



# ANALYSIS, MANIFOLDS AND PHYSICS

## Part I: Basics

YVONNE CHOQUET-BRUHAT, C. DeWITT-MORETTE, M. DILLARD-BLEICK

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# ANALYSIS, MANIFOLDS AND PHYSICS

## Part I: Basics

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## INTRODUCTION

All too often in physics familiarity is a substitute for understanding, and the beginner who lacks familiarity wonders which is at fault: physics or himself. Physical mathematics provides well defined concepts and techniques for the study of physical systems. It is more than mathematical techniques used in the solution of problems which have already been formulated; it helps in the very formulation of the laws of physical systems and brings a better understanding of physics. Thus physical mathematics includes mathematics which gives promise of being useful in our analysis of physical phenomena. Attempts to use mathematics for this purpose may fail because the mathematical tool is too crude; physics may then indicate along which lines it should be sharpened. In fact, the analysis of physical systems has spurred many a new mathematical development.

Considerations of relevance to physics underlie the choice of material included here. Any choice is necessarily arbitrary; we included first the topics which we enjoy most but we soon recognized that instant gratification is a short range criterion. We then included material which can be appreciated only after a great deal of intellectual asceticism but which may be farther reaching. Finally, this book gathers the starting points of some great currents of contemporary mathematics. It is intended for an advanced physical mathematics course.

Chapters I and II are two preliminary chapters included here to spare the reader the task of looking up in several specialized books the definitions and the basic theorems used in the subsequent chapters. Chapter I is merely a review of fundamental notions of algebra, topology, integration, and analysis. Chapter II treats the essentials of differential calculus and calculus of variations on Banach spaces. Each of the following chapters introduces a mathematical structure and exploits it until it is sufficiently familiar to become an "instrument de pensée"; Chapter III, differentiable manifolds, tangent bundles and their use in Lie groups; Chapter IV, exterior derivation and the solutions of exterior differential systems; Chapter V, Riemannian structures which, together with the previous structures provide the basic geometric notions needed in physics; Chapter VI, distributions and the Sobolev spaces with recent applications to the theory of partial differential equations. The last

chapter covers some selected topics in the theory of infinite dimensional manifolds.

At the end of each chapter, several problems are worked out. Most of them show how the concepts and the theorems introduced in the text can be used in physics. They should be of interest both to physicists and mathematicians. A sentence like "The Lagrangian is a function defined on the tangent bundle of the configuration space" helps explain to the physicist what a tangent bundle is and tells a mathematician what a Lagrangian is. A sentence like "The strain tensor is the Lie derivative of the metric with respect to the deformation" helps a physicist to understand the concept of Lie derivatives and defines the strain tensor to a mathematician. To both, they bring an added pleasure.

The pleasure of physical mathematics is well described by Hilbert: learning that some genetic laws of the fruit fly had been derived by the application of a certain set of axioms he exclaimed "So simple and precise and at the same time so miraculous that no daring fantasy could have imagined it"<sup>2</sup>.

#### PREFACE TO THE SECOND EDITION

We are happy that the success of the first edition gave us a chance to prepare a revised edition. We have made numerous changes and added exercises with their solutions to ease the study of several chapters. The major addition is a chapter "Connections on principal fibre bundles" which includes sections on holonomy, characteristic classes, invariant curvature integrals and problems on the geometry of gauge fields, monopoles, instantons, spin structure and spin connections. Other additions include a section on the second fundamental form, a section on almost complex and kählerian manifolds, and a problem on the method of stationary phase. More than 150 entries have been added to the index.

Can this book, now polished by usage, serve as a text for an advanced physical mathematics course? This question raises another question: What is the function of a text book for graduate studies? In our times of rapidly expanding knowledge, a teacher looks for a book which will provide a broader base for future developments than can be covered in one or two semesters of lectures and a student hopes that his purchase will serve him for many years. A reference book which can be used as a text is an answer to their needs. This is what this book is intended to be, and thanks to a publishing company which keeps it moderately priced, this is what we hope it will be.

