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Abstract

Motivated by a question of Vincent Lafforgue, we study the Banach spaces X satisfying the following property: there is a function $\varepsilon \to \Delta_X(\varepsilon)$ tending to zero with $\varepsilon > 0$ such that every operator $T\colon L_2 \to L_2$ with $\|T\| \le \varepsilon$ that is simultaneously contractive (i.e. of norm ≤ 1) on L_1 and on L_∞ must be of norm $\le \Delta_X(\varepsilon)$ on $L_2(X)$. We show that $\Delta_X(\varepsilon) \in O(\varepsilon^\alpha)$ for some $\alpha > 0$ iff X is isomorphic to a quotient of a subspace of an ultraproduct of θ-Hilbertian spaces for some $\theta > 0$ (see Corollary 6.7), where θ-Hilbertian is meant in a slightly more general sense than in our previous paper (1979). Let $B_r(L_2(\mu))$ be the space of all regular operators on $L_2(\mu)$. We are able to describe the complex interpolation space

$$(B_r(L_2(\mu)), B(L_2(\mu)))^\theta.$$

We show that $T\colon L_2(\mu) \to L_2(\mu)$ belongs to this space iff $T \otimes id_X$ is bounded on $L_2(X)$ for any θ-Hilbertian space X.

More generally, we are able to describe the spaces

$$(B(\ell_{p_0}), B(\ell_{p_1}))^\theta \text{ or } (B(L_{p_0}), B(L_{p_1}))^\theta$$

for any pair $1 \le p_0, p_1 \le \infty$ and $0 < \theta < 1$. In the same vein, given a locally compact Abelian group G, let $M(G)$ (resp. $PM(G)$) be the space of complex measures (resp. pseudo-measures) on G equipped with the usual norm $\|\mu\|_{M(G)} = |\mu|(G)$ (resp.

$$\|\mu\|_{PM(G)} = \sup\{|\hat{\mu}(\gamma)| \mid \gamma \in \widehat{G}\}).$$

We describe similarly the interpolation space $(M(G), PM(G))^\theta$. Various extensions and variants of this result will be given, e.g. to Schur multipliers on $B(\ell_2)$ and to operator spaces.

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Introduction

This paper is a contribution to the study of the complex interpolation method. The latter originates in 1927 with the famous Marcel Riesz theorem which says that, if $1 \le p_0 < p_1 \le \infty$, and if (a_{ij}) is a matrix of norm ≤ 1 simultaneously on $\ell_{p_0}^n$ and $\ell_{p_1}^n$, then it must be also of norm ≤ 1 on ℓ_p^n for any $p_0 < p < p_1$, and similarly for operators on L_p-spaces. Later on in 1938, Thorin found the most general form using a complex variable method; see **[2]** for more on this history.

Then around 1960, J.L. Lions and independently A. Calderón **[12]** invented the complex interpolation method, which may be viewed as a far reaching "abstract" version of the Riesz–Thorin theorem, see **[2, 36]**. There the pair (L_{p_0}, L_{p_1}) can be replaced by a pair (B_0, B_1) of Banach spaces (assumed compatible in a suitable way). One then defines for any $0 < \theta < 1$ the complex interpolation space $B_\theta = (B_0, B_1)_\theta$ which appears as a continuous deformation of B_0 into B_1 when θ varies from 0 to 1. In many ways, the unit ball $\mathcal{B}_\theta$ of the space B_θ looks like the "geometric mean" of the respective unit balls $\mathcal{B}_0$ and $\mathcal{B}_1$ of B_0 and B_1, i.e. it seems to be the multiplicative analogue of the Minkowski sum $(1-\theta)\mathcal{B}_0 + \theta\mathcal{B}_1$. The main result of this paper relates directly to the sources of interpolation theory: we give a description of the space $B_\theta = (B_0, B_1)_\theta$ when $B_0 = B(\ell_{p_0}^n)$ and $B_1 = B(\ell_{p_1}^n)$, or more generally for the pair $B_0 = B(L_{p_0}(\mu))$, $B_1 = B(L_{p_1}(\mu))$.

Although our description of the norm of B_θ for these pairs is, admittedly, rather "abstract" it shows that the problem of calculating B_θ is equivalent to the determination of a certain class of Banach spaces

$$(SQ(p_0), SQ(p_1))_\theta$$

roughly interpolated between the classes $SQ(p_0)$ and $SQ(p_1)$ where $SQ(p)$ denotes the class of subspaces of quotients (subquotients in short) of L_p-spaces.

