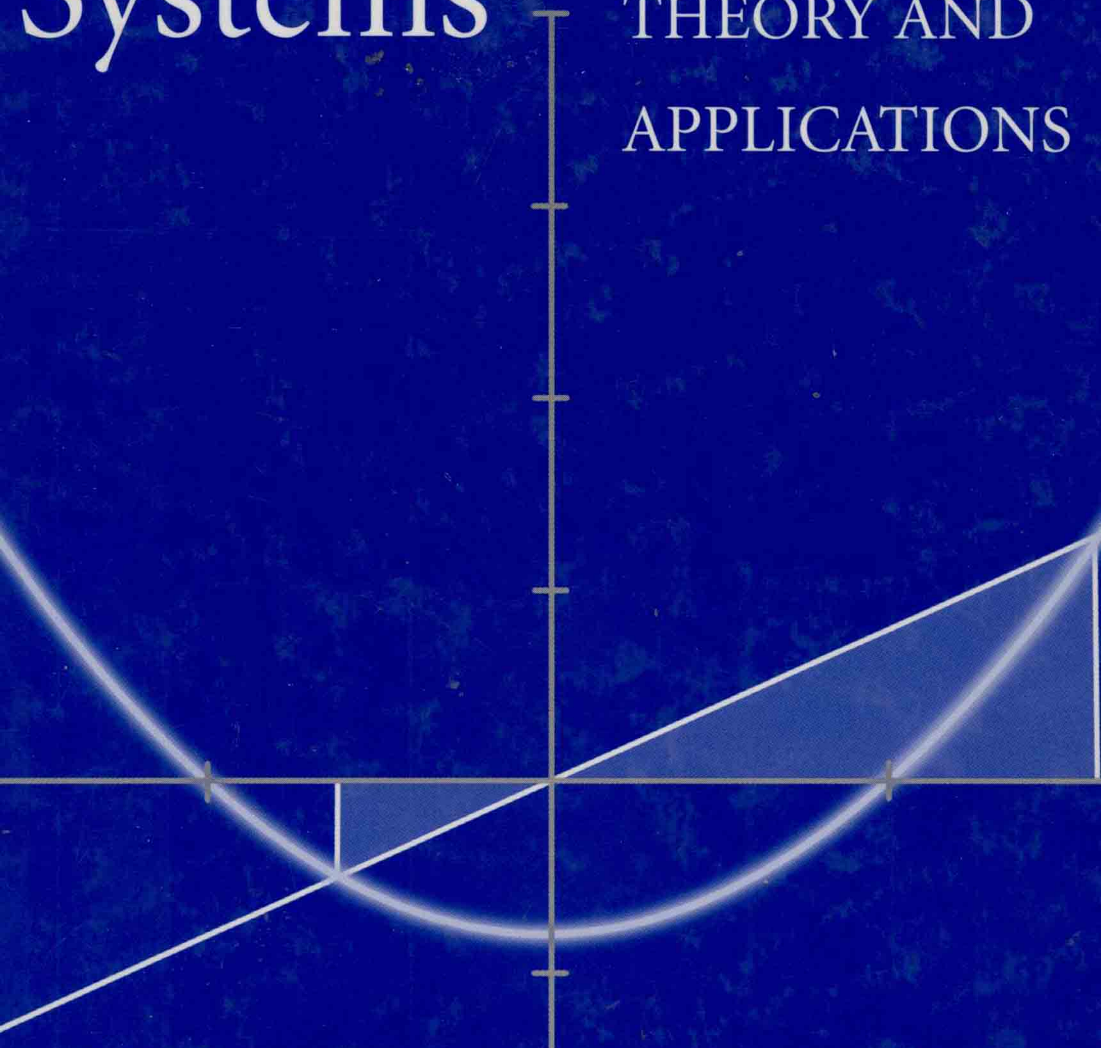
Random Dynamical Systems

THEORY AND APPLICATIONS



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Theory and Applications

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Preface

The scope of this book is limited to the study of discrete time dynamic processes evolving over an infinite horizon. Its primary focus is on models with a one-period lag: "tomorrow" is determined by "today" through an exogenously given rule that is itself stationary or time-independent. A finite lag of arbitrary length may sometimes be incorporated in this scheme. In the deterministic case, the models belong to the broad mathematical class, known as dynamical systems, discussed in Chapter 1, with particular emphasis on those arising in economics. In the presence of random perturbations, the processes are random dynamical systems whose long-term stability is our main quest. These occupy a central place in the theory of discrete time stochastic processes.

