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Discriminants, Resultants, and Multidimensional Determinants

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CHAPTER 9

A-Discriminants

We now introduce the second main object of study: the A-discriminant Δ_A.

1. Basic definitions and examples

A. Definitions and first examples

Our setup now will be the same as in Section 1, Chapter 5. Namely, we choose a finite subset A in the integral lattice $\mathbf{Z}^{k-1}$ whose elements ω correspond to Laurent monomials $x^\omega = x_1^{\omega_1} \cdots x_{k-1}^{\omega_{k-1}}$ in $k-1$ variables. We consider the space $\mathbf{C}^A$ of Laurent polynomials of the form $f(x) = \sum_{\omega \in A} a_\omega x^\omega$.

We let $\nabla_0 \subset \mathbf{C}^A$ denote the set of all f for which there exists $x^{(0)} \in (\mathbf{C}^*)^{k-1}$ such that

$$f(x^{(0)}) = (\partial f/\partial x_i)(x^{(0)}) = 0 \ for\ all\ i. \tag{1.1}$$

Let ∇_A be the closure of ∇_0. It is not hard to see that ∇_A is an irreducible variety defined over $\mathbf{Q}$. Indeed, let X_A be the toric variety associated to A (Section 1B Chapter 5). Then we have the following fact, which is obvious from the definitions.

Proposition 1.1. *The variety ∇_A is conical, i.e., it is invariant under the multiplication by scalars. Its projectivization $P(\nabla_A)$ is the variety projectively dual to X_A.*

Now we can give the definition of the A-discriminant.

Definition 1.2. If the set $A \subset \mathbf{Z}^{k-1}$ has the property that $\nabla_A \subset \mathbf{C}^A$ is a subvariety of codimension 1, then by the *A-discriminant* we mean an irreducible integral polynomial $\Delta_A(f)$ in the coefficients a_ω, $\omega \in A$ of $f \in \mathbf{C}^A$ which vanishes on ∇_A. Such a polynomial is uniquely determined up to sign. If $\operatorname{codim} \nabla_A > 1$, we set $\Delta_A = 1$.

Thus Δ_A is a particular case of the general discriminants defined in Chapter 1: under the notation of that chapter, we have $\Delta_A = \Delta_{X_A}$. We start with some simple properties and then give basic examples.

Proposition 1.3. *The polynomial Δ_A is homogeneous. In addition, it satisfies $(k-1)$ quasi-homogeneity conditions: for all monomials $\prod a_\omega^{m(\omega)}$ in Δ_A, the vector $\sum m(\omega) \cdot \omega \in \mathbf{Z}^{k-1}$ is the same.*

Proof. If $f \in \nabla_A$ and $\lambda_0, \lambda_1, \dots, \lambda_{k-1}$ are nonzero numbers, then the polynomial

$$g(x_1, \dots, x_{k-1}) = \lambda_0 f(\lambda_1 x_1, \dots, \lambda_{k-1} x_{k-1})$$

is obviously in ∇_A as well. This implies our proposition.

In fact, the variety ∇_A, and hence also the polynomial Δ_A, depend only on the affine geometry of $A \subset \mathbf{Z}^{k-1}$.

Proposition 1.4. *Let $A \subset \mathbf{Z}^{k-1}$, $B \subset \mathbf{Z}^{m-1}$ be two finite subsets and $T : \mathbf{Z}^{k-1} \to \mathbf{Z}^{m-1}$ be an integral affine transformation which is injective and such that $T(A) = B$. Then under the corresponding identification of $\mathbf{C}^A$ and $\mathbf{C}^B$, the variety ∇_A is identified with ∇_B and the polynomial Δ_A is identified with Δ_B.*

Proof. In this case X_A and X_B are naturally identified (Proposition 1.2, Chapter 5). Hence the projectively dual varieties are also identified.

Remark 1.5. The above proposition means, in particular, that we can shrink, if necessary, the affine lattice $\mathbf{Z}^{k-1}$ containing A to the lattice $\mathrm{Aff}_{\mathbf{Z}}(A)$, affinely generated over $\mathbf{Z}$ by A (see Definition 1.3, Chapter 5).

Examples 1.6. We consider the same choices of sets A which were discussed in Examples 1.1, Chapter 5.

(a) Let A consist of all monomials of degree $\le d$ in $k-1$ variables $x_1, \dots, x_{k-1}$. The space $\mathbf{C}^A$ consists of all polynomials $f(x_1, \dots, x_{k-1})$ of degree $\le d$. Equivalently, let $\tilde{A}$ consist of all homogeneous monomials in k variables $x_1, \dots, x_k$ of degree exactly d. Then $\mathbf{C}^{\tilde{A}}$ is the space $S^d \mathbf{C}^k$ of forms of degree d in k variables. There is an obvious identification

$$\mathbf{C}^{\tilde{A}} \to \mathbf{C}^A, \quad f(x_1, \dots, x_k) \mapsto f(x_1, \dots, x_{k-1}, 1),$$

which takes $\Delta_{\tilde{A}}$ to Δ_A. The polynomial $\Delta_{\tilde{A}}$ is the classical discriminant of a form of degree d in k variables, discussed in Example 4.15 Chapter 1. Recall, in particular, Boole's formula $\deg \Delta = k(d-1)^{k-1}$.

(b) Let A consist of bilinear monomials $x_i \cdot y_j$, where the x_i and y_j ($i = 1, \dots, m$; $j = 1, \dots, n$) are two sets of variables. Then $\mathbf{C}^A$ consists of bilinear forms $f(x, y) = \sum a_{ij} x_i y_j$ and is identified with the space of $m \times n$ matrices $\|a_{ij}\|$. The A-discriminant $\Delta_A(f) = \Delta_A(\|a_{ij}\|)$ is identically equal to 1 unless $m = n$, and in this case it is the determinant of the square matrix $\|a_{ij}\|$. The monomials in $\Delta_A(f)$ have an obvious combinatorial significance: they correspond to the permutations of the set of m elements, and the coefficients are the signs of the permutations. This explains our interest in the monomials appearing in the A-discriminant in other cases.