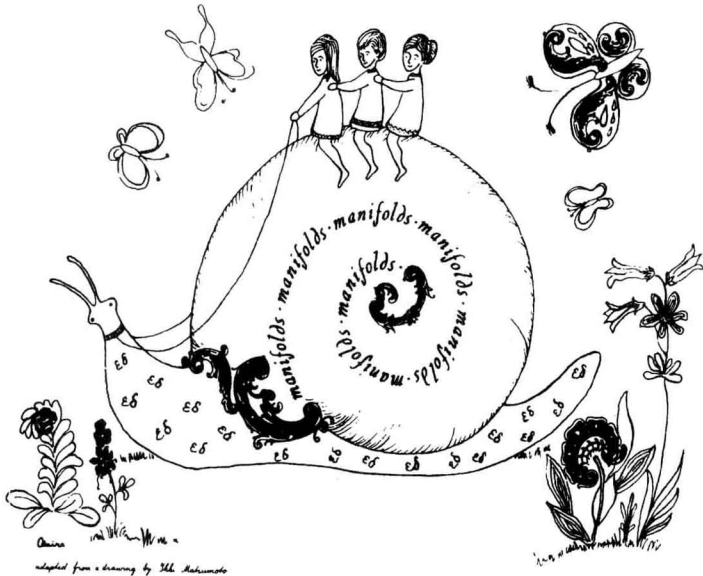
<sup>2</sup>"Hilbert" Constance Reid, Springer Verlag, 1970.

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We regret that one of us who is now engaged in other pursuits could not contribute to the revised edition. The core of the book remains nevertheless under the trademark of our collaboration.

Y. Choquet-Bruhat  
C. DeWitt-Morette



## CONTENTS

I. Review of Fundamental Notions of Analysis	1
A. <i>Set Theory, Definitions</i>	1
1. Sets	1
2. Mappings	2
3. Relations	5
4. Orderings	5
B. <i>Algebraic Structures, Definitions</i>	6
1. Groups	7
2. Rings	8
3. Modules	8
4. Algebras	9
5. Linear spaces	9
C. <i>Topology</i>	11
1. Definitions	11
2. Separation	13
3. Base	14
4. Convergence	14
5. Covering and compactness	15
6. Connectedness	16
7. Continuous mappings	17
8. Multiple connectedness	19
9. Associated topologies	20
10. Topology related to other structures	21
11. Metric spaces	23
metric spaces	23
Cauchy sequence; completeness	25
12. Banach spaces	26
normed vector spaces	27
Banach spaces	28
strong and weak topology; compactedness	29
13. Hilbert spaces	30
D. <i>Integration</i>	31
1. Introduction	32
2. Measures	33
3. Measure spaces	34



4. Measurable functions	40
5. Integrable functions	41
6. Integration on locally compact spaces	46
7. Signed and complex measures	49
8. Integration of vector valued functions	50
9. $L^1$ space	52
10. $L^p$ space	53
 E. Key Theorems in Linear Functional Analysis	57
1. Bounded linear operators	57
2. Compact operators	61
3. Open mapping and closed graph theorems	63
 Problems and Exercises	64
Problem 1: Clifford algebra; Spin(4)	64
Exercise 2: Product topology	68
Problem 3: Strong and weak topologies in $L^2$	69
Exercise 4: Hölder spaces	70
See Problem VI 4: Application to the Schrödinger equation	
 II. Differential Calculus on Banach Spaces	71
A. Foundations	71
1. Definitions. Taylor expansion	71
2. Theorems	73
3. Diffeomorphisms	74
4. The Euler equation	76
5. The mean value theorem	78
6. Higher order differentials	79
B. Calculus of Variations	82
1. Necessary conditions for minima	82
2. Sufficient conditions	83
3. Lagrangian problems	86
C. Implicit Function Theorem. Inverse Function Theorem	88
1. Contracting mapping theorems	88
2. Inverse function theorem	90
3. Implicit function theorem	91
4. Global theorems	92
D. Differential Equations	94
1. First order differential equation	94
2. Existence and uniqueness theorems for the lipschitzian case	95