When $p_0 = 1$ or $= \infty$, this class $SQ(p_0)$ is the class of all Banach spaces while when $p_1 = 2$, $SQ(p_1)$ is the class of all Hilbert spaces. In that case, the class $(SQ(p_0), SQ(p_1))_\theta$ is the class of all the Banach spaces B which can be written (isometrically) as $B = (B_0, B_1)_\theta$ for some compatible pair (B_0, B_1) with

$$B_j \in SQ(p_j) \qquad (j = 0, 1).$$

We already considered this notion in a previous paper **[58]**. There we called θ-Hilbertian the resulting spaces. However, in the present context we need to slightly extend the notion of θ-Hilbertian, so we decided to rename "strictly θ-Hilbertian" the spaces called θ-Hilbertian in **[58]**. In our new notion of "θ-Hilbertian" we found it necessary to use the complex interpolation method for "families" $\{B_z \mid z \in \partial D\}$ defined on the boundary of a complex domain D and not only pairs of Banach spaces

This generalization was developed around 1980 in a series of papers mainly by Coifman, Cwikel, Rochberg, Sagher, Semmes and Weiss (cf. [**13, 14, 15, 73, 16**]). There ∂D can be the unit circle and, restricting to the n-dimensional case for simplicity, we may take $B_z = (\mathbb{C}^n, \| \ \|_z)$ where $\{\| \ \|_z \mid z \in \partial D\}$ is a measurable family of norms on $\mathbb{C}^n$ (with a suitable nondegeneracy). The interpolated spaces now consist in a family $\{B(\xi) \mid \xi \in D\}$ which extends the boundary data $\{B_z \mid z \in \partial D\}$ in a specific way reminiscent of the harmonic extension. When $B_z = \ell^n_{p(z)}$ with $1 \le p(z) \le \infty$ $(z \in \partial D)$ one finds $B(\xi) = \ell^n_{p(\xi)}$ where $p(\xi)$ is determined by

$$p(\xi)^{-1} = \int_{\partial D} p(z)^{-1} \mu_\xi(dz)$$

where μ_ξ is the harmonic (probability) measure of $\xi \in D$ relative to ∂D.

Consider then an $n \times n$ matrix $a = [a_{ij}]$, let $\beta_z = B(\ell^n_{p(z)})$ for $z \in \partial D$ and let $\beta(\xi)$ $(\xi \in D)$ be the resulting interpolation space.

One of our main results is the equality

$$\|a\|_{\beta(\xi)} = \sup\{\|a_X\colon \ \ell^n_{p(\xi)}(X) \to \ell^n_{p(\xi)}(X)\} \tag{0.1}$$

where a_X is the matrix $[a_{ij}]$ viewed as acting on X^n in the natural way and where the supremum runs over all the n-dimensional Banach spaces X in the class $\mathcal{C}(\xi)$. The class $\mathcal{C}(\xi)$ consists of all the spaces X which can be written as $X = X(\xi)$ for some (compatible) family $\{X(z) \mid z \in \partial D\}$ such that $X(z) \in SQ(p(z))$ for all z in ∂D and of all ultraproducts of such spaces.

By a sort of "duality," this also provides us with a characterization of this class $\mathcal{C}(\xi)$, or more precisely of the class of subspaces of quotients of spaces in $\mathcal{C}(\xi)$: a Banach space X belongs to the latter class (resp. is C-isomorphic to a space in that class) iff for any n

$$\sup_{\|a\|_{\beta(\xi)} \le 1} \|a_X\colon \ \ell^n_{p(\xi)}(X) \to \ell^n_{p(\xi)}(X)\| \le 1 \quad (\text{resp. } \le C). \tag{0.2}$$

Consider for example the case when $p(z)$ takes only two values $p(z) = 1$ and $p(z) = 2$ with measure respectively $1-\theta$ and θ. Then $\beta(0) = (B(\ell^n_1), B(\ell^n_2))_\theta$ and $\mathcal{C}(0)$ is the class of all the spaces which can be written as $X(0)$ for some boundary data $\partial D \ni z \longmapsto X(z)$ such that $X(z)$ is Hilbertian on a subset of normalized Haar measure $\ge \theta$ (and is Banach on the complement). We call these spaces θ-Euclidean and we call θ-Hilbertian all ultraproducts of θ-Euclidean spaces of arbitrary dimension.

Actually, our result can be formulated in a more general framework: we give ourselves classes of Banach spaces $\{\mathcal{C}(z) \mid z \in \partial D\}$ with minimal assumptions and we set by definition

$$\|a\|_{\beta(z)} = \sup_{X \in \mathcal{C}(z)} \|a_X\colon \ \ell^n_{p(z)}(X) \to \ell^n_{p(z)}(X)\|.$$

Then (0.1) and (0.2) remain true with $\mathcal{C}(z)$ in the place of $SQ(p(z))$. In particular, we may now restrict to the case when $p(z) = 2$ for all z in ∂D. Consider for instance the case when $\mathcal{C}(z) = \ell^n_2$ for z in a subset (say an arc) of ∂D of normalized Haar measure θ and let $\mathcal{C}(z)$ be the class of all n-dimensional Banach spaces on the complement. Then (0.1) yields a description of the space $(B_0, B_1)_\theta$ when B_0, B_1 is the following pair of normed spaces consisting of $n \times n$ matrices:

$$\|a\|_{B_0} = \|[|a_{ij}|]\|_{B(\ell_2^n)}$$
$$\|a\|_{B_1} = \|[a_{ij}]\|_{B(\ell_2^n)}.$$

More generally, if $B_1 = B(L_2(\mu))$ and if B_0 is the Banach space $B_r(L_2(\mu))$ of all regular operators T on $L_2(\mu)$ (i.e. those T with a kernel $(T(s,t))$ such that $|T(s,t)|$ is bounded on $L_2(\mu)$), then we are able to describe the space $(B_r(L_2(\mu)), B(L_2(\mu)))^\theta$. By [**1**] this also yields $(B_0, B_1)_\theta$ as the closure of $B_0 \cap B_1$ in $(B_0, B_1)^\theta$.