(b′) Let A consist of trilinear monomials

$$x_i \cdot y_j \cdot z_l, \ i = 1, \dots, m_1, \ j = 1, \dots, m_2, \ l = 1, \dots, m_3.$$

The previous example makes it natural to refer to the A-discriminant of a polynomial

$$\sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \sum_{l=1}^{m_3} a_{ijk} x_i y_j z_l \ \in \ \mathbf{C}^A$$

as the *hyperdeterminant* of the three-dimensional "matrix" $\|a_{ijl}\|$. This concept (and its obvious generalization to higher-dimensional "matrices") was introduced by Cayley [Ca1] at almost the same time as that of the determinant of a square matrix. Hyperdeterminants were later studied by Schläfli [Schl] and, after the break of almost 150 years, by the authors [GKZ3]. In view of the previous example, the monomials in the hyperdeterminant form a "higher analog" of the symmetric group. We shall present our treatment of the hyperdeterminants in Chapter 14. (We note that there is another notion of the "determinant" of a multidimensional matrix which is different from ours (see, e.g., [P] [So]). It is based on a direct generalization of the determinant formula for a square matrix and includes a summation over the product of several symmetric groups.)

(c) Let A consists of monomials

$$1, x, x^2, \dots, x^p, \ y, yx, yx^2, \dots, yx^q.$$

Then $\mathbf{C}^A$ consists of polynomials $\Phi(x, y) = f(x) + yg(x)$ where f, g are polynomials in one variable x of degrees not greater than p or q, respectively. In this case $\Delta_A(\Phi) = R(f, g)$ is the classical resultant of f and g (see Example 3.6 Chapter 3). This relation holds in a more general context.

Let $A_1, \dots, A_k \subset \mathbf{Z}^{k-1}$ be finite subsets satisfying the assumptions of Section 1A, Chapter 8. Let $R_{A_1,\dots,A_k}$ be the $(A_1, \dots, A_k)$-resultant; it is a function of k polynomials $f_i \in \mathbf{C}^{A_i}$. Let $A \subset \mathbf{Z}^{2k-2} = \mathbf{Z}^{k-1} \times \mathbf{Z}^{k-1}$ be the following set:

$$A = (A_1 \times \{e_1\}) \cup \dots \cup (A_{k-1} \times \{e_{k-1}\}) \cup (A_k \times \{0\}), \tag{1.2}$$

where the e_i are the standard basis vectors of $\mathbf{Z}^{k-1}$. Then $\mathbf{C}^A$ is the space of polynomials of the form

$$f_k(x) + \sum_{i=1}^{k-1} y_i f_i(x),$$

where $f_i \in \mathbf{C}^{A_i}$. We have the following statement ("Cayley trick").

Proposition 1.7. *We have*

$$R_{A_1,\ldots,A_k}(f_1,\ldots,f_k)=\Delta_A\left(f_k(x)+\sum_{i=1}^{k-1}y_i f_i(x)\right).$$

Proof. This follows from Corollary 3.5, Chapter 3 about the relation between projectively dual and associated varieties. The proof is so simple that we repeat it here for our particular case. If $x^{(0)}$ is a common root of $(f_1,\ldots,f_k)$, then we can find $y_1^{(0)},\ldots,y_{k-1}^{(0)}$ such that the polynomial $f_k+\sum y_i f_i$ vanishes at the point $y_1^{(0)},\ldots,y_{k-1}^{(0)},x_1^{(0)},\ldots,x_k^{(0)}$ along with its first derivatives. We do this by solving a linear system. Conversely, if $(y^{(0)},x^{(0)})$ is such a point, then

$$f_i(x^{(0)})=\left.\frac{\partial(f_k+\sum_{i=1}^{k-1}y_i f_i)}{\partial y_i}\right|_{(x^{(0)},y^{(0)})}=0$$

for $1\le i\le k-1$, and so $f_k(x^{(0)})=0$ as well.

B. The case of a circuit

Let $A\subset \mathbf{Z}^{k-1}$ be a circuit. This means (see Section 1B, Chapter 7) that A is affinely dependent, but any proper subset of A is affinely independent. In this case the A-discriminant Δ_A can be calculated explicitly. We present this formula, following Kouchnirenko [Kou].

We can assume that A generates $\mathbf{Z}^{k-1}$ as an affine lattice. So $\#(A)=k+1$. There is, up to scaling, just one affine relation between elements of A:

$$\sum_{\omega\in A}m_\omega\cdot\omega=0,\quad \sum_{\omega\in A}m_\omega=0. \tag{1.3}$$

We normalize such a relation uniquely up to sign by requiring that all m_ω be integers with the greatest common divisor equal to 1. Note that

$$|m_\omega|=\mathrm{Vol}_{\mathbf{Z}^{k-1}}(\mathrm{Conv}(A-\{\omega\})).$$

Let $A_+,A_-\subset A$ be the sets of ω such that m_ω is positive (resp. negative).

Proposition 1.8. *Suppose that $A\subset\mathbf{Z}^{k-1}$ is a circuit which generates $\mathbf{Z}^{k-1}$ as an affine lattice. Let $f=\sum_{\omega\in A}a_\omega x^\omega$ be an indeterminate polynomial from $\mathbf{C}^A$. Then the A-discriminant of f is a non-zero scalar multiple of the polynomial*

$$\left(\prod_{\omega\in A_+}m_\omega^{m_\omega}\right)\prod_{\omega\in A_-}a_\omega^{-m_\omega}-\left(\prod_{\omega\in A_-}m_\omega^{-m_\omega}\right)\prod_{\omega\in A_+}a_\omega^{m_\omega}, \tag{1.4}$$

where the m_ω are defined as above.

