College Algebra

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To

Rebecca and Catherine

Preface

This text is intended for the standard course in college algebra. Although all of the essential topics for further mathematical courses are carefully covered, no assumption of specific mathematical training is made beyond a one year course in high school algebra. Necessary background material is introduced as it is required, and the basic algebraic manipulations are extensively reviewed in the early chapters. This is in keeping with the recent C.U.P.M. recommendations for a course designed to meet the needs of students not equipped to begin an intensive calculus preparatory course.

There are over 200 worked examples in the text. Full solutions to nearly half of the exercises in the book form a wealth of additional examples to further illustrate the material. Each section closes with a set of exercises and a true-false quiz which is aimed at developing the student's reasoning capacities and testing his understanding of the definitions and theorems.

Chapter 1 begins slowly and with an intuitive point of view, employing Venn diagrams to discuss the concepts of union and intersection, and beginning with the integers in the study of sets of numbers. The standard material on inequalities and absolute value is covered, and complex numbers are discussed in terms of their algebraic properties and geometric representations. In Chapter 2, the fundamental concepts of relations and functions are introduced, and an entire section is devoted to graphs of functions to provide the student with a more intuitive approach. Polynomials and polynomial functions are discussed in Chapter 3; included are remedial material and a study of divisibility.

Chapter 4 covers the important material on systems of linear equations, and contains an elementary introduction to matrix theory. Matrix manipulations are motivated by a discussion of n-vectors as solutions to systems of equations. Chapter 5 is an elementary introduction to the analytic geometry of the line, circle, and parabola. In Chapter 6 the exponential function and its inverse, the logarithmic function, are introduced and their basic properties carefully discussed. Included is a detailed description of the use of common logarithms in numerical computations. In Chapter 7 we introduce the principle of mathematical induction and use it to solve a number of interesting problems in combinatorics.

There are a number of paths which may be taken through the book. For instance, a very short and elementary course presupposing almost no mathematical background would include the following:

Chapter 1 (Sets and Numbers)
Chapter 2 (Relations and Functions)
Chapter 3 Section 3.1 only (Introduction to Polynomials)
Chapter 5 (Linear and Quadratic Graphs)
Chapter 7 (Mathematical Induction)

A course assuming somewhat more knowledge, say one and a half years of high school algebra, could easily begin with Chapter 3 and cover the material in the remainder of the book.

The present book differs substantially from our *Elementary Functions and Coordinate Geometry*. Topics not essential to a pre-calculus course, such as groups, rings, and fields, have been omitted as has a discussion of trigonometry. Included in their place are reviews of background material and an introduction to linear algebra.

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1 Sets and Numbers

1.1 Sets and Subsets

A *set* is one of the most elementary and primitive concepts in mathematics. The word "set" will not be formally defined in terms of more elementary ideas. However, some synonyms for the word are collection, aggregate, totality, class. Consider the following examples:

(a) the set consisting of the numbers 0, 1, 2;
(b) the set of all blue-eyed coeds at the University of California at Santa Barbara;
(c) the set of all letters appearing in the word "banana";
(d) the set of all even numbers between 2 and 10, including 2 and 10;
(e) the set of all grains of sand on Goleta Beach;
(f) the set of all electrons in the universe;
(g) the set of all whole numbers.

In order to discuss these examples, we introduce notation that will be fundamental throughout this book. First is the notation for set membership. If S is a set and x is an item that belongs to this set, we write

$$x \in S. \tag{1}$$

The formula (1) is read "x is a member of S," or "x belongs to S," or "x is an *element* of S." It means, then, that x is one of the items in the set S. It is clear that we can understand a particular set if we are able to write down explicitly each one of the elements (members) in S. In example (a) above, where S is the set consisting of the numbers 0, 1, 2, we write

$$0 \in S, 1 \in S, 2 \in S.$$

In example (b), we can list all the elements of the set if we can write down the names of all blue-eyed coeds at U.C.S.B. If the set of letters in the word "banana" is denoted by S, then

$$b \in S, a \in S, n \in S. \tag{2}$$

Of course, a and n both appear more than once in the word "banana," but we are interested only in the distinct letters in the word, and unless we wish to specify also the number of times each of these letters occurs, we would not write them down more than once. The set S in (d) is completely specified by

(3) $$2 \in S, 4 \in S, 6 \in S, 8 \in S, 10 \in S.$$

The sets described in (e) and (f) defy any kind of explicit enumeration of the type that we used in (3). In principle we could label each one of the grains of sand on Goleta Beach in some way and thereby describe the set. In the case of all electrons in the universe, however, such a labeling process is completely out of the question, even though we can still believe that this set is a comprehensible thing.