The origin of this paper is a question raised by Vincent Lafforgue: what are the Banach spaces X satisfying the following property: there is a function $\varepsilon \to \Delta_X(\varepsilon)$ tending to zero with $\varepsilon > 0$ such that every operator $T\colon\ L_2 \to L_2$ with $\|T\| \le \varepsilon$ that is simultaneously contractive (i.e. of norm ≤ 1) on L_1 and on L_∞ must be of norm $\le \Delta_X(\varepsilon)$ on $L_2(X)$?

We show that $\Delta_X(\varepsilon) \in O(\varepsilon^\alpha)$ for some $\alpha > 0$ iff X is isomorphic to a subspace of a quotient of a θ-Hilbertian space for some $\theta > 0$ (see Corollary 6.7). We also give a sort of structural, but less satisfactory, characterization of the spaces X such that $\Delta_X(\varepsilon) \to 0$ when $\varepsilon \to 0$ (see Theorem 9.2).

V. Lafforgue's question is motivated by the (still open) problem whether expanding graphs can be coarsely embedded into uniformly convex Banach spaces; he observed that such an embedding is impossible into X if $\Delta_X(\varepsilon) \to 0$ when $\varepsilon \to 0$. See §3 for more on this.

The preceding results all have analogues in the recently developed theory of operator spaces ([**18, 66**]). Indeed, the author previously introduced and studied mainly in [**64, 63**] all the necessary ingredients, notably complex interpolation and operator space valued non-commutative L_p-spaces. With these tools, it is an easy task to check the generalized statements, so that we merely review them, giving only indications of proofs. In addition, in the last section, we include an example hopefully demonstrating that interpolation of *families* (i.e. involving more than a pair) of operator spaces, appears very naturally in harmonic analysis on the free group.

Let us now describe the contents, section by section. In §1, we review some background on regular operators. An operator T on $L_p(\mu)$ is called regular if there is a positive operator S, still bounded on $L_p(\mu)$, such that

$$\forall f \in L_p(\mu) \qquad |Tf| \le S(|f|).$$

The regular norm of T is equal to the infimum of $\|S\|$. These operators can be characterized in many ways. They play an extremal role in Banach space valued analysis because they are precisely the operators on $L_p(\mu)$ that extend (with the same norm) to $L_p(\mu; X)$ for any Banach space X.

In §2 we use the fact that regular operators on $L_p(\mu)$ $(1 < p < \infty)$ with regular norm ≤ 1 are closely related (up to a change of density) to what we call fully contractive operators, i.e. operators that are of norm ≤ 1 simultaneously on L_1 and L_∞.

This allows us to rewrite the definition of $\Delta_X(\varepsilon)$ in terms of regular operators.

In §4, we describe a certain duality between, on one hand, classes of Banach spaces, and on the other one, classes of operators on L_p. Although these ideas already appeared (cf. [**40, 41, 29, 35**]), the viewpoint we emphasize was left sort of implicit. We hope to stimulate further research on the list of related questions that we present in this section.

In §5, we present background on the complex interpolation method for families (or "fields") of Banach spaces. This was developed mainly by Coifman, Cwikel, Rochberg, Sagher, Semmes, and Weiss cf. [**13, 14, 15, 16, 73**].

In §6, we generalize the notion of θ-Hilbertian Banach space from our previous paper [**58**]. We first call θ-Euclidean any n-dimensional space which can be obtained as the interpolation space at the center of the unit disc D associated to a family of n-dimensional spaces $\{X(z) \mid z \in \partial D\}$ such that $X(z)$ is Hilbertian for a set of z with (Lebesgue) measure $\geq \theta$. Then we call θ-Hilbertian all ultraproducts of θ-Euclidean spaces. In our previous definition (now called strictly θ-Hilbertian), we only considered a two-valued family $\{X(z) \mid z \in \partial D\}$. We are then able to describe the interpolation space

$$(B_r, B)^\theta$$

where B_r and B denote respectively the regular and the bounded operators on ℓ_2. We then characterize the Banach spaces X such that $\Delta_X(\varepsilon) \in O(\varepsilon^\alpha)$ for some $\alpha > 0$ as the subspaces of quotients of θ-Hilbertian spaces.

In §7, we briefly compare our notion of θ-Hilbertian with the corresponding "arcwise" one, where the set of z's for which $X(z)$ is Hilbertian is required to be an arc.