Proof. First, we show that the polynomial (1.4) vanishes for $f \in \nabla_0$, i.e., when the system (1.1) has a solution $x^{(0)} \in (\mathbf{C}^*)^{k-1}$. Indeed, (1.1) can be written as

$$\sum_{\omega \in A} a_\omega (x^{(0)})^\omega = 0, \ \sum_{\omega \in A} a_\omega (x^{(0)})^\omega \cdot \omega = 0. \tag{1.5}$$

Comparing (1.5) with (1.3) we conclude that the vectors $(a_\omega(x^{(0)})^\omega)_{\omega \in A}$ and $(m_\omega)_{\omega \in A}$ are proportional to each other. To eliminate $x^{(0)}$, we apply to both vectors the function $(y_\omega)_{\omega \in A} \mapsto \prod_{\omega \in A} y_\omega^{m_\omega}$ (since $\sum_{\omega \in A} m_\omega = 0$, this function takes the same value at proportional vectors). We obtain the equality

$$\prod_{\omega \in A} \left(\frac{a_\omega}{m_\omega} \right)^{m_\omega} = 1; \tag{1.6}$$

its polynomial form is exactly the vanishing of (1.4).

Conversely, suppose (1.4) vanishes at some $f \in \mathbf{C}^A$. We can assume that f is generic, so that all a_ω are non-zero. This implies (1.6), i.e., that the vector with components $(y_\omega = \frac{a_\omega}{m_\omega})_{\omega \in A}$ satisfies the relation

$$\prod_{\omega \in A} y_\omega^{m_\omega} = 1. \tag{1.7}$$

Using the fact that A affinely spans $\mathbf{Z}^{k-1}$, it is easy to see that every solution of (1.7) has the form $(y_\omega = c(x^{(0)})^{-\omega})$ for some non-zero constant c and some $x^{(0)} \in (\mathbf{C}^*)^{k-1}$. This, in turn, implies $f \in \nabla_0$.

The above arguments show that (1.4) defines ∇_A set-theoretically. So, up to a scalar multiple, it must be a power of Δ_A. But, since it is a sum of only two monomials, we conclude that it can be only the first power of Δ_A, which completes the proof.

2. The discriminantal complex

The problem of finding the A-discriminant Δ_A, raised in Section 1, is a special case of a more general problem addressed in Chapter 1: finding the equation of $X^\vee$, the projectively dual variety to a given projective variety $X \subset P^{n-1}$. In our present case X is the toric variety X_A.

Suppose that a projective variety X is smooth. In Theorem 2.5 of Chapter 2 we have represented the equation of $X^\vee$ as the determinant of the so-called discriminantal complex. The terms of this complex consist of some differential forms on the affine variety $Y \subset \mathbf{C}^n$ which is the cone over X (see Section 4, Chapter 2). In our present situation we can make the complexes more explicit using a description of differential forms on tori and toric varieties going back to Danilov [D], see also [O]. For the convenience of the reader we recall this description from scratch, starting with the case of a torus.

A. Differential forms on a torus

Let $H = (\mathbf{C}^*)^k$ be a torus and let $\Xi = \mathrm{Hom}(H, \mathbf{C}^*) = \mathbf{Z}^k$ be its character lattice. The ring $\mathbf{C}[H]$ of regular functions on H is identified with the group algebra $\mathbf{C}[\Xi]$ of Ξ. This means that we can regard a Laurent polynomial

$$f(x) = \sum_{\omega \in \mathbf{Z}^k} a_\omega x^\omega \in \mathbf{C}[H]$$

as a complex-valued function $\omega \mapsto a_\omega$ with finite support on $\Xi = \mathbf{Z}^k$. In this language the multiplication in $\mathbf{C}[H]$ is given by the convolution product: if (a_ω) and (b_ω) are elements of $\mathbf{C}[H]$ then their product (c_ω) is given by

$$c_\omega = \sum_{\omega' + \omega'' = \omega} a_{\omega'} b_{\omega''}.$$

We want to describe in similar terms the de Rham complex formed by the $\Omega^i(H)$, the spaces of regular differential i-forms on H.

Let $\Xi_{\mathbf{C}} = \Xi \otimes \mathbf{C}$ be the complexification of the free Abelian group Ξ. By a *discrete vector field* on Ξ we shall mean an assignment $\omega \mapsto v_\omega$ which takes any $\omega \in \Xi$ to a vector $v_\omega \in \Xi_{\mathbf{C}}$ such that $v_\omega = 0$ for all but finitely many ω. For $\omega \in \Xi$, $v \in \Xi_{\mathbf{C}}$, we denote by (ω, v) the discrete vector field equal to v at ω and 0 elsewhere.

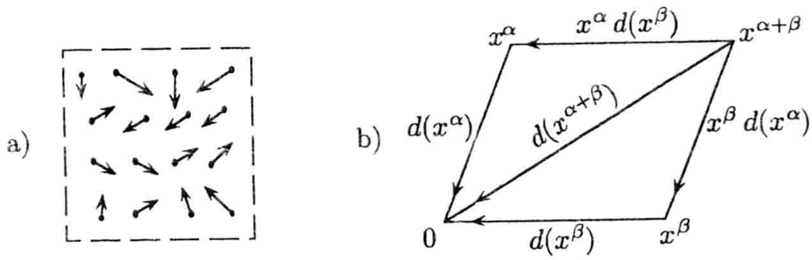

Figure 41. (a) A discrete vector field on $\mathbf{Z}^2$
(b) The proof of the Leibniz rule

Denote by $\mathrm{Vec}^1(\Xi)$ the space of all discrete vector fields on Ξ. Clearly $\mathrm{Vec}^1(\Xi)$ is a free $\mathbf{C}[\Xi]$-module of rank k under the obvious convolution product

$$\mathbf{C}[\Xi] \otimes \mathrm{Vec}^1(\Xi) \to \mathrm{Vec}^1(\Xi), \quad x^\alpha \otimes (\omega, v) \mapsto (\omega + \alpha, v). \tag{2.1}$$

Proposition 2.1. *There is a canonical isomorphism* $\varphi : \Omega^1(H) \to \mathrm{Vec}^1(\Xi)$ *of* $\mathbf{C}[\Xi]$*-modules. This isomorphism takes* $d(x^\gamma) \in \Omega^1(H)$ *into the discrete vector field* $(\gamma, -\gamma)$.

Geometrically, the field $(\gamma, -\gamma)$ is just one vector at the point γ which joins this point with 0 (see Figure 41 b).

