Integer and Mixed Programming

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and linear values. For the latter, the storehouse is well stocked with algorithms, but the same does not yet apply for problems with integer values, although considerable progress has been made, especially during the past five years. The reason for this lies in the fact that diophantine mathematics contains combinatorial difficulties that do not occur with continuous values. This is a situation that cannot be altered, but considerable progress has nevertheless been made and some essential results are now available.

As all mathematicians concerned are aware, the subject of this volume is a mathematically difficult one, but we have endeavoured to balance the strictness of the theory with the instructional needs of our readers. Among the more useful methods of procedure are some very difficult algorithms such as Gomory's asymptotic algorithm as well as the methods of Benders and Trubin. These have been grouped in a Supplement, but they have still been given the same instructional presentation.

The largest category of programs and the one involving the greatest difficulties, that of nonlinear programs, will be treated in a fourth volume now in preparation. I have again asked A. Henry-Labordère to be my collaborator, while we have been joined by my friend and former pupil at Grenoble, M. D. Coster, who is currently a consultant in informatics and operations research. During recent years he has acquired a wide knowledge of these nonlinear problems.

Returning to the present volume, I would like to outline my attitude toward the publication of new material in the series MMOR (Methods and Models of Operations Research) as they are now known by a wide circle of engineers. Instead of bringing the volumes up to date with each new edition I have preferred to leave them as published and to publish the new material every five or six years in fresh works that will not render the earlier ones obsolete.

In writing these MMOR volumes we have often recalled one of the rules of St. Benoît: "Encourage the strong without discouraging the weak." By means of this precept each student can progress according to the individual's mental speed and available resources. What is needed is to progress, slowly and surely or quickly and dangerously, according to one's wishes and ability, as long as progress is made. It is not in human nature not to advance or to attempt, since this is reserved for the negligent and the idle, for those who do not wish to confer any benefit on their fellows but merely to live for themselves. The latter are those to whom I scathingly referred in one of my books[1] as "subhumans," and this is the lot of far too many who refuse to realize that self-improvement at all levels is the object of existence.

The conquest of knowledge and of mental, moral, and emotional equilibrium is the basic adventure of our species; and if in this respect it has

[1] A. Kaufmann and J. Pezé, "Des sous-hommes et des super-machines," Albin Michel.

PREFACE

In this third volume of Methods and Models of Operations Research our loyal readers will discover that the same organization has been adopted as in the first two volumes: a first part in which mathematics is subordinated to the practical aspects of the concepts to be studied and a second part devoted to the mathematical side of the various problems. This method of presentation in the earlier volumes has been widely welcomed, as shown by the numerous editions and by translations into a variety of languages.

This volume deals with integer programs and programs with mixed values and will complete a small library for engineers and specialist groups. Operations research is now a part of their equipment, but advances in this field take place every year and it is necessary that they should become acquainted with them.

For the present volume I have had the collaboration of my friend A. Henry-Labordère, Engineer in Arts and Manufacturing, Master of Science, and Ph.D. He is an engineer with a wide reputation in operations research, an advisor to a very important firm of European consultants, and has also taught mathematical programming at l'École Centrale des Arts et Manufactures in Paris for several years.[1] The latter experience has assisted him in presenting numerous sections of this work in an instructional form. We have shared the production between us, with the author of the previous volumes retaining the responsibility for its coordination.

Integer programming is a subject that is of ever-increasing interest to engineers, economists, and informaticians since problems with integer solutions occur in every field of science and technological research. Such problems are, as a rule, appreciably more difficult to solve than those with continuous

[1] At present, Dr. Henry-Labordère is teaching at l'École Nationale des Ponts et Chaussées in Paris.

something of the tortoise and the hare, its only real goal is that of self-mastery.

I wish to thank our friends: Hervé Thiriez, Professor at the Centre d'Enseignement Supérieur des Affaires at Jouy-en-Sosas, and Michel Gondran, Research Engineer with Électricité de France, who have taken meticulous care in rereading and finalizing the manuscript. We are additionally indebted to them for a number of constructive suggestions about the models and the proofs.

My son Alain has also had an important part in checking the manuscript and the proofs.