<i>Problems and Exercises</i>	98
Problem 1: Banach spaces, first variation, linearized equation	98
Problem 2: Taylor expansion of the action; Jacobi fields; the Feynman-Green function; the Van Vleck matrix; conjugate points; caustics	100
Problem 3: Euler-Lagrange equation; the small disturbance equation; the soap bubble problem; Jacobi fields	105
 III. Differentiable Manifolds, Finite Dimensional Case	 111
A. Definitions	111
1. Differentiable manifolds	111
2. Diffeomorphisms	115
3. Lie groups	116
B. Vector Fields; Tensor Fields	117
1. Tangent vector space at a point	117
tangent vector as a derivation	118
tangent vector defined by transformation properties	120
tangent vector as an equivalence class of curves	121
images under differentiable mappings	121
2. Fibre bundles	124
definition	125
bundle morphisms	127
tangent bundle	127
frame bundle	128
principal fibre bundle	129
3. Vector fields	132
vector fields	132
moving frames	134
images under diffeomorphisms	134
4. Covariant vectors; cotangent bundles	135
dual of the tangent space	135
space of differentials	137
cotangent bundle	138
reciprocal images	138
5. Tensors at a point	138
tensors at a point	138
tensor algebra	140
6. Tensor bundles; tensor fields	142
C. Groups of Transformations	143
1. Vector fields as generators of transformation groups	143
2. Lie derivatives	147
3. Invariant tensor fields	150

<i>D. Lie Groups</i>	152
1. Definitions; notations	152
2. Left and right translations; Lie algebra; structure constants	155
3. One-parameter subgroups	158
4. Exponential mapping; Taylor expansion; canonical coordinates	160
5. Lie groups of transformations; realization	162
6. Adjoint representation	166
7. Canonical form, Maurer–Cartan form	168
<i>Problems and Exercises</i>	169
Problem 1: Change of coordinates on a fiber bundle, configuration space, phase space	169
Problem 2: Lie algebras of Lie groups	172
Problem 3: The strain tensor	177
Problem 4: Exponential map; Taylor expansion; adjoint map; left and right differentials; Haar measure	178
Problem 5: The group manifolds of $SO(3)$ and $SU(2)$	181
Problem 6: The 2-sphere	190
 IV. Integration on Manifolds	 195
<i>A. Exterior Differential Forms</i>	195
1. Exterior algebra	195
exterior product	196
local coordinates; strict components	197
change of basis	199
2. Exterior differentiation	200
3. Reciprocal image of a form (pull back)	203
4. Derivations and antiderivations	205
definitions	206
interior product	207
5. Forms defined on a Lie group	208
invariant forms	208
Maurer–Cartan structure equations	208
6. Vector valued differential forms	210
<i>B. Integration</i>	212
1. Integration	212
orientation	212
odd forms	212
integration of $n$ -forms in $\mathbb{R}^n$	213
partitions of unity	214
properties of integrals	215
2. Stokes' theorem	216
<p><math>p</math>-chains</p>	217
integrals of $p$ -forms on $p$ -chains	217
boundaries	218

mappings of chains	219
proof of Stokes' theorem	221
3. Global properties	222
homology and cohomology	222
0-forms and 0-chains	223
Betti numbers	224
Poincaré lemmas	224
de Rham and Poincaré duality theorems	226
<i>C. Exterior Differential Systems</i>	229
1. Exterior equations	229
2. Single exterior equation	229
3. Systems of exterior equations	232
ideal generated by a system of exterior equations	232
algebraic equivalence	232
solutions	233
examples	235
4. Exterior differential equations	236
integral manifolds	236
associated Pfaff systems	237
generic points	238
closure	238
5. Mappings of manifolds	239
introduction	239
immersion	241
embedding	241
submersion	242
6. Pfaff systems	242
complete integrability	243
Frobenius theorem	243
integrability criterion	245
examples	246
dual form of the Frobenius theorem	248
7. Characteristic system	250
characteristic manifold	250
example: first order partial differential equations	250
complete integrability	253
construction of integral manifolds	254
Cauchy problem	256
examples	259
8. Invariants	261
invariant with respect to a Pfaff system	261
integral invariants	263
9. Example: Integral invariants of classical dynamics	265
Liouville theorem	266
canonical transformations	267