Proof. By associating to $x^\alpha d(x^\beta)$ the discrete vector field $(\alpha + \beta, -\beta)$ we get the required isomorphism. The inverse isomorphism is defined as follows:

$$(\omega, v) \mapsto -x^\omega \cdot \frac{d(x^v)}{x^v} = -x^\omega \cdot d\log(x^v), \tag{2.2}$$

where $v \in \Xi$. This correspondence is additive in v and hence extends to any $v \in \Xi_{\mathbf{C}}$ by linearity.

It is instructive to see the validity of the Leibniz rule $d(x^{\alpha+\beta}) = x^\alpha d(x^\beta) + x^\beta d(x^\alpha)$ in the language of discrete vector fields (Figure 41 b).

Let us introduce the space $\mathrm{Vec}^i(\Xi)$ of *discrete* i*-vector fields* on Ξ whose elements are finitely supported functions $\omega \mapsto \lambda_\omega$ mapping Ξ to $\bigwedge^i(\Xi_{\mathbf{C}})$. For any $\omega \in \Xi$ and $\lambda \in \bigwedge^i(\Xi_{\mathbf{C}})$ we shall denote by (ω, λ) the discrete i-vector field on Ξ equal to λ at ω and 0 elsewhere.

The space $\mathrm{Vec}^0\Xi$ is just the group algebra $\mathbf{C}[\Xi]$. We have multiplication

$$\mathrm{Vec}^i\,\Xi \otimes \mathrm{Vec}^j\,\Xi \to \mathrm{Vec}^{i+j}\,\Xi, \quad (\omega, \lambda) \otimes (\eta, \mu) \mapsto (\omega + \eta, \lambda \wedge \mu). \tag{2.3}$$

Proposition 2.2. *The space* $\Omega^i(H)$ *of regular differential* i*-forms on* H *is naturally identified (as a* $\mathbf{C}[H]$*-module) with* $\mathrm{Vec}^i(\Xi)$*. Under this identification the exterior product in* $\Omega^\bullet(H)$ *corresponds to the product (2.3) on* $\mathrm{Vec}^\bullet(\Xi)$*. The exterior derivative of a form represented by a discrete* i*-vector field* $\omega \mapsto \lambda_\omega$ *on* Ξ *is represented by the discrete* $(i+1)$*-vector field* $\omega \mapsto \lambda_\omega \wedge (-\omega)$.

Proof. The isomorphism in question takes

$$x^{\alpha_0} d(x^{\alpha_1}) \wedge \cdots \wedge d(x^{\alpha_i}) \mapsto (-1)^i(\alpha_0 + \cdots + \alpha_i, \alpha_1 \wedge \cdots \wedge \alpha_i).$$

The inverse isomorphism takes

$$(\alpha, \beta_1 \wedge \cdots \wedge \beta_i) \mapsto (-1)^i x^\alpha \frac{d(x^{\beta_1})}{x^{\beta_1}} \wedge \cdots \wedge \frac{d(x^{\beta_i})}{x^{\beta_i}} \tag{2.4}$$

where $\alpha, \beta_j \in \Xi$. The correspondence (2.4) is additive in the β's and extends to the arbitrary $\beta_j \in \Xi_{\mathbf{C}}$ by linearity.

B. Differential forms on an affine toric variety

Let $\Xi = \mathbf{Z}^k$ be a free Abelian group of rank k. Let $S \subset \Xi$ be a finitely generated semigroup containing 0 and generating Ξ as a group. Let Ξ_R be the real vector space $\Xi \otimes R$ and let $K \subset \Xi_R$ be the convex hull of S. This is a convex polyhedral cone with apex 0. For any face $\Gamma \subset K$, we denote by $\mathrm{Lin}_{\mathbf{C}}(\Gamma)$ the smallest $\mathbf{C}$-vector subspace in $\Xi_{\mathbf{C}}$ containing Γ. Clearly, the complex dimension of $\mathrm{Lin}_{\mathbf{C}}(\Gamma)$ equals the dimension of Γ in the usual sense. For example, $\mathrm{Lin}_{\mathbf{C}}(K)$ is the whole $\Xi_{\mathbf{C}}$. For any $\omega \in S$, let $\Gamma(\omega)$ be the smallest face of K containing ω. In particular, if ω lies in the interior of K then $\Gamma(\omega) = K$.

Definition 2.3. Denote by $\mathrm{Vec}^i(S)$ the space of discrete i-vector fields (λ_ω) on Ξ with the properties:
(a) $v_\omega = 0$ if $\omega \notin S$;
(b) for any $\omega \in S$ we have $v_\omega \in \bigwedge^i \mathrm{Lin}_{\mathbf{C}}(\Gamma(\omega))$.

Figure 42 illustrates this definition.

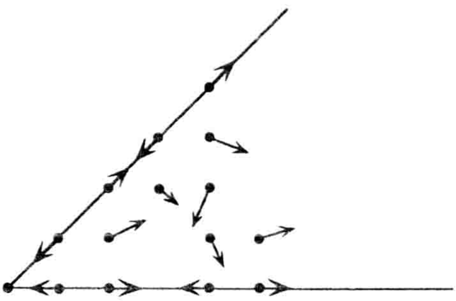

Figure 42. A discrete vector field from $\mathrm{Vec}^1(S)$

As in subsection A, for $\omega \in S$ and $\lambda \in \bigwedge^i \mathrm{Lin}_{\mathbf{C}}(\Gamma(\omega))$, we denote by (ω, λ) the discrete i-vector field equal to λ at ω and 0 elsewhere.

Clearly, $\mathrm{Vec}^0(S)$ is the semigroup algebra of S. The multiplication (2.3) restricts to the multiplication

$$\mathrm{Vec}^i S \otimes \mathrm{Vec}^j S \longrightarrow \mathrm{Vec}^{i+j} S. \tag{2.5}$$

With respect to (2.5), $\mathrm{Vec}^i S$ is a finitely generated module over $\mathbf{C}[S] = \mathrm{Vec}^0 S$. Let us also define the "exterior derivative"

$$d : \mathrm{Vec}^i S \to \mathrm{Vec}^{i+1} S, \quad d(\,(\omega, \lambda)\,) = (\omega, -\omega \wedge \lambda). \tag{2.6}$$

Proposition 2.4.