In §8, we turn to Fourier and Schur multipliers: we can describe analogously the complex interpolation spaces $(B_0, B_1)^\theta$ when B_0 (resp. B_1) is the space of measures (resp. pseudo-measures) on a locally compact Abelian group G (and similarly on an amenable group). We also treat the case when B_0 (resp. B_1) is the class of bounded Schur multipliers on $B(\ell_2)$ (resp. on the Hilbert–Schmidt class S_2 on ℓ_2). In the latter case, B_1 can be identified with the space of bounded functions on $\mathbb{N} \times \mathbb{N}$.

In §9, we give a characterization of "uniformly curved" spaces, i.e. the Banach spaces X such that $\Delta_X(\varepsilon) \to 0$ when $\varepsilon \to 0$. This appears as a real interpolation result, but is less satisfactory than in the case $\Delta_X(\varepsilon) \in O(\varepsilon^\alpha)$ for some $\alpha > 0$ and many natural questions remain open.

In §10, we generalize an extension property of regular operators from [**61**] which may be of independent interest. See [**46**] for related questions. This result will probably be relevant if one tries, in analogy with [**41**], to characterize the subspaces or the complemented subspaces of θ-Hilbertian spaces. In particular we could not distinguish any of the two latter classes from that of subquotients of θ-Hilbertian spaces. The paper [**21**] contains useful related information. We should mention that an extension property similar to ours appears in [**35**, 1.3.2].

In §11, we describe the complex interpolation spaces $(B_0, B_1)^\theta$ when $B_0 = B(L_{p_0}(\mu))$ and $B_1 = B(L_{p_1}(\mu))$ with $1 \leq p_0, p_1 \leq \infty$. Actually, the right framework seems to be here again the interpolation of families $\{B_z \mid z \in \partial D\}$ where $B_z = B(\ell^n_{p(z)})$. We treat this case and an even more general one related to the "duality" discussed in §3, see Theorem 11.1 for the most general statement.

In §12 and §13, we turn to the analogues of the preceding results in the operator space framework. There operators on $L_p(\mu)$ are replaced by mappings acting on "non-commutative" L_p-spaces associated to a trace. The main results are entirely parallel to the ones obtained in §6 and §11 in the commutative case.

Lastly, in §14, we describe a family of operator spaces closely connected to various works on the "non-commutative Khintchine inequalities" for homogeneous polynomials of degree d (see e.g. [**54**]). Here we specifically need to consider a family $\{X(z) \mid z \in \partial D\}$ taking $(d+1)$-values but we are able to compute precisely the interpolation at the center of D (or at any point inside D).

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CHAPTER 1

Preliminaries. Regular operators

Let $1 \le p < \infty$ throughout this section. For operators on L_p it is well known that the notions of "regular" and "order bounded" coincide, so we will simply use the term regular. We refer to [**50, 72**] for general facts on this. The results of this section are all essentially well known, we only recall a few short proofs for the reader's convenience and to place them in the context that is relevant for us.

1.1. We say that an operator $T\colon\ L_p(\mu) \to L_p(\nu)$ is regular if there is a constant C such that for all n and all $x_1, \ldots, x_n$ in $L_p(\mu)$ we have

$$\| \sup |Tx_k| \ \|_p \le C \| \sup |x_k| \ \|_p.$$

We denote by $\|T\|_{\text{reg}}$ the smallest C for which this holds and by $B_r(L_p(\mu), L_p(\nu))$ (or simply $B_r(L_p(\mu))$ if $\mu = \nu$) the Banach space of all such operators equipped with the norm $\| \ \|_{\text{reg}}$.

Clearly this definition makes sense more generally for operators $T\colon\ \Lambda_1 \to \Lambda_2$ between two Banach lattices Λ_1, Λ_2.

1.2. It is known that $T\colon\ L_p(\mu) \to L_p(\nu)$ is regular iff $T \otimes id_X\colon\ L_p(\mu; X) \to L_p(\nu; X)$ is bounded for *any* Banach space X and

$$\|T\|_{\text{reg}} = \sup_X \|T \otimes id_X\colon\ L_p(\mu; X) \to L_p(\nu; X)\|. \tag{1.1}$$

This assertion follows from the fact that any finite dimensional subspace $Y \subset X$ can be embedded almost isometrically into ℓ_∞^n for some large enough n. See 1.7 below. The preceding definition corresponds to ℓ_∞^n for all n, or equivalently to $X = c_0$.

Actually, $T\colon\ L_p(\mu) \to L_p(\nu)$ is regular iff there is a constant C such that for all n and all $x_1, \ldots, x_n$ in $L_p(\mu)$ we have

$$\| \sum |Tx_k| \ \|_p \le C \| \sum |x_k| \ \|_p,$$

and the smallest such C is equal to $\|T\|_{\text{reg}}$. This follows from the fact that any finite dimensional space X is almost isometric to a quotient of ℓ_1^n for some large enough n.