10. Symplectic structures and hamiltonian systems	267
<i>Problems and Exercises</i>	270
Problem 1: Compound matrices	270
Problem 2: Poincaré lemma, Maxwell equations, wormholes	271
Problem 3: Integral manifolds	271
Problem 4: First order partial differential equations, Hamilton–Jacobi equations, lagrangian manifolds	272
Problem 5: First order partial differential equations, catastrophes	277
Problem 6: Darboux theorem	281
Problem 7: Time dependent hamiltonians	283
See Problem VI 11 paragraph c: Electromagnetic shock waves	
 V. Riemannian Manifolds. Kählerian Manifolds	285
<i>A. The Riemannian Structure</i>	285
1. Preliminaries	285
metric tensor	285
hyperbolic manifold	287
2. Geometry of submanifolds, induced metric	290
3. Existence of a riemannian structure	292
proper structure	292
hyperbolic structure	293
Euler–Poincaré characteristic	293
4. Volume element. The star operator	294
volume element	294
star operator	295
5. Isometries	298
 <i>B. Linear Connections</i>	300
1. Linear connections	300
covariant derivative	301
connection forms	301
parallel translation	302
affine geodesic	302
torsion and curvature	305
2. Riemannian connection	308
definitions	309
locally flat manifolds	310
3. Second fundamental form	312
4. Differential operators	316
exterior derivative	316
operator $\delta$	317
divergence	317
laplacian	318

<i>C. Geodesics</i>	320
1. Arc length	320
2. Variations	321
Euler equations	323
energy integral	324
3. Exponential mapping	325
definition	325
normal coordinates	326
4. Geodesics on a proper riemannian manifold	327
properties	327
geodesic completeness	330
5. Geodesics on a hyperbolic manifold	330
<i>D. Almost Complex and Kählerian Manifolds</i>	330
<i>Problems and Exercises</i>	336
Problem 1: Maxwell equation; gravitational radiation	336
Problem 2: The Schwarzschild solution	341
Problem 3: Geodetic motion; equation of geodetic deviation; exponentiation; conjugate points	344
Problem 4: Causal structures; conformal spaces; Weyl tensor	350
 Vbis. Connections on a Principal Fibre Bundle	 357
<i>A. Connections on a Principal Fibre Bundle</i>	357
1. Definitions	357
2. Local connection 1-forms on the base manifold	362
existence theorems	362
section canonically associated with a trivialization	363
potentials	364
change of trivialization	364
examples	366
3. Covariant derivative	367
associated bundles	367
parallel transport	369
covariant derivative	370
examples	371
4. Curvature	372
definitions	372
Cartan structural equation	373
local curvature on the base manifold	374
field strength	375
Bianchi identities	375
5. Linear connections	376
definition	376
soldering form, torsion form	376