Finally, we wish to thank the editor and his collaborators for their usual care in the publication of this series, as well as the Director of the Collection, Professor Ad. André-Brunet who has always given me his sincere encouragement and support.

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CONTENTS

Part 1. METHODS AND MODELS

Chapter I. PROGRAMS WITH INTEGER AND MIXED VALUES

Section 1. Introduction

In this chapter we shall consider such practical problems as can be expressed in the form of mathematical programs, which are similar to those of linear programming as discussed in the first volume,[1] except for the requirement that the variables must be integers such as 0, 1, 2, 3, ... The reader will already have been convinced as to the practical importance of problems defined by linear programs. In operations research and econometrics we are often aware that the choices are discrete, in other words, that they can only assume definite and not closely contiguous values, that this or that has to be done, a factory has to be built or not built. Consequently, for practical purposes, problems of linear programming with integer solutions are of an even greater importance than the classic problems of linear programming. We shall see that choices for investment and problems for the engineer and even for the plumber can be expressed in this form.

It may well be asked, therefore, why the interest in programming with integers is so recent, dating from some fifteen years only, if it can be so widely applied. Paradoxically, discrete mathematics, which originated with the arithmetic of the Greeks and Arabs, has over recent centuries occupied the position of a poor relation in the field of research. From many points of the scientific spectrum, logic, algebra, operations research, information, humane sciences, and the arts, interest in them has awakened to such a degree that at a

[1] *Note to Reader*: Throughout the present work, Volume 1 refers to A. Kaufmann, "Methods and Models of Operations Research," Prentice-Hall, Englewood Cliffs, New Jersey, 1963.

recent congress of pure mathematics more than half the discussion was devoted to the subject of discrete mathematics. It is but recently that effective methods have been discovered for solving such problems; easy to formulate, they possess the disadvantage of extensive calculations, containing, as they do, numerous variables and constraints.

In this chapter we shall give practical cases that can be expressed as problems with integer variables. Brief statements about the main properties will be given, and methods will be outlined. In the second part of this work the reader will, as usual, find the requisite theoretical analyses. In particular, he will find those dealing with the problems of programs with mixed numbers in which some variables must be integers and others may be continuous, as in classic linear programming. We shall observe that the latter type of problem is specially important.

Section 2. **Some Examples of Problems with Integer Solutions**

1. Characteristics of Problems with Integer Solutions

Let us consider the set **S** containing the solutions of a linear program and let $[x] = [x_1, x_2, ..., x_n]$ be one of the solutions belonging to **S**. If we now impose the constraint that the components of $[x]$ must be natural numbers (integer and nonnegative) we can state that $[x]$ is an integer solution. Thus, in a case where $n = 5$,

(2.1) $$[x] = [x_1, x_2, x_3, x_4, x_5] = [3, 0, 1, 9, 0] ,$$

$[x]$ will be an integer solution. This will not be the case for

(2.2) $$[x] = [x_1, x_2, x_3, x_4, x_5] = [3 ,1.08, 0, 5.7, 1] ,$$

nor for

(2.3) $$[x] = [-1, 0, -3, 2, 9]$$

and

(2.4) $$[x] = [6, 1, 9/2, 2/3, 0] .$$

Let us examine a simple example of linear programming of which we will temporarily ignore the economic function to be optimized.

Let

(2.5) $$\begin{aligned} 6x_1 + 9x_2 &\leqslant 54 , \\ 7x_1 + 6x_2 &\leqslant 42 , \\ x_1 &\leqslant 4 , \\ x_1 &\geqslant 0 , \\ x_2 &\geqslant 0 . \end{aligned}$$