(a) *The maps d satisfy $d^2 = 0$ and the Leibniz rule thus making the direct sum* $\mathrm{Vec}^\bullet S = \bigoplus_i \mathrm{Vec}^i S$ *into a supercommutative differential graded algebra.*

(b) *If $S = \mathbf{Z}_+^a \times \mathbf{Z}^b$ then* $\mathrm{Vec}^i S$ *is identified with the space of regular differential i-forms on $\mathbf{C}^a \times (\mathbf{C}^*)^b$, and (2.5) and (2.6) coincide with the usual exterior multiplication and differential.*

Proof. Obvious.

Let $Y = \mathrm{Spec}\,\mathbf{C}[S]$ be the affine toric variety corresponding to S. By Serre's theorem [Hart] any module over the coordinate ring of an affine algebraic variety gives rise to a coherent sheaf on this variety. In our case we consider the $\mathbf{C}[S]$-module $\mathrm{Vec}^i S$. The coherent sheaf on Y corresponding to this module will be denoted by $\tilde{\Omega}_Y^i$ and called the sheaf of *Danilov i-forms* on Y.

Theorem 2.5.

(a) *Let Y_{sm} be the smooth locus of Y. Then the restriction of $\tilde{\Omega}_Y^i$ to Y_{sm} is naturally identified with the sheaf of regular differential i-forms on Y_{sm}.*

(b) *The maps (2.3) and (2.6) extend to morphisms of sheaves on Y*

$$\tilde{\Omega}_Y^i \otimes \tilde{\Omega}_Y^j \to \tilde{\Omega}_Y^{i+j}; \qquad d : \tilde{\Omega}_Y^i \to \tilde{\Omega}_Y^{i+1} \tag{2.7}$$

which, after restriction to Y_{sm}, coincide with the usual exterior multiplication and differential on forms.

Proof. Clearly, Y_{sm} is a union of open subsets of the form $\mathbf{C}^a \times (\mathbf{C}^*)^b$ invariant under the torus action. The coordinate ring of any such subset in Y has the form $\mathbf{C}[S']$ where S' is obtained from S by inverting some elements. Our statement now follows from Proposition 2.4 (b) and the identification

$$\mathrm{Vec}^i(S') = \mathrm{Vec}^i S \otimes_{\mathbf{C}[S]} \mathbf{C}[S'],$$

which can be verified immediately.

C. Combinatorial description of the discriminantal complex

Let $A \subset \mathbf{Z}^{k-1}$ be a finite set of n lattice points (=Laurent monomials) and let $X_A \subset P^{n-1}$ be the corresponding toric variety, see Section 1B, Chapter 5. Let us assume that X_A is smooth and has dimension $k-1$. In this case the formalism of Chapter 2 is applicable and we can represent the A-discriminant, i.e., the equation of the variety projectively dual to X_A, as the determinant of the discriminantal complex

$$\Delta_X(f) = const \cdot \det(C_+^\bullet(X_A, \mathcal{M}), \partial_f, e)^{(-1)^k} \tag{2.8}$$

where $\mathcal{M}$ is a sufficiently ample invertible sheaf on X_A (Theorem 2.5, Chapter 2). We take $\mathcal{M} = \mathcal{O}(l)$, $l \gg 0$. We shall use the results of the previous subsection to describe this complex quite explicitly.

Shrinking, if necessary, the lattice $\mathbf{Z}^{k-1}$, we can assume that it is affinely generated by A. As before, we embed $\mathbf{Z}^{k-1}$ into a free Abelian group $\Xi = \mathbf{Z}^k = \mathbf{Z}^{k-1} \times \mathbf{Z}$ as the set of lattice points with the last component equal to 1. We denote by $h : \Xi \to \mathbf{Z}$ the projection given by this last component.

Let $S \subset \Xi$ be the semigroup generated by A and 0. The semigroup algebra $\mathbf{C}[S]$ is graded by means of h (i.e., the degree of the monomial t^u, $u \in S$, is set to be $h(u)$). Under this grading $\mathbf{C}[S]$ is the homogeneous coordinate ring of the projective toric variety X_A and also the affine coordinate ring of the affine toric variety Y_A (the cone over X_A). The space $\mathbf{C}^A$ is embedded into $\mathbf{C}[S]$ as the graded component of degree 1.

Let l be an integer. Consider the graded vector space $C^\bullet(A, l)$, where

$$C^i(A, l) = \bigoplus_{u \in S, h(u) = l+i} \bigwedge^i \mathrm{Lin}_{\mathbf{C}} \Gamma(u). \tag{2.9}$$

Let us denote a typical element in the u-th summand of $C^i(A, l)$ by (u, λ) where $\lambda \in \bigwedge^i \mathrm{Lin}_{\mathbf{C}} \Gamma(u)$. For any $f(x) = \sum_{\omega \in A} a_\omega x^\omega \in \mathbf{C}^A$, we define the differential $\partial_f : C^i(A, l) \to C^{i+1}(A, l)$ by

$$\partial_f(u, \lambda) = -\sum_{\omega \in A} a_\omega \cdot (\omega + u, \omega \wedge \lambda). \tag{2.10}$$

It is straightforward to see that $\partial_f^2 = 0$.

Theorem 2.6. *Assume that X_A is smooth. For $l \gg 0$ the complex $(C^\bullet(A, l), \partial_f)$ coincides with the discriminantal complex $(C_+^\bullet(X_A, \mathcal{O}(l)), \partial_f)$.*

Proof. By Corollary 4.2, Chapter 2, the space $C_+^i(X_A, \mathcal{O}(l))$ (denoted there by $C^i(X_A, l)$) is identified with the space of differential i-forms on $Y - \{0\}$ homogeneous of degree $i + l$. On the other hand, (2.9) is the $(i+l)$-th graded component of the $\mathbf{C}[S]$-module $\mathrm{Vec}^i S$ with respect to the following grading: $\deg(u, \lambda) = h(u)$. This grading is obviously compatible with the similar grading on $\mathbf{C}[S]$ defined above.

