Since a standard trigonometry course is a prerequisite for a program of study in calculus, we pay special attention to the development of the concepts needed for calculus. We hope that the student using this text will become proficient in setting up the type of geometric diagrams needed to solve an applied calculus problem and will attain an appreciation and understanding of the identities that are so important to calculus. To address these goals we have given special emphasis to geometrical constructions throughout the first two chapters, and we present most of the major identities at least twice. Many of the identities have very simple geometric derivations; we develop a number of them in Chapter 2 along with some of their geometric applications. In Chapter 3 we present a unified treatment of the identities so the student will learn how to derive them from the sum and difference formulas.

Chapter 1 contains the necessary geometric background. In Section 1.1 we summarize basic definitions and facts. In Section 1.2 we present the degree and radian systems of angular measurement, and we discuss the methods for converting from one system to the other. Section 1.3 reviews the Theorem of Pythagoras as well as its converse via the geometric versions of the Law of Cosines. In Section 1.4 we define angles of arbitrary measurement in terms of a rectangular coordinate system. Finally, Section 1.5 reviews the concept of similarity; it also discusses the Theorem of Ptolemy, which is later used to give a geometric proof of the Sum Identity for the sine and cosine functions. For students who have adequate preparation in geometry, Chapter 1 may be treated quickly as a review in geometry. Exercises in Chapter 1 that are referred to later in the text are marked with an asterisk.

Our treatment of trigonometry itself progresses from the concrete to the abstract in stages. The trigonometric functions for acute angles are defined in Section 2.1 in terms of right triangles, and these functions are given their usual geometric interpretations in terms of the unit circle. Section 2.2 is devoted to elementary identities whose proofs are based on algebraic manipulations. Geometric derivations of several important identities are given in Section 2.3. The Laws of Cosines and

Sines and the use of the calculator for finding inverse cosines and inverse sines when the desired angles are in a triangle are presented in Sections 2.4 and 2.5. Chapter 2 ends with applications of trigonometry to areas—notably Heron's Formula—and vectors.

Chapter 3 is devoted to analytic trigonometry and the unified derivation of all the trigonometric identities. The trigonometric functions are developed both in terms of a point on an angle in standard position and in terms of a unit circle. Chapter 4 studies the graphs of the trigonometric functions and Chapter 5 presents the inverse trigonometric functions and their graphs along with techniques for solving trigonometric equations. Finally, Chapter 6 contains sections showing how trigonometric functions are used in the study of complex numbers and polar coordinates.

Chapters 1 and 2 form a complete course in right triangle trigonometry. Chapters 3, 4, 5, and 6 along with parts of Chapters 1 and 2 form a complete course in analytic trigonometry. If time is a problem or if a less geometrically developed course is desired, Sections 2.3, 2.6, and 2.7 may be treated as optional; and any topics of Chapter 1 may be presented as needed. Also, the Laws of Cosines and Sines may be postponed until the analytic trigonometry is fully developed.

The text is supplemented with over 400 accurately scaled figures, numerous illustrative examples, and remarks that draw special attention to certain ideas. All major theorems and definitions are set off by boxes for emphasis. Many sections end with a historical note designed to enrich the student's understanding and appreciation of the material.

We have tried to spread out the applications by including some of them, as well as other types of word problems, in most of the exercise sets. Many of these problems are related to geometric constructions and derivations of identities. Each section also generally includes many exercises requiring computations. We expect the student to make full use of a scientific calculator throughout the text. Trigonometric tables and their uses are discussed in an appendix. For the student having access to a computer, we have included computer programs at the end of many of the exercise sets. Following each chapter is a chapter summary, a set of miscellaneous exercises that may be used to enhance the material discussed in that chapter, and a sample test.

The following supplementary materials are available from the publisher: a *Student Solutions Manual,* which contains detailed solutions to selected exercises; an *Instructor's Solutions Manual,* which provides solutions for all the exercises; and IBM PC software. The software includes *EXP*TEST with a computerized test bank and *TrueBASIC*™ *Pre-Calculus,* which is ideal for classroom demonstrations and student exploration.

Our thanks go to the students at Miami-Dade Community College who classroom-tested the entire text. Also, we wish to thank the following reviewers who helped refine this text: Arthur P. Dull, Diablo

Valley College; Michael D. Eurgubian, Santa Rosa Junior College; Ferdinald Haring, North Dakota State University; William L. Hinrichs, Rock Valley College; Daniel H. Lee, Southwest Texas State University; Rebecca F. Porter, Truckee Meadows Community College; Karen Robinson, Aims Community College; Dorothy Schwellenbach, Hartnell College; Ken Seydel, Skyline College; John Snyder, Sinclair Community College; Donna M. Szott, South Campus—Community College of Allegheny County; and Ann Thorn, College of DuPage.

The organization of this text is due in part to these experiences along with our desire to enrich the student's understanding and appreciation of geometrical constructions and trigonometric identities.

Charles W. Gantner
Thomas E. Gantner

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Introduction

The word *trigonometry* is derived from two Greek words, *trigonon* and *metria,* that together mean triangle-measurement. As its name implies, trigonometry was developed from the study of the relationships among the sides and angles of a triangle. Triangles are a geometric concept, but trigonometric relationships are numerical quantities; thus the study of trigonometry involves both geometry and arithmetic.

Initial developments in trigonometry occurred during the period 250 B.C. to A.D. 150 through the efforts of Greek scholars who were using geometrical methods in the study of astronomy. As early as 250 B.C. Eratosthenes of Cyrene, the head librarian at Alexandria, used geometrical methods to estimate the circumference of the earth. Around 140 B.C. Hipparchus of Nicaea began to use trigonometric methods in a systematic way to study astronomy and hence he is regarded as the founder of the subject. The early development of trigonometry reached a climax around A.D. 150, when Claudius Ptolemy completed his work, the *Almagest,* which was regarded as the major work on astronomy and trigonometry for over a thousand years. However, the roots of trigonometry extend back to around 3000 B.C., when the great empire societies of Babylon and Egypt began to evolve. Let us briefly trace some of these roots of the development of trigonometry.

In response to the needs of their social systems, Babylonian and Egyptian mathematicians developed an extensive and sophisticated body of arithmetic and geometric facts. In social systems involving millions of people, land measurements had to be made for the purpose of taxation; large public buildings and monuments had to be designed and erected; and astronomical calculations had to be made in order to develop an accurate calender. The problems of this period, which extended roughly from 3000 B.C. to around 600 B.C., were very specific in nature, and they required numerical answers. Our knowledge of the