Aside from the appearance of many examples from economics, there is a significant distinction between the presentation in this book and that found in standard texts on Markov processes. Following the exposition in Chapter 2 of the basic theory of irreducible processes, especially Markov chains, much of Chapters 3–5 deals with the problem of stability of random dynamical systems which may not, in general, be irreducible. The latter models arise, for example, if the random perturbation is limited to a finite or countable number of choices. Quite a bit of this theory is of relatively recent origin and appears especially relevant to economics because of underlying structures of monotonicity or contraction. But it is useful in other contexts as well.

In view of our restriction to discrete time frameworks, we have not touched upon powerful techniques involving deterministic and stochastic differential equations or calculus of variations that have led to significant advances in many disciplines, including economics and finance.

It is not possible to rely on the economic data to sift through various possibilities and to compute estimates with the degrees of precision that natural or biological scientists can often achieve through controlled experiments. We duly recognize that there are obvious limits to the lessons that formal models with exogenously specified laws of motion can offer.

The first chapter of the book presents a treatment of deterministic dynamical systems. It has been used in a course on dynamic models in economics, addressed to advanced undergraduate students at Cornell. Supplemented by appropriate references, it can also be part of a graduate course on dynamic economics. It requires a good background in calculus and real analysis.

Chapters 2–6 have been used as the core material in a graduate course at Cornell on Markov processes and their applications to economics. An alternative is to use Chapters 1–3 and 5 to introduce models of intertemporal optimization/equilibrium and the role of uncertainty. Complements and Details make it easier for the researchers to follow up on some of the themes in the text.

In addition to numerous examples illustrating the theory, many exercises are included for pedagogic purposes. Some of the exercises are numbered and set aside in paragraphs, and a few appear at the end of some chapters. But quite a few exercises are simply marked as (Exercise), in the body of a proof or an argument, indicating that a relatively minor step in reasoning needs to be formally completed.

Given the extensive use of the techniques that we review, we are unable to provide a bibliography that can do justice to researchers in many disciplines. We have cited several well-known monographs, texts, and review articles which, in turn, have extended lists of references for curious readers.

The quote attributed to Toni Morrison in Chapter 1 is available on the Internet from Simpson's Contemporary Quotations, compiled by J. B. Simpson.

The quote from Shizuo Kakutani in Chapter 2 is available on the Internet at www.uml.edu/Dept/Math/alumni/tangents/tangents_Fall2004/MathInTheNews.htm. Endnote 1 of the document describes it as "a joke by Shizuo Kakutani at a UCLA colloquium talk as attributed in Rick Durrett's book *Probability: Theory and Examples*." The other quote in this chapter is adapted from Bibhuti Bandyopadhyay's original masterpiece in Bengali.

The quote from Gerard Debreu in Chapter 4 appeared in his article in *American Economic Review* (Vol. 81, 1991, pp. 1–7).

The quote from Patrick Henry in Chapter 5 is from Bartlett's Quotations (no. 4598), available on the Internet.

The quote attributed to Freeman J. Dyson in the same chapter appeared in the circulated abstract of his Nordlander Lecture ("The Predictable and the Unpredictable: How to Tell the Difference") at Cornell University on October 21, 2004.

The quote from Kenneth Arrow at the beginning of Chapter 6 appears in Chapter 2 of his classic *Essays in the Theory of Risk-Bearing.*

Other quotes are from sources cited in the text.

Acknowledgment

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Two collections of published articles have played an important role in our exposition: a symposium on *Chaotic Dynamical Systems* (edited by Mukul Majumdar) and a symposium on *Dynamical Systems Subject to Random Shocks* (edited by Rabi Bhattacharya and Mukul Majumdar) that appeared in *Economic Theory* (the first in Vol. 4, 1995, and the second in Vol. 23, 2004). We acknowledge the enthusiastic support of Professor C. D. Aliprantis in this context.

Finally, thanks are due to Scott Parris, who initiated the project.