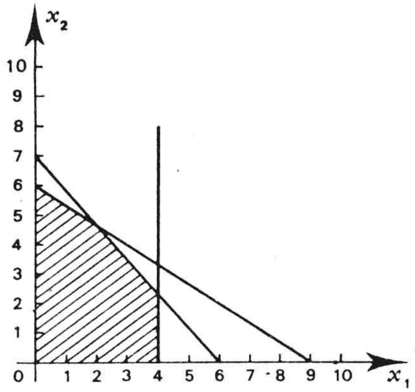

FIG. 2.1

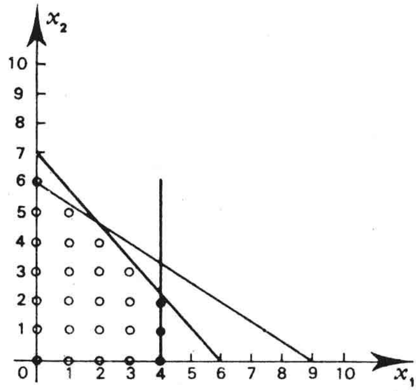

FIG. 2.2

The set **S** of the solutions of (2.5) is represented by the hachured area of Fig. 2.1. Let us now introduce the constraint of only accepting as solutions those of which the components x_1 and x_2 are nonnegative integers: the set Σ of the corresponding solutions is represented in Fig. 2.2.

This subset Σ of **S** consists of

$$\Sigma = \{[0, 0], [0, 1], [0, 2], [0, 3], [0, 4], [0, 5], [0, 6], [1, 0], [1, 1], [1, 2], [1, 3], [1, 4], [1, 5], [2, 0], [2, 1], [2, 2], [2, 3], [2, 4], [3, 0], [3, 1], [3, 2], [3, 3], [4, 0], [4, 1], [4, 2]\} . \tag{2.6}$$

Here the number of integer solutions is finite; in other cases it might be infinite.

Let us now suppose that the economic function of the linear program (2.5) is

$$[\text{MAX}]z = 7x_1 + 5x_2 . \tag{2.7}$$

From Fig. 2.3 it can be seen that the maximal solution of the linear program (2.5), (2.7) is

$$[x_1, x_2] = [4, 7/3] . \tag{2.8}$$

This is not an integer solution, but let us nevertheless calculate the corresponding value of z:

$$\begin{aligned} z &= (7).(4)+(5).(7/3) \\ &= 39\tfrac{2}{3} = 39.66\ldots \end{aligned} \tag{2.9}$$

Let us now impose the constraint that the solution of this program is to be integer. With the very simple problem that we are considering, it is sufficient to determine which will be the first point (or points) representing an integer value encountered after entering the polygon of solutions when the straight

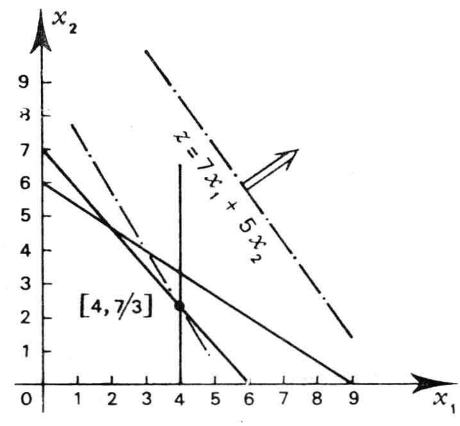

FIG. 2.3

line $7x_1 + 5x_2 = z$, has undergone a parallel displacement. It can be seen by inspecting Figs. 2.2 and 2.3 that this point will be

(2.10) $[x_1, x_2] = [4, 2]$,

for which we have

(2.11) $z = (7).(4)+(5).(2) = 38$.

The next point with integer values that we encounter is

(2.12) $[x_1, x_2] = [3, 3]$,

and for this we obtain

(2.13) $z = (7).(3)+(5).(3) = 36$.

It is advisable to clarify at once for the reader that the maximal solution with integer values is not always obtained by taking the maximal solution of the program for continuous values and by then suppressing the decimal portion of it. In this context, the reader should study the linear program represented in Fig. 2.4. The maximal solution of this program is [2.8; 4.3] and the maximal solution for integer values is not [2.4] or [3.4] but [3.3], as can be verified by sliding the straight line representing the function z parallel to itself. The same remark applies when we consider a minimal solution with integer values. This is not always obtained from the minimal solution for the corresponding program with continuous values. For example, if [3.17; 2.92] is the minimal solution of a given program, it is perfectly possible that neither [3.3] nor [4.3] is a minimal solution for integer values.