By our assumption X_A is smooth and hence $Y_A - \{0\}$ is the smooth locus of Y_A. So the sheaf $\tilde{\Omega}^i_{Y_A}$ on Y_A corresponding to $\mathrm{Vec}^i S$ is identified, after restriction to $Y_A - \{0\}$, with the sheaf of i-forms $\Omega^i_{Y_A - \{0\}}$. Note that Theorem 2.5 gives a natural homomorphism of vector spaces

$$\Phi : \mathrm{Vec}^i S \longrightarrow \{i\text{-forms on } Y_A - \{0\}\}. \tag{2.11}$$

Let $j : Y_A - \{0\} \hookrightarrow Y_A$ be the embedding. Since the point 0 has codimension at least 2 in Y_A, the direct image under j of any coherent sheaf on $Y_A - \{0\}$, in particular, of the sheaf $\Omega^i_{Y_A - \{0\}}$, is a coherent sheaf on Y. The homomorphism Φ comes (by taking global sections) from a morphism

$$\varphi : \tilde{\Omega}^i_{Y_A} \to j_* \Omega^i_{Y_A - \{0\}}$$

of coherent sheaves on Y_A whose existence follows from Theorem 2.5. Since φ is an isomorphism outside 0, it follows that Ker φ and Coker φ are coherent sheaves on Y supported at 0. Therefore, by taking global sections, we find that Ker Φ and Coker Φ are finite-dimensional vector spaces. Since both spaces in (2.11) are graded, we conclude that, for $l \gg 0$, the induced map of $(i + l)$-th graded components is an isomorphism, i.e.,

$$C^i_+(X_A, \mathcal{O}(l)) \cong C^i(A, l).$$

The fact that the differentials in these complexes agree under this isomorphism follows from Theorem 2.5 (b). Theorem 2.6 is proved.

The above theorem implies that (in the case of smooth X_A) the A-discriminant equals $const \cdot D^{(-1)^k}$ where D is the determinant of $C^\bullet(A, l)$. We shall give a more precise formula, valid up to sign.

Namely, consider the following system of bases in the vector spaces $C^i(A, l)$. For any $u \in S$ the vector space $\mathrm{Lin}_{\mathbf{C}}\Gamma(u)$ contains a $\mathbf{Z}$-lattice $\mathrm{Lin}_{\mathbf{C}}\Gamma(u) \cap \Xi$. We choose any $\mathbf{Z}$-basis in this lattice as a basis of $\mathrm{Lin}_{\mathbf{C}}\Gamma(u)$. Correspondingly, we choose the basis in $\bigwedge^i(\mathrm{Lin}_{\mathbf{C}}\Gamma(u))$ formed by exterior products of basis vectors in $\mathrm{Lin}_{\mathbf{C}}\Gamma(u)$. Finally, we choose as a basis in the direct sum (2.9), the union of the chosen bases in the summands. Let e be the resulting system of bases in the terms $C^i(A, l)$.

Theorem 2.7. *Assume that X_A is smooth. Then for $l \gg 0$ we have*

$$\Delta_A(f) = \pm \det(C^\bullet(A, l), \partial_f, e)^{(-1)^k}.$$

Proof. Up to a constant factor, the statement follows from Theorem 2.5, Chapter 2. To prove it up to a sign, we proceed as in the proof of Theorem 2.5, Chapter 8. Namely, we denote by $C^\bullet_{\mathbf{Z}}(A, l)$ the natural $\mathbf{Z}$-form of $C^\bullet(A, l)$:

$$C^i_{\mathbf{Z}}(A, l) = \bigoplus_{u \in S, h(u) = l + i} \bigwedge\nolimits^i_{\mathbf{Z}} (\mathrm{Lin}_{\mathbf{C}}\Gamma(u) \cap \Xi).$$

Now let p be an arbitrary prime number, let F be the algebraic closure of $\mathbf{Z}/p\mathbf{Z}$ and consider the graded F-vector space

$$C_F^\bullet(A,l) = C_\mathbf{Z}^\bullet(A,l) \otimes_\mathbf{Z} F.$$

For any $f \in F^A$ we get a differential ∂_f in $C_F^\bullet(A,l)$. As in the proof of Theorem 2.5 Chapter 8, it suffices for our purposes to show that, for generic $f \in F^A$, the complex $(C_F^\bullet(A,l), \partial_f)$ is exact. The description of differential forms on a smooth toric variety given in subsection B, remains valid over a field of any characteristic. So our statement is proved in the same way as generic exactness of the discriminantal complex over $\mathbf{C}$ (Theorem 2.3, Chapter 2). This completes the proof of Theorem 2.7.

D. The degree of the A-discriminant

We continue to assume that X_A is smooth and that A affinely spans $\mathbf{Z}^{k-1}$ over $\mathbf{Z}$. Let $Q \subset \mathbf{R}^{k-1}$ be the convex hull of A. For each face $\Gamma \subset Q$, let $\mathrm{Aff}_\mathbf{R}(\Gamma)$ be the smallest real affine subspace containing Γ. This space comes equipped with an affine $\mathbf{Z}$-lattice $\mathrm{Aff}_\mathbf{Z}(\Gamma \cap A)$ generated by $\Gamma \cap A$. Since we assume X_A to be smooth, this lattice coincides with $\mathrm{Aff}_\mathbf{R}(\Gamma) \cap \mathbf{Z}^{k-1}$ (Corollary 3.2, Chapter 5).

As in Section 4D, Chapter 5, the above lattice gives rise to a volume form on $\mathrm{Aff}_\mathbf{R}(\Gamma)$ normalized so that the volume of an elementary lattice simplex equals 1. Let us denote this form as Vol_Γ.

Theorem 2.8. *Suppose X_A is smooth. Then the degree of homogeneity of the A-discriminant equals*

$$\sum_{\Gamma \subset Q} (-1)^{\mathrm{codim}\,\Gamma} (\dim \Gamma + 1) \cdot \mathrm{Vol}_\Gamma(\Gamma).$$

In particular, this sum is always non-negative; it equals zero if and only if $\Delta_A = 1$.