Notation

$\mathbb{Z}$	set of *all* integers.
$\mathbb{Z}_+(\mathbb{Z}_{++})$	set of all *nonnegative* (*positive*) integers.
$\mathbb{R}$	set of *all* real numbers.
$\mathbb{R}_+(\mathbb{R}_{++})$	set of all *nonnegative* (*positive*) real numbers.
$\mathbb{R}^\ell$	set of all ℓ-vectors.
$\mathbf{x} = (x_i) = (x_1, \ldots, x_\ell)$	an element of $\mathbb{R}^\ell$.
$\mathbf{x} \geq 0$	$x_i \geq 0$ for $i = 1, 2, \ldots, \ell$; [$\mathbf{x}$ is *nonnegative*].
$\mathbf{x} > 0$	$x_i \geq 0$ for all i; $x_i > 0$ for some i; [$\mathbf{x}$ is *positive*].
$\mathbf{x} \gg 0$	$x_i > 0$ for all i; [$\mathbf{x}$ is *strictly positive*].
$(S, \mathcal{S})$	a measurable space [when S is a metric space, $\mathcal{S} = \mathcal{B}(S)$ is the Borel sigmafield unless otherwise specified].

Contents

1

Dynamical Systems

Not only in research, but also in the everyday world of politics and economics, we would all be better off if more people realized that simple nonlinear systems do not necessarily possess simple dynamical properties.

Robert M. May

There is nothing more to say – except why. But since why is difficult to handle, one must take refuge in how.

Toni Morrison

1.1 Introduction

There is a rich literature on discrete time models in many disciplines – including economics – in which dynamic processes are described formally by first-order difference equations (see (2.1)). Studies of dynamic properties of such equations usually involve an appropriate definition of a steady state (viewed as a dynamic equilibrium) and conditions that guarantee its existence and local or global stability. Also of importance, particularly in economics following the lead of Samuelson (1947), have been the problems of comparative statics and dynamics: a systematic analysis of how the steady states or trajectories respond to changes in some parameter that affects the law of motion. While the dynamic properties of linear systems (see (4.1)) have long been well understood, relatively recent studies have emphasized that "the very simplest" nonlinear difference equations can exhibit "a wide spectrum of qualitative behavior," from stable steady states, "through cascades of stable cycles, to a regime in which the behavior (although fully deterministic) is in many respects chaotic or indistinguishable from the sample functions of a random process" (May 1976, p. 459). This chapter is not intended to be a comprehensive review of the properties of complex dynamical systems, the study of which has benefited from a collaboration between the more "abstract" qualitative analysis of difference and differential equations, and a careful exploration of "concrete" examples through increasingly sophisticated computer experiments. It does recall some of the basic results on dynamical systems, and draws upon a variety of examples from economics (see Complements and Details).

There is by now a plethora of definitions of "chaotic" or "complex" behavior, and we touch upon a few properties of chaotic systems in Sections 1.2 and 1.3. However, the map (2.3) and, more generally, the quadratic family discussed in Section 1.7 provide a convenient framework for understanding many of the definitions, developing intuition and achieving generalizations (see Complements and Details). It has been stressed that the qualitative behavior of the solution to Equation (2.5) depends crucially on the initial condition. Trajectories emanating from initial points that are very close may display radically different properties. This may mean that small changes in the initial condition "lead to predictions so different, after a while, that prediction becomes in effect useless" (Ruelle 1991, p. 47). Even within the quadratic family, complexities are not "knife-edge," "abnormal," or "rare" possibilities. These observations are particularly relevant for models in social sciences, in which there are obvious limits to gathering data to identify the initial condition, and avoiding computational errors at various stages.

In Section 1.2 we collect some basic results on the existence of fixed points and their stability properties. Of fundamental importance is the contraction mapping theorem (Theorem 2.1) used repeatedly in subsequent chapters. Section 1.3 introduces complex dynamical systems, and the central result is the Li–Yorke theorem (Theorem 3.1). In Section 1.4 we briefly touch upon linear difference equations. In Section 1.5 we explore in detail dynamical systems in which the state space is $\mathbb{R}_+$, the set of nonnegative reals, and the law of motion α is an increasing function. Proposition 5.1 is widely used in economics and biology: it identifies a class of dynamical systems in which all trajectories (emanating from initial x in $\mathbb{R}_{++}$) converge to a unique fixed point. In contrast, Section 1.6 provides examples in which the long-run behavior depends on initial conditions. In the development of complex dynamical systems, the "quadratic family" of laws of motion (see (7.11)) has played a distinguished role. After a review of some results on this family in Section 1.7, we turn to examples of dynamical systems from economics and biology.

We have selected some descriptive models, some models of optimization with a single decision maker, a dynamic game theoretic model, and an example of intertemporal equilibrium with overlapping generations. An interesting lesson that emerges is that variations of some well-known models that generate monotone behavior lead to dynamical systems exhibiting Li–Yorke chaos, or even to systems with the quadratic family as possible laws of motion.