In addition, when the number of variables in the program exceeds two, it may prove very difficult to determine the solutions with integer values without enumerating and verifying all the solutions by means of the constraints. Such a process, useful as it may be for certain particular cases, is not generally

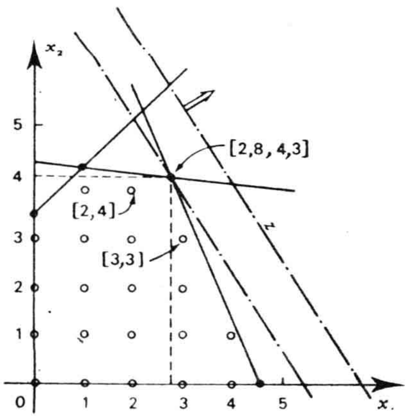

FIG. 2.4

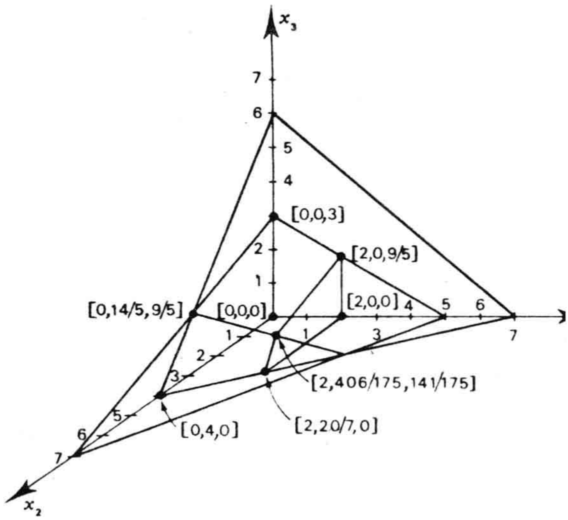

FIG. 2.5

practical because of the large number of integer solutions to be considered. Even in a program with three variables and three constraints (Fig. 2.5),

$$\frac{x_1}{7} + \frac{x_2}{4} + \frac{x_3}{6} \leqslant 1,$$

$$\frac{x_1}{5} + \frac{x_2}{7} + \frac{x_3}{3} \leqslant 1, \tag{2.14}$$

$$x_1 \leqslant 2,$$

$$x_1 \geqslant 0, \qquad x_2 \geqslant 0, \qquad x_3 \geqslant 0.$$

it is by no means easy to discover the integer solutions; to obtain the set that contains them, it is necessary to verify some thirty points.

Except for very simple problems, we are therefore obliged to make use of special algorithms for programs with integer values. The various principles underlying them will be very briefly discussed in the present chapter, and their fuller explanation and proofs will be given in the second part.

Let us, however, first consider some very simple examples.

2. Some Preliminary Examples

A Problem Dealing with the Transportation of School Children[1]

In a village A there is a school attended by some hundred children, 72 of whom live a certain distance away, whence the need to arrange their trans-

[1] This problem is given by Mlle. Edith Heurgon in her thesis, "Programming with integer numbers. Arborescent method of Robert Faure and Yves Malgrange." Faculté des Sciences de Paris, 1967. We have slightly modified the terms to satisfy the requirements of the present work.

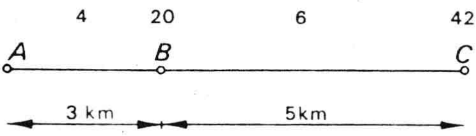

FIG. 2.6

portation by bus. There are two main collection points B and C (B being situated between A and C) (Fig. 2.6). The number of pupils to be collected is as follows: 42 at C, six between C and B, 20 at B, and four between B and A. The firm that can provide the transport owns two types of bus: one with 35 seats and another with 50 seats. The prices charged by the firm are as follows for each journey and for each kind of bus:

	Type of Bus	
	35 seats	*50 seats*
BA	39 F	50.50 F
CA	54 F	68 F
CB	45 F	57.50 F.

We must not be surprised that the proposed charges are not proportional to the distances, since the fixed costs of such an operation generally exceed the variable ones.

The problem is to decide which type of bus should be used on each of the sections in order to minimize the total outlay.