1.3. A (bounded) positive (meaning positivity preserving) operator T is regular and $\|T\|_{\text{reg}} = \|T\|$. More precisely, it is a classical fact that T is regular iff there is a bounded positive operator $S\colon\ L_p(\mu) \to L_p(\nu)$ (here $1 \le p < \infty$) such that $|T(x)| \le S(|x|)$ for any x in $L_p(\mu)$. Moreover, there is a smallest S with this property, denoted by $|T|$, and we have:

$$\|T\|_{\text{reg}} = \| \ |T| \ \|.$$

In case $L_p(\mu) = L_p(\nu) = \ell_p$, the operator T can be described by a matrix $T = [t_{ij}]$. Then

$$|T| = [|t_{ij}|].$$

Similarly, if T is given by a nice kernel $(K(s,t))$ then $|T|$ corresponds to the kernel $(|K(s,t)|)$.

1.4. In this context, although we will not use this, we should probably mention the following identities (see [**58**]) that are closely related to Schur's criterion for boundedness of a matrix on ℓ_2 and its (less well known) converse:

$$(B(\ell_1^n), B(\ell_\infty^n))_\theta = B_r(\ell_p^n, \ell_p^n)$$
$$(B(\ell_1), B(c_0))^\theta = B_r(\ell_p, \ell_p).$$

These are isometric isomorphisms with p defined as usual by $p^{-1} = (1-\theta)$.

More explicitly, a matrix $b = (b_{ij})$ is in the unit ball of $B_r(\ell_p^n)$ iff there are matrices b^0 and b^1 satisfying

$$|b_{ij}| \le |b_{ij}^0|^{1-\theta} |b_{ij}^1|^\theta$$

and such that

$$\sup_i \sum\nolimits_j |b_{ij}^0| \le 1 \quad \text{and} \quad \sup_j \sum\nolimits_i |b_{ij}^1| \le 1.$$

The "if" direction boils down to Schur's well known classical criterion when $p = 2$ (see also [**38**]).

1.5. We will now describe the unit ball of the dual of $B_r(\ell_2^n)$.

LEMMA 1.1. *Consider an $n \times n$ matrix $\varphi = (\varphi_{ij})$. Then*

$$\|\varphi\|_{B_r(\ell_2^n)^*} = \inf \left\{ \left(\sum\nolimits_1^n |x_i|^2 \sum\nolimits_1^n |y_j|^2 \right)^{1/2} \right\} \tag{1.2}$$

where the infimum runs over all x, y in ℓ_2^n such that

$$\forall i,j \qquad |\varphi_{ij}| \le |x_i|\ |y_j|.$$

PROOF. Let C be the set of all φ for which there are x, y in the unit ball of ℓ_2^n such that $|\varphi_{ij}| \le |x_i|\ |y_j|$. Clearly we have for all a in $B(\ell_2^n)$

$$\|a\|_{B_r(\ell_2^n)} = \|[|a_{ij}|]\| = \sup_{\varphi \in C} \left| \sum \varphi_{ij} a_{ij} \right|.$$

Therefore, to prove the Lemma it suffices to check that C is convex (since the right-hand side of (1.2) is the gauge of C). This is easy to check: consider φ, φ' in C and $0 < \theta < 1$ then assuming

$$|\varphi_{ij}| \le |x_i|\ |y_j| \quad \text{and} \quad |\varphi'_{ij}| \le |x'_i|\ |y'_j|$$

with x, y, x', y' all in the Euclidean unit ball, we have by Cauchy-Schwarz

$$|(1-\theta)\varphi_{ij} + \theta\varphi'_{ij}| \le ((1-\theta)|x_i|^2 + \theta|x'_i|^2)^{1/2}((1-\theta)|y_j|^2 + \theta|y'_j|^2)^{1/2},$$

which shows that $(1-\theta)\varphi + \theta\varphi'$ is in C. □

Let C be as above. Then $\varphi \in C$ iff there are h_i, k_j in $\mathbb{C}^n$ such that $\varphi_{ij} = \langle h_i, k_j\rangle$ and

$$\sum \|h_i\|^2_{\ell_1^n} \le 1, \qquad \sum \|k_j\|^2_{\ell_\infty^n} \le 1.$$

Indeed, if this holds we can write

$$|\varphi_{ij}| \le \sum\nolimits_m |h_i(m)|\ |k_j(m)| \le \|h_i\|_{\ell_1^n} \|k_j\|_{\ell_\infty^n}$$

from which $\varphi \in C$ follows. Conversely, if $\varphi \in C$, we may assume $\varphi_{ij} = x_i y_j \gamma_{ij}$ with $|\gamma_{ij}| \le 1$, $\|x\|_2 \le 1$, $\|y\|_2 \le 1$. Let (e_m) denote the canonical basis of $\mathbb{C}^n$. Then, letting

$$h_i = x_i e_i \quad \text{and} \quad k_j = y_j \quad \sum\nolimits_m \gamma_{mj} e_m$$

we obtain the desired representation.