Proof. We retain the notation of the previous subsections. Thus $S \subset \mathbf{Z}^k$ is the semigroup generated by A and 0; it is graded by $h : S \to \mathbf{Z}_+$. Denote by S_l the graded component $\{u \in S : h(u) = l\}$. We also need the convex hull K of S. This is a polyhedral cone with apex 0 whose base is Q. We extend h to a linear functional (denoted also h) from K to $\mathbf{R}$ and denote $K_l = \{u \in K : h(u) = l\}$. Since X_A is assumed to be smooth and, in particular, normal, the intersection $K \cap \mathbf{Z}^k$ differs from S in finitely many points only.

Let $\Gamma \subset Q$ be a face. Then the cone $\mathbf{R}_+\Gamma$ generated by Γ is a face of K. Let Γ^0 be the interior of Γ. Consider the set $S_l \cap \mathbf{R}_+\Gamma^0$. By the above, for $l \gg 0$ this set coincides with $K_l \cap \mathbf{R}_+\Gamma^0$. Thus, for large l, the number $\#\left(S_l \cap \mathbf{R}_+\Gamma^0\right)$

coincides with the number of integer points in the l times dilated open polytope Γ^0. It is known [D] that for $l \gg 0$ this number is given by a polynomial in l which we denote by $p_\Gamma(l)$ (it is closely related to the so-called Ehrhart polynomial counting the number of integer points in the dilations of Γ, not Γ^0). The leading term of this polynomial is, by Proposition 3.7, Chapter 5,

$$\frac{\mathrm{Vol}_\Gamma(\Gamma)}{(\dim \Gamma)!} l^{\dim \Gamma}.$$

By Corollary 14 from Appendix A, the degree of Δ_A is equal to

$$\deg(\Delta_A) = \sum_{i=0}^{k} (-1)^{k-i} i \cdot \dim C^i(A, l).$$

By representing $C^i(A, l)$ as a direct sum over $u \in S_{i+l}$ (see (2.9)) and separating the u's lying in different faces of K, we find that, for $l \gg 0$,

$$\dim C^i(A, l) = \sum_{\Gamma \subset Q} \binom{\dim \Gamma + 1}{i} p_\Gamma(l + i).$$

Substituting this expression into the above formula for $\deg(\Delta_A)$ we obtain, after some easy algebraic transformations, that

$$\deg(\Delta_A) = \sum_{\Gamma \subset Q} (-1)^{\mathrm{codim}\, \Gamma} (\dim \Gamma + 1) \sum_{i=0}^{\dim \Gamma} (-1)^{\dim \Gamma - i} \binom{\dim \Gamma}{i} \cdot p_\Gamma(l + 1 + i).$$

To deduce Theorem 2.8 from this expression, we have only to show that the inner sum is equal to $\mathrm{Vol}_\Gamma(\Gamma)$. We shall use the following elementary lemma.

Lemma 2.9. *Let $p(t) = a_0 t^r + \cdots + a_r$ be a polynomial of degree r. Then for any value of t the sum $\sum_{i=0}^{r} (-1)^i \binom{r}{i} p(t - i)$ is equal to $r! a_0$.*

The lemma is well-known; the easiest way to prove it is to observe that the sum in question is the iterated difference $\Delta^r p(t)$, where $\Delta p(t) = p(t) - p(t-1)$.

To complete the proof of Theorem 2.8, it is enough to apply Lemma 2.9 to each polynomial $p(t) = p_\Gamma(t + \dim \Gamma + 1)$.

Examples 2.10. Let us illustrate Theorem 2.8 for the sets A in Examples 1.6.

(a) Let A consist of all monomials in $x_1, \ldots, x_{k-1}$ of degree $\leq d$ (or, equivalently, of all homogeneous monomials in $x_1, \ldots, x_k$ of degree d). Since $X_A = P^{k-1}$ is smooth, Theorem 2.8 is applicable. The polytope Q is the simplex

$$\left\{ (t_1, \ldots, t_{k-1}) \in \mathbf{R}^{k-1} : t_i \geq 0, \sum t_i \leq d \right\}$$

of dimension $k-1$. For each $i = 0, \ldots, k-1$, this simplex has exactly $\binom{k}{i+1}$ faces of dimension i, and each of them has the normalized volume equal to d^i. By Theorem 2.8,

$$\deg(\Delta_A) = \sum_{i=0}^{k-1} (-1)^{k-1-i}(i+1)\binom{k}{i+1} d^i$$

$$= k\sum_{i=0}^{k-1}(-1)^{k-1-i}\binom{k-1}{i} d^i k(d-1)^{k-1}, \qquad (2.12)$$

the last equality being the binomial formula. By Lemma 2.9, $\deg(\Delta_A)$ is equal to $k(d-1)^{k-1}$. This is Boole's formula (see Examples 1.6 above).

(b) Let A consist of bilinear monomials $x_i \cdot y_j$, $i = 1, \ldots, m$; $j = 1, \ldots, n$. The polytope Q is the product of two simplices $\Delta^{m-1} \times \Delta^{n-1}$. A face of Q is given by a pair of non-empty subsets $I \subset \{1, \ldots, m\}$, $J \subset \{1, \ldots, n\}$ and is itself a product $\Delta^{i-1} \times \Delta^{j-1}$ where $i = \#(I)$, $j = \#(J)$. The normalized volume of such a face is $\binom{i+j-2}{i-1}$. Since $X_A = P^{m-1} \times P^{n-1}$ is smooth, the degree of Δ_A is given by Theorem 2.8. We obtain

$$\deg(\Delta_A) = \sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{m+n-i-j}(i+j-1)\binom{m}{i}\binom{n}{j}\binom{i+j-2}{i-1}. \qquad (2.13)$$

Without loss of generality, we can assume that $n \le m$. To simplify (2.13) we rewrite it as

$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}p(m-i),$$

where $p(t)$ is a polynomial of degree n given by

$$p(t) = \sum_{j=1}^{n}(-1)^{n-j}\frac{1}{(j-1)!}\binom{n}{j}t(t+1)\cdots(t+j-1).$$

Using Lemma 2.9, we see that $\deg(\Delta_A) = 0$ unless $m = n$, and in the latter case $\deg(\Delta_A) = n$. This is in accordance with the fact that Δ_A is identically 1 for $m \ne n$ and coincides with the determinant of a $n \times n$ matrix if $m = n$.