Let us use the following symbols for the variables representing the number of buses to be considered in each case:

	Buses	
	35 seats	*50 seats*
BA	x	x'
CA	y	y'
CB	z	z'

The linear program with integer numbers is easily obtained:

$$[\text{MIN}]f = 39x + 54y + 45z + 50.5x' + 68y' + 57.5z',$$

(2.15)
$$35y + 35z + 50y' + 50z' \geqslant 48\,,$$
$$35x + 35y + 50x' + 50y' \geqslant 72\,,$$
$$x \geqslant 0,\quad y \geqslant 0,\quad z \geqslant 0,\quad x' \geqslant 0,\quad y' \geqslant 0,\quad z' = 0\,.$$

The first line of the program (2.15) expresses the economic function, the total cost. The second line represents the constraint imposed by the different possibilities that the buses must provide when they start their collection of pupils at C, bring them to B and finally to A. The third line represents the buses that finish at A.

Resolved into continuous variables, the linear program (2.15) provides an optimal solution of

$$x = 0, \quad y = 0, \quad z = 0, \quad x' = 12/25, \quad y' = 24/25, \quad z' = 0, \quad \min f = 89.52. \tag{2.16}$$

Resolved into integer variables by means of one of the algorithms described in the second part, or by enumeration (which is easy in this case), we then obtain as the minimal solution

$$x = 1, \quad y = 0, \quad z = 0, \quad x' = 0, \quad y' = 1, \quad z' = 0, \quad \min f = 107\,. \tag{2.17}$$

It will be observed that this solution cannot be obtained by rounding off the solution of (2.16) to the integer immediately below or above it.

The Problem of the Knapsack. A Problem of Investment

A hiker wishes to carry a certain number of articles $X_1, X_2, \ldots, X_n$ in his knapsack. He knows the weight $P_1, P_2, \ldots, P_n$ of each of the articles, as well as their respective volumes[1] $V_1, V_2, \ldots, V_n$. He is unable to carry a total load in excess of P, and his knapsack cannot contain a volume greater than V. The hiker allots values $k_1, k_2, \ldots, k_n$ to each of the articles according to its intrinsic utility. Which objects should he take with him to maximize their total utility?

This problem will be represented by the following linear program with integer values, in which x_1 is the number of the articles X_1 to be carried:

$$\begin{aligned} &[\mathrm{MAX}]z = k_1x_1 + k_2x_2 + \ldots + k_nx_n\,, \\ &P_1x_1 + P_2x_2 + \ldots + P_nx_n \leqslant P\,, \\ &V_1x_1 + V_2x_2 + \ldots + V_nx_n \leqslant V\,, \\ &x_1 \geqslant 0, \quad x_2 \geqslant 0, \ldots, x_n \geqslant 0\,. \end{aligned} \tag{2.18}$$

A variation of this problem plays an interesting part in a number of algorithms. Let us suppose that our aim is to maximize V and to take P as a constraint (which would not make much sense for the bearer of the knapsack, but makes sense for other concepts). We should then write

$$\begin{aligned} &[\mathrm{MAX}]\,V = V_1x_1 + V_2x_2 + \ldots + V_nx_n\,, \\ &P_1x_1 + P_2x_2 + \ldots + P_nx_n \leqslant P\,, \\ &x_1 \geqslant 0, \quad x_2 \geqslant 0, \ldots, x_n \geqslant 0\,. \end{aligned} \tag{2.19}$$

A concrete and practical problem can be envisaged in the form of (2.19).

[1] It would be strictly more fitting to speak of cumbersomeness rather than of volume. The introduction of volumes (unless the articles are soft ones) is clearly open to criticism, and we must ask indulgence for the somewhat theoretical nature of the term.

A capital sum K is available and can be used to construct units of production in different localities L_1, L_2, L_3, and L_4, the installation costs C_1, C_2, C_3, and C_4 varying according to the locality selected. Let us use B_1, B_2, B_3, and B_4 to represent the unit profits derived from investments in the corresponding localities. The problem is which localities to choose and how many units of production to build in each of them in order to maximize the total profit.