1.6. The predual of $B(L_2(\mu), L_2(\mu'))$ is classically identified with the projective tensor product $L_2(\mu)\widehat{\otimes}L_2(\mu')$, i.e. the completion of the algebraic tensor product $L_2(\mu) \otimes L_2(\mu')$ with respect to the norm

$$\|T\|_\wedge = \inf \sum \|x_m\| \|y_m\|$$

where the infimum runs over all representations of T as a sum $T = \Sigma x_m \otimes y_m$ of rank one tensors. Let $T(s,t) = \Sigma x_m(s) y_m(t)$ be the corresponding kernel in $L_2(\mu \times \mu')$. An easy verification shows that

$$\|T\|_\wedge = \inf\{\|h\|_{L_2(\ell_2)} \|k\|_{L_2(\ell_2)}\}$$

where the infimum runs over all h, k in $L_2(\ell_2)$ such that $T(s,t) = \langle h(s), k(t)\rangle$.

We now describe a predual of $B_r(L_2(\mu), L_2(\mu'))$. For any T in $L_2(\mu) \otimes L_2(\mu')$, let

$$N_r(T) = \inf\{\|x\|_2 \|y\|_2\} \tag{1.3}$$

where the infimum runs over all x in $L_2(\mu)$ and all y in $L_2(\mu')$ such that

$$|T(s,t)| \le x(s) y(t)$$

for almost all s, t. Equivalently, we have

$$N_r(T) = \inf\left\{\left\|\sum\nolimits_1^n |h_i|\right\|_2 \|\sup |k_j|\|_2\right\} = \inf\{\|h\|_{L_2(\ell_1^n)} \|k\|_{L_2(\ell_\infty^n)}\} \tag{1.4}$$

where the infimum runs over all n and all $h = (h_1, \dots, h_n)$ $k = (k_1, \dots, k_n)$ in $(L_2)^n$ such that

$$T(s,t) = \sum\nolimits_1^n h_i(s) k_i(t). \tag{1.5}$$

Indeed, it is easy to show that the right-hand sides of both (1.3) and (1.4) are convex functions of T and moreover (recalling 1.1, 1.2 and 1.3) that for any b in $B_r(L_2(\mu), L_2(\mu'))$

$$\|b\|_{\text{reg}} = \sup\{|\langle b, T\rangle|\}$$

where the supremum runs over T such that the right-hand side of either (1.3) or (1.4) is ≤ 1. This implies that (1.3) and (1.4) are equal. Let $L_2(\mu)\widehat{\otimes}_r L_2(\mu')$ be the completion of $L_2(\mu) \otimes L_2(\mu')$ with respect to this norm. Then there is an isometric isomorphism

$$(L_2(\mu)\widehat{\otimes}_r L_2(\mu'))^* \simeq B_r(L_2(\mu), L_2(\mu'))$$

associated to the duality pairing

$$\forall b \in B_r(L_2(\mu), L_2(\mu')) \qquad \langle b, x \otimes y\rangle = \langle b(x), y\rangle$$

1.7. More generally, a predual of $B_r(L_p(\mu), L_p(\mu'))$ can be obtained as the completion of $L_{p'}(\mu) \otimes L_p(\mu')$ for the norm

$$(1.6) \qquad \forall T \in L_{p'}(\mu') \otimes L_p(\mu) \qquad N_r(T) = \inf\left\{\left\|\sum\nolimits_1^n |f_i|\right\|_{p'} \|\sup_{i\leq n} |g_i|\|_p\right\}$$

where the supremum runs over all decompositions of the kernel of T as $T(s,t) = \sum_1^n f_i(s)g_i(t)$. To verify that (1.6) is indeed a norm, we will first show that (1.6) coincides with

$$(1.7) \qquad M_r(T) = \inf\{\|\xi\|_{L_{p'}(Y^*)}\|\eta\|_{L_p(Y)}\}$$

where the infimum runs over all finite dimensional normed spaces Y and all pairs $(\xi, \eta) \in L_{p'}(\mu'; Y^*) \times L_p(\mu, Y)$ such that $T(s,t) = \langle \xi(s), \eta(t)\rangle$.

Clearly $M_r(T) \leq N_r(T)$. Conversely, given Y as in (1.7), for any $\varepsilon > 0$ there is n and an embedding $j\colon\ Y \to \ell_\infty^n$ such that $\|y\| \leq \|j(y)\| < (1+\varepsilon)\|y\|$ for all y in Y. Let (ξ, η) be as in (1.7). Let $\hat\eta = j\eta \in L_p(\ell_\infty^n)$. Note $\|\hat\eta\| \leq (1+\varepsilon)\|\eta\|$. Let $q = j^*\colon\ \ell_1^n \to Y^*$ be the corresponding surjection. By an elementary lifting, there is $\hat\xi$ in $L_{p'}(\ell_1^n)$ with $\|\hat\xi\| \leq (1+\varepsilon)\|\xi\|$ such that $\xi = q\hat\xi$.