The above two examples have a common generalization to the case when A consists of all multihomogeneous monomials of a given multidegree in several groups of variables. An application of Theorem 2.8 to this case will be discussed in Section 2, Chapter 13.

(c) Let A consist of monomials

$$1, x, x^2, \ldots, x^m, y, yx, yx^2, \ldots, yx^n.$$

The polytope Q is the trapezoid depicted in Figure 43.

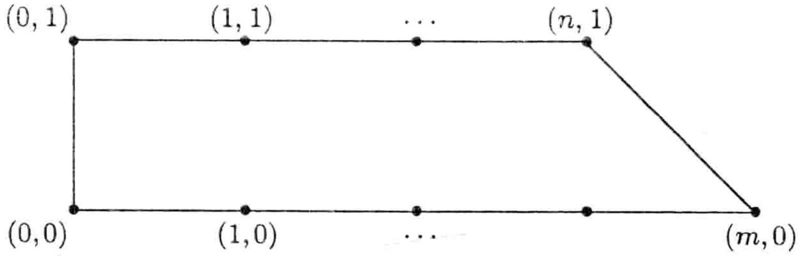

Figure 43.

Its area normalized with respect to $\mathbf{Z}^2$ is $m+n$. Under the normalizations used in Theorem 2.8, the horizontal sides have lengths m and n, and two other sides are of length 1. Four vertices of Q each should be ascribed the "volume" 1. Hence Theorem 2.8 gives

$$\deg(\Delta_A) = 3(m+n) - 2(m+n+2) + 1 \cdot 4 = m+n.$$

This is in accordance with the interpretation of Δ_A as the resultant of two polynomials of degrees m and n in one variable, see Example 1.6 (c).

3. A differential-geometric characterization of A-discriminantal hypersurfaces

In this section, which is based on [Ka1], we exhibit one characteristic property of discriminantal hypersurfaces regarded as hypersurfaces in tori. As we have seen in Section 1, Chapter 6, the geometry of a hypersurface in a torus is closely related to the Newton polytope of the Laurent polynomial defining this hypersurface. The differential-geometric property described in this section can be compared with the description of the Newton polytope of the A-discriminant given in Chapter 11 below.

A. The Gauss map in an algebraic group

Let G be an algebraic group. For each $g \in G$, let $l_g : G \to G$ be the left multiplication by g. Let $\mathfrak{g}$ be the Lie algebra of G. Let $Z \subset G$ be an irreducible algebraic hypersurface (possibly with singularities). The (left) *Gauss map* of Z is the rational map $\gamma_Z : Z \to P(\mathfrak{g}^*)$ which takes a smooth point $z \in Z$ into $d(l_z^{-1})(T_z Z)$, i.e., to the translation to unity of the tangent hyperplane to Z at z. This translation is a hyperplane in $T_e G = \mathfrak{g}$, i.e., a point in $P(\mathfrak{g}^*)$.

Note that both varieties Z and $P(\mathfrak{g}^*)$ have the same dimension. This raises the following natural problem.

Problem. *Classify algebraic hypersurfaces $Z \subset G$ such that $\gamma_Z : Z \to P(\mathfrak{g}^*)$ is a birational isomorphism.*

In what follows we shall consider only the case when $G = (\mathbf{C}^*)^m$ is an algebraic torus. In this case we shall refer to γ_Z as the *logarithmic Gauss map* since explicit formulas for it involve logarithmic derivatives.

It turns out that the above problem for tori can be completely solved and the class of hypersurfaces in question essentially coincides with the class of A-discriminantal hypersurfaces.

B. The reduced A-discriminantal variety

Let $A \subset \mathbf{Z}^{k-1}$ be a finite subset of cardinality n, which generates $\mathbf{Z}^{k-1}$ as an affine lattice. Let $\nabla_A \subset \mathbf{C}^A$ be the corresponding discriminantal variety. We consider the action of $(\mathbf{C}^*)^k$ on $\mathbf{C}^A$ given by

$$(t_1, \ldots, t_k) : f(x_1, \ldots, x_{k-1}) \mapsto t_k f(t_1 x_1, \ldots, t_{k-1} x_{k-1}). \tag{3.1}$$

This action preserves ∇_A. Consider the subset $(\mathbf{C}^*)^A \subset \mathbf{C}^A$ as an algebraic torus acting on $\mathbf{C}^A$ componentwise. A point of $(\mathbf{C}^*)^A$ will be denoted as $(z_\omega)_{\omega \in A}$ where $z_\omega \in \mathbf{C}^*$. Then (3.1) comes from a homomorphism of tori

$$\varphi : (\mathbf{C}^*)^k \to (\mathbf{C}^*)^A, \quad \varphi(t_1, \ldots, t_k)_\omega = t_k t_1^{\omega_1} \cdots t_{k-1}^{\omega_{k-1}}.$$

Let $\varphi^* : \mathbf{Z}^A \to \mathbf{Z}^k$ be the dual homomorphism of character lattices. Since we assume that A generates $\mathbf{Z}^{k-1}$ as an affine lattice, φ^* is surjective. Set $L_A = \operatorname{Ker} \varphi^*$. Clearly, L_A consists of all affine relations between elements of A, i.e., of families $(c_\omega)_{\omega \in A}$, $c_\omega \in \mathbf{Z}$ such that

$$\sum_{\omega \in A} c_\omega \cdot \omega = 0 \quad \text{and} \quad \sum_{\omega \in A} c_\omega = 0.$$

Let $H(L_A) = \operatorname{Spec} \mathbf{C}[L_A] = \operatorname{Hom}(L_A, \mathbf{C}^*)$ be the algebraic torus whose character lattice is L_A. Then we have an exact sequence of tori

$$1 \to (\mathbf{C}^*)^k \xrightarrow{\varphi} (\mathbf{C}^*)^A \xrightarrow{p} H(L_A) \to 1, \tag{3.2}$$

where p is the natural projection. Since ∇_A is $(\mathbf{C}^*)^k$-invariant, the intersection $\nabla_A \cap (\mathbf{C}^*)^A$ is the inverse image, under p, of some subvariety $\tilde{\nabla}_A \subset H(L_A)$ which