1.2 Basic Definitions: Fixed and Periodic Points

We begin with some formal definitions. A dynamical system is described by a pair (S, α) where S is a nonempty set (called the *state space*) and α is a function (called the *law of motion*) from S into S. Thus, if x_t is the state of the system in period t, then

$$x_{t+1} = \alpha(x_t) \tag{2.1}$$

is the state of the system in period $t + 1$.

In this chapter we always assume that the state space S *is a (nonempty) metric space (the metric is denoted by d).* As examples of (2.1), take S to be the set $\mathbb{R}$ of real numbers, and define

$$\alpha(x) = ax + b, \tag{2.2}$$

where a and b are real numbers.

Another example is provided by $S = [0, 1]$ and

$$\alpha(x) = 4x(1 - x). \tag{2.3}$$

Here in (2.3), $d(x, y) \equiv |x - y|$.

The evolution of the dynamical system $(\mathbb{R}, \alpha)$ where α is defined by (2.2) is described by the difference equation

$$x_{t+1} = ax_t + b. \tag{2.4}$$

Similarly, the dynamical system $([0, 1], \alpha)$ where α is defined by (2.3) is described by the difference equation

$$x_{t+1} = 4x_t(1 - x_t). \tag{2.5}$$

Once the initial state x (i.e., the state in period 0) is specified, we write $\alpha^0(x) \equiv x$, $\alpha^1(x) = \alpha(x)$, and for every positive integer $j \geq 1$,

$$\alpha^{j+1}(x) = \alpha(\alpha^j(x)). \tag{2.6}$$

We refer to α^j as the *jth iterate* of α. For any initial x, the *trajectory* from x is the sequence $\tau(x) = \{\alpha^j(x)_{j=0}^{\infty}\}$. The *orbit* from x is the set $\gamma(x) = \{y\colon y = \alpha^j(x) \text{ for some } j \geq 0\}$. The limit set $w(x)$ of a trajectory $\tau(x)$ is defined as

$$w(x) = \bigcap_{j=1}^{\infty} \overline{[\tau(\alpha^j(x))]}, \tag{2.7}$$

where $\bar{A}$ is the closure of A.

Fixed and periodic points formally capture the intuitive idea of a *stationary* state or an *equilibrium* of a dynamical system. In his *Foundations*, Samuelson (1947, p. 313) noted that "*Stationary* is a descriptive term characterizing the behavior of an economic variable over time; it usually implies constancy, but is occasionally generalized to include behavior periodically repetitive over time."

A point $x \in S$ is a *fixed point* if $x = \alpha(x)$. A point $x \in S$ is a *periodic point* of *period* $k \geq 2$ if $\alpha^k(x) = x$ and $\alpha^j(x) \neq x$ for $1 \leq j < k$. Thus, to prove that x is a periodic point of period, say, 3, one must prove that x is a fixed point of α^3 and that it is *not* a fixed point of α and α^2. *Some writers consider a fixed point as a periodic point of period* 1.

Denote the set of all periodic points of S by $\wp(S)$. We write $\aleph(S)$ to denote the set of nonperiodic points.

We now note some useful results on the existence of fixed points of α.

Proposition 2.1 *Let $S = \mathbb{R}$ and α be continuous. If there is a (nondegenerate) closed interval $I = [a, b]$ such that (i) $\alpha(I) \subset I$ or (ii) $\alpha(I) \supset I$, then there is a fixed point of α in I.*

Proof.

(i) If $\alpha(I) \subset I$, then $\alpha(a) \geq a$ and $\alpha(b) \leq b$. If $\alpha(a) = a$ or $\alpha(b) = b$, the conclusion is immediate. Otherwise, $\alpha(a) > a$ and $\alpha(b) < b$. This means that the function $\beta(x) = \alpha(x) - x$ is positive at a and negative at b. Using the intermediate value theorem, $\beta(x^*) = 0$ for some x^* in (a, b). Then $\alpha(x^*) = x^*$.

(ii) By the Weierstrass theorem, there are points x_m and x_M in I such that $\alpha(x_m) \leq \alpha(x) \leq \alpha(x_M)$ for all x in I. Write $\alpha(x_m) = m$ and $\alpha(x_M) = M$. Then, by the intermediate value theorem, $\alpha(I) = [m, M]$.