We have then $T(s,t) = \langle \xi(s), \eta(t)\rangle = \langle q\hat\xi(s), \eta(t)\rangle = \langle \hat\xi(s), \hat\eta(t)\rangle$ and $\|\hat\xi\|_{L_{p'}(\ell_1^n)}\|\hat\eta\|_{L_p(\ell_\infty^n)} \leq (1+\varepsilon)^2\|\xi\|_{L_{p'}(Y^*)}\|\eta\|_{L_p(Y)}$. Thus we conclude that $M_r(T) \leq N_r(T)$ and hence $M_r(T) = N_r(T)$.

To check that N_r is a norm, we will prove it for M_r. This is very easy. Consider T_1, T_2 with $M_r(T_j) < 1$, $(j = 1,2)$ and let $0 \leq \theta \leq 1$. We can write

$$T_1(s,t) = \langle \xi_1(s), \eta_1(t)\rangle$$

$$T_2(s,t) = \langle \xi_2(s), \eta_2(t)\rangle$$

with $(\xi_j, \eta_j) \in L_p(Y_j) \times L_{p'}(Y_j^*)$. Then

$$(1-\theta)T_1(s,t) + \theta T_2(s,t) = \langle \xi(s), \eta(t)\rangle$$

where $(\xi, \eta) \in L_p(Y) \times L_{p'}(Y^*)$ with

$$Y = Y_1 \oplus_p Y_2 \qquad\qquad Y^* = Y_1^* \oplus_{p'} Y_2^*$$

$$\xi = ((1-\theta)^{1/p}\xi_1 \oplus \theta^{1/p}\xi_2), \qquad \eta = ((1-\theta)^{1/p'}\eta_1 \oplus \theta^{1/p'}\eta_2).$$

We conclude that

$$M_p(T_1 + T_2) \leq \|\xi\|_{L_p(Y)}\|\eta\|_{L_{p'}(Y^*)} \leq 1.$$

Now that we know that (1.6) is indeed a norm, it is clear (either by 1.1 or 1.2) that the completion $L_p(\mu)\widehat{\otimes}_r L_{p'}(\mu')$ of $(L_p(\mu) \otimes L_{p'}(\mu), N_r)$ is isometrically a predual of $B_r(L_p(\mu), L_{p'}(\mu'))$.

1.8. We refer e.g. to [**59**] for more information and references on all this subsection (see also [**18, 66**] for the operator space analogue). The original ideas can be traced back to [**23**].

An operator $v\colon\ E \to F$ between Banach spaces is called nuclear if it can be written as an absolutely convergent series of rank one operators, i.e. there are $x_n^* \in E^*$, $y_n \in F$ with $\sum \|x_n^*\|\|y_n\| < \infty$ such that

$$v(x) = \sum \langle x_n^*, x\rangle y_n \quad \forall x \in E.$$

The space of such maps is denoted by $N(E,F)$. The nuclear norm $N(v)$ is defined as

$$N(v) = \inf \sum \|x_n^*\| \|y_n\|,$$

where the infimum runs over all possible such representations of v. Equipped with this norm, $N(E,F)$ is a Banach space.

If E and F are finite dimensional, it is well known that we have isometric identities

$$B(E,F)^* = N(F,E) \quad \text{and} \quad N(E,F)^* = B(F,E)$$

with respect to the duality defined for $u: E \to F$ and $v: F \to E$ by

$$\langle u, v\rangle = \mathrm{tr}(uv).$$

We will denote by $\Gamma_H(E,F)$ the set of operators $u\colon E \to F$ that factorize through a Hilbert space, i.e. there are bounded operators $u_1\colon H \to F$, $u_2\colon E \to H$ such that $u = u_1 u_2$. We equip this space with the norm $\gamma_H(.)$ defined by

$$\gamma_H(u) = \inf\{\|u_1\| \; \|u_2\|\}$$

where the infimum runs over all such factorizations.

We will denote by $\gamma_H^*(.)$ the norm that is dual to $\gamma_H(.)$ in the above duality, i.e. for all $v\colon F \to E$ we set

$$\gamma_H^*(v) = \sup\{|\mathrm{tr}(uv)| \mid u \in \Gamma_H(E,F), \gamma_H(u) \le 1.\}$$

PROPOSITION 1.2. *Consider $v\colon \ell_\infty^n \to \ell_1^n$.*

(i) *$\gamma_H^*(v) \le 1$ iff there are λ, μ in the unit ball of ℓ_2^n and (a_{ij}) in the unit ball of $B(\ell_2^n)$ such that $v_{ij} = \lambda_i a_{ij} \mu_j$.*
(ii) *$N(v) \le 1$ iff there are λ', μ' in the unit ball of ℓ_2^n and (b_{ij}) in the unit ball of $B_r(\ell_2^n)$ such that $v_{ij} = \lambda_i' b_{ij} \mu_j'$.*