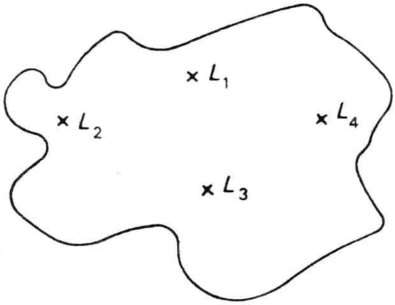

FIG. 2.7

Taking as variables x_1, x_2, x_3, x_4 to represent the number of units to be built in the various localities, we obtain as a model one in all respects similar to (2.19).

$$
\begin{aligned}
&[\mathrm{MAX}]\, z = B_1 x_1 + B_2 x_2 + B_3 x_3 + B_4 x_4 \,, \\
(2.20) \qquad &C_1 x_1 + C_2 x_2 + C_3 x_3 + C_4 x_4 \leqslant K \,, \\
&x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad x_3 \geqslant 0, \quad x_4 \geqslant 0 \,.
\end{aligned}
$$

The reader will have learned in Volume 2[1] (Section 12, page 86) how to resolve this problem by means of dynamic programming. Some problems with integer values can, indeed, be resolved by this method, but, in cases where there are a greater number of constraints, the method cannot easily be employed and may even have to be discarded from the outset, since the problem cannot be reduced, after it has been transformed, into a sequential form.

3. Another Well-Known Problem

In Volume 1 (page 64) and in Volume 2 (page 265) we gave a problem known in mathematical parlance as a *problem of assignment* but which is equally a

[1] *Note to Reader*: Throughout the present work, Volume 2 refers to A. Kaufmann, "Graphs, Dynamic Programming, and Finite Games," Academic Press, New York, 1967.

linear program with integer values. Here these values are bivalent; that is to say that, in such problems, they can only assume the values of 0 or 1.

Let us recall this problem.[1] We have to consider n workmen $X_1, X_2, \ldots, X_n$ and n positions of employment $Y_1, Y_2, \ldots, Y_n$. To each assignment (X_i, Y_j) a cost is attached (Fig. 2.8):

$$c_{ij} \geqslant 0, \qquad i, j = 1, 2, \ldots, n. \tag{2.21}$$

Some of the c_{ij} may be infinite (which means that the corresponding assignment is impossible).

We are required to assign the n workmen to n positions in such a manner that each workman will have one and only one position and that the total cost of the assignments will be minimal. This gives the following program:

$$[\text{MIN}]\ z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij},$$

$$\begin{aligned} &\sum_{i=1}^{n} x_{ij} = 1, \qquad j = 1, 2, \ldots, n, \\ &\sum_{j=1}^{n} x_{ij} = 1, \qquad i = 1, 2, \ldots, n, \\ &x_{ij}^2 = x_{ij}, \qquad i, j = 1, 2, \ldots, n. \end{aligned} \tag{2.22}$$

The relation $x_{ij}^2 = x_{ij}$ imposes the constraint on each variable x_{ij} that it cannot be equal to a number other than 0 or 1. An assignment is represented by a table (Fig. 2.9) containing a single and only a single 1 in each line and also in each column. Various special methods exist for the solution of such problems, as can be discovered from our references [K74]–[K76].

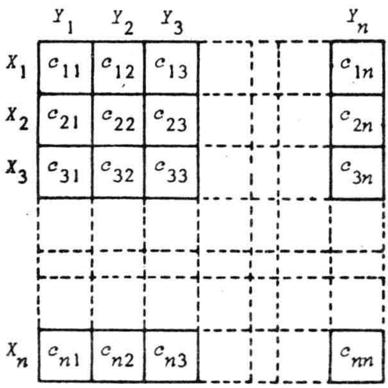

FIG. 2.8

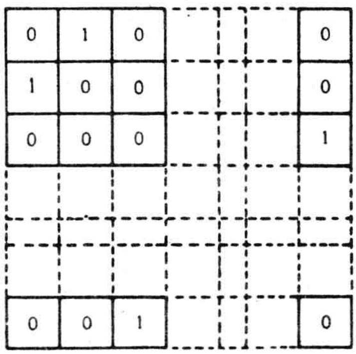

FIG. 2.9

[1]This problem is given by M. R. de Grove, Revue Française de Recherche Opérationnelle, No. 39, pp. 171–183.