torsion structural equation	376
standard horizontal (basic) vector field	378
curvature and torsion on the base manifold	378
bundle homomorphism	380
metric connection	381
<b>B. Holonomy</b>	381
1. Reduction	381
2. Holonomy groups	386
<b>C. Characteristic Classes and Invariant Curvature Integrals</b>	390
1. Characteristic classes	390
2. Gauss–Bonnet theorem and Chern numbers	395
3. The Atiyah–Singer index theorem	396
<b>Problems and Exercises</b>	401
Problem 1: The geometry of gauge fields	401
Problem 2: Charge quantization. Monopoles	408
Problem 3: Instanton solution of euclidean $SU(2)$ Yang–Mills theory (connection on a non-trivial $SU(2)$ bundle over $S^4$ )	411
Problem 4: Spin structure; spinors; spin connections	415
<b>VI. Distributions</b>	423
<b>A. Test Functions</b>	423
1. Seminorms	423
definitions	423
Hahn–Banach theorem	424
topology defined by a family of seminorms	424
2. $\mathcal{D}$ -spaces	427
definitions	427
inductive limit topology	429
convergence in $\mathcal{D}^m(U)$ and $\mathcal{D}(U)$	430
examples of functions in $\mathcal{D}$	431
truncating sequences	434
density theorem	434
<b>B. Distributions</b>	435
1. Definitions	435
distributions	435
measures; Dirac measures and Leray forms	437
distribution of order $p$	439
support of a distribution	441
distributions with compact support	441

2. Operations on distributions	444
sum	444
product by $C^\infty$ function	444
direct product	445
derivations	446
examples	447
inverse derivative	450
3. Topology on $\mathcal{D}'$	453
weak star topology	453
criterion of convergence	454
4. Change of variables in $\mathbb{R}^n$	456
change of variables in $\mathbb{R}^n$	456
transformation of a distribution under a diffeomorphism	457
invariance	459
5. Convolution	459
convolution algebra $L^1(\mathbb{R}^n)$	459
convolution algebra $\mathcal{D}'^+$ and $\mathcal{D}'^-$	462
derivation and translation of a convolution product	464
regularization	465
support of a convolution	465
equations of convolution	466
differential equation with constant coefficients	469
systems of convolution equations	470
kernels	471
6. Fourier transform	474
Fourier transform of integrable functions	474
tempered distributions	476
Fourier transform of tempered distributions	476
Paley-Wiener theorem	477
Fourier transform of a convolution	478
7. Distribution on a $C^\infty$ paracompact manifold	480
8. Tensor distributions	482
C. Sobolev Spaces and Partial Differential Equations	486
1. Sobolev spaces	486
properties	487
density theorems	488
$W_p^m$ spaces	489
Fourier transform	490
Plancherel theorem	490
Sobolev's inequalities	491
2. Partial differential equations	492
definitions	492
Cauchy-Kovalevski theorem	493
classifications	494
3. Elliptic equations; laplacians	495



elementary solution of Laplace's equation	495
subharmonic distributions	496
potentials	496
energy integral, Green's formula, unicity theorem	499
Liouville's theorem	500
boundary-value problems	502
Green function	503
introduction to hilbertian methods; generalized	
Dirichlet problem	505
hilbertian methods	507
example: Neumann problem	509
4. Parabolic equations	510
heat diffusion	510
5. Hyperbolic equation; wave equations	511
elementary solution of the wave equation	511
Cauchy problem	512
energy integral, unicity theorem	513
existence theorem	515
6. Leray theory of hyperbolic systems	516
7. Second order systems; propagators	522
 <i>Problems and Exercises</i>	525
Problem 1: Bounded distributions	525
Problem 2: Laplacian of a discontinuous function	527
Exercise 3: Regularized functions	528
Problem 4: Application to the Schrödinger equation	528
Exercise 5: Convolution and linear continuous responses	530
Problem 6: Fourier transforms of $\exp(-x^2)$ and $\exp(ix^2)$	531
Problem 7: Fourier transforms of Heaviside functions and $P_v(1/x)$	532
Problem 8: Dirac bitensors	533
Problem 9: Legendre condition	533
Problem 10: Hyperbolic equations; characteristics	534
Problem 11: Electromagnetic shock waves	535
Problem 12: Elementary solution of the wave equation	538
Problem 13: Elementary kernels of the harmonic oscillator	538
 VII. Differentiable Manifolds, Infinite Dimensional Case	543
A. Infinite-Dimensional Manifolds	543
1. Definitions and general properties	543
E-manifolds	543
differentiable functions	544
tangent vector	544
vector and tensor field	545
differential of a mapping	546