PROOF. (i) is a classical fact (cf. e.g. [**65**, Prop. 5.4]). To verify (ii), note that $N(v) = \sum\limits_{ij} |v_{ij}|$. Assume $N(v) = 1$. Let then $\lambda_i' = (\sum_j |v_{ij}|)^{1/2}$ and $\mu_j' = \left(\sum\limits_i |v_{ij}|\right)^{1/2}$ and $b_{ij} = v_{ij}(\lambda_i'\mu_j')^{-1}$. We have then (with the convention $\frac{0}{0} = 0$)

$$|b_{ij}| \le |b_{ij}^0|^{1/2} |b_{ij}^1|^{1/2}$$

with $b_{ij}^0 = |v_{ij}|(\sum_j |v_{ij}|)^{-1}$ and $b_{ij}^1 = |v_{ij}|(\sum_i |v_{ij}|)^{-1}$. Since $\sup\limits_i \sum_j |b_{ij}^0| \le 1$ and $\sup\limits_j \sum_i |b_{ij}^1| \le 1$, by 1.4 we have $\|b\|_{\mathrm{reg}} \le 1$. □

PROPOSITION 1.3. *Consider $\varphi\colon \ell_2^n \to \ell_2^n$.*

(i) *$\|\varphi\|_{B(\ell_2^n)^*} \le 1$ iff there are λ, μ in the unit ball of ℓ_2^n and $v\colon \ell_1^n \to \ell_\infty^n$ with $\gamma_H(v) \le 1$ such that $\varphi_{ij} = \lambda_i v_{ij} \mu_j$ for all i, j.*
(ii) *$\|\varphi\|_{B_r(\ell_2^n)^*} \le 1$ iff there are λ, μ in the unit ball of ℓ_2^n and $v\colon \ell_1^n \to \ell_\infty^n$ with $\|v\| \le 1$ such that $\varphi_{ij} = \lambda_i v_{ij} \mu_j$ for all i, j.*

PROOF. (i) If φ factors as indicated we have $v = v_1 v_2$ with $v_1\colon H \to \ell_\infty^n$ and $v_2\colon \ell_1^n \to H$ such that $\|v_1\|\|v_2\| \le 1$. Let D_λ and D_μ denote the diagonal operators with coefficients (λ_i) and (μ_j). We have then $\varphi = D_\lambda v_1 v_2 D_\mu$, hence using the Hilbert–Schmidt norm $\|\cdot\|_{HS}$ we find that $\|\varphi\|_{B(\ell_2^n)^*}$ (which is the trace class norm of φ) is $\le \|D_\lambda v_1\|_{HS} \|v_2 D_\mu\|_{HS} \le 1$. Conversely, if the trace class norm of φ is ≤ 1, then for some H Hilbert (actually $H = \ell_2^n$) we can write $\varphi = \varphi_1\varphi_2$, $\varphi_2\colon \ell_2^n \to H$

and $\varphi_1\colon\ H \to \ell_2^n$ such that $\|\varphi_1\|_{HS}\|\varphi_2\|_{HS} \le 1$. Let $v_2\colon\ \ell_1^n \to H$ and $v_1\colon\ H \to \ell_\infty^n$ be the maps defined by $v_2 e_j = (\varphi_2 e_j)\|\varphi_2 e_j\|^{-1}$ and $v_1^* e_i = (\varphi_1^* e_i)\|\varphi_1^* e_i\|^{-1}$. Note that $\|v_1\| = \|v_2\| = 1$. Let $v = v_1 v_2$ and $\lambda_i = \|\varphi_1^* e_i\|$, $\mu_j = \|\varphi_2 e_j\|$. We have then $\|v\| \le 1$ and $\varphi_{ij} = \langle \varphi e_j, e_i\rangle = \lambda_i v_{ij} \mu_j$, which verifies (i).

(ii) By Lemma 1.1, $\|\varphi\|_{B_r(\ell_2^n)^*} \le 1$ iff there are λ, μ in the unit ball of ℓ_2^n and $v\colon\ \ell_1^n \to \ell_\infty^n$ with $\|v\| = \sup_{ij} |v_{ij}| \le 1$ such that $\varphi_{ij} = \lambda_i v_{ij} \mu_j$. □

1.9. In the sequel, we will invoke several times "a measurable selection argument." Each time, the following well known fact will be sufficient for our purposes. Consider a continuous surjection $f\colon\ K \to L$ from a compact metric space K onto another one L. Then there is a Borel measurable map $g\colon\ L \to K$ lifting f, i.e. such that $f \circ g$ is the identity on L. This (now folkloric) fact essentially goes back to von Neumann. The references [**33**, p. 9] or [**76**, chap. 5] contain considerably more sophisticated results.

1.10. Throughout this memoir (at least until we reach §12), given an operator $T\colon\ L_p(\mu) \to L_p(\nu)$ such that $T \otimes id_X$ extends to a bounded operator from $L_p(\mu; X)$ to $L_p(\nu; X)$, we will denote for short by

$$T_X\colon\ L_p(\mu; X) \to L_p(\nu; X)$$

the resulting operator. In §12, this notation will be extended to the non-commutative setting.

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