In example (g) there is no way, even in principle, in which we could write down all the whole numbers explicitly. It is true, though, that given any object, most of us can decide whether or not it is a whole number and therefore an element of the set of whole numbers.

We see from these examples that there are essentially two ways to describe a set. First, we can explicitly name all the elements of the set, or second, we can describe the set in terms of some common defining property of all the elements. These two methods of defining sets lead us to introduce some standard useful notation. Curly brackets will be used to denote sets in the first instance. For example, in (a) we can write

(4) $$S = \{0, 1, 2\}.$$

The set in (c) above can be written

(5) $$S = \{b, a, n\}.$$

The set in (d) can be written

(6) $$S = \{2, 4, 6, 8, 10\}.$$

No order is implied in this notation. Thus (6) could also have been written

$$S = \{4, 6, 8, 10, 2\},$$

and (5) could have been written

$$S = \{a, b, n\}.$$

The second way of describing a set is as follows:

(7) $$S = \{x \mid x \text{ satisfies the defining property}\}.$$

The formula in (7) is read "S is equal to the set of all x such that x satisfies the defining property." Thus the set S in (a) could also have been denoted

$$S = \{x \mid x = 0 \quad \text{or} \quad x = 1 \quad \text{or} \quad x = 2\}.$$

We could write the set in (d) as follows:

(8) $$S = \{x \mid x = 2k,\ k = 1, 2, 3, 4, 5\}.$$

The formula (8) is read, "S is equal to the set of all x which satisfy $x = 2k$, where k is any one of the numbers 1, 2, 3, 4, or 5."

There is nothing mutually exclusive about the two methods of describing a set. We can use either one depending on circumstances; but as we have seen, it may not always be possible to describe a set by the first method.

We shall also find it convenient to have a notation which tells us when an element is *not* a member of a given set. For instance, $\frac{1}{2}$ is not a member of the set of all whole numbers, and we can indicate this notationally by putting a vertical stroke through the set membership symbol,

(9) $$\tfrac{1}{2} \notin S.$$

The formula (9) is therefore read "$\frac{1}{2}$ is not an element of S."

We say that two sets are equal if they are the same set. Thus S and T are equal if S and T consist of precisely the same objects. Of course, a set may be described in various ways, and it is not always trivial to assert the equality of two sets. For example, consider the following two descriptions of the same set:

$$S = \{x \mid x \text{ is a whole number between 1 and 50 and } x \text{ is not divisible by any whole number except 1 and } x\}.$$

It takes a moment's reflection to see that

$$S = \{1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47\}.$$

The set T consisting of the integers 2, 8, 10 is part of the set S in (6). We denote this situation notationally as follows:

(10) $$T \subset S.$$

The formula (10) is read "T is contained in S," or "T is a *subset* of S," and the above symbol is called an *inclusion* sign. It means simply that each member of the set T is a member of the set S; equivalently, for each x, if $x \in T$, then $x \in S$. There is an important distinction in meaning between the set membership sign $\in$ and the inclusion symbol $\subset$. Certainly we could not write $T \in S$, for T itself is not any one of the numbers 2, 4, 6, 8, 10. Nor could we write $2 \subset S$, for 2 is not a subset of S. This second distinction is somewhat more subtle, for we have distinguished between the elements of a set and the set consisting of those elements. Thus it is perfectly clear and indeed correct to write

$$\{2\} \subset S, \tag{11}$$

for, each member of the set on the left of the inclusion sign is a member of the set on the right of the inclusion sign in (11). Of course, the set on the left in (11) has only one member, namely 2.

As stated above, two sets S and T are equal if and only if they consist of precisely the same objects. Equivalently, to say that S is equal to T means that every element in S is in T and every element in T is in S: $S \subset T$ and $T \subset S$.

Consider the following phrase which purports to describe a set: "The set of all whole numbers no larger than 10, but greater than 15." While it makes sense to talk about the set of all whole numbers less than 10 and the set of all numbers greater than 15, there are no numbers which belong to both of these sets. Nevertheless, in mathematics it is convenient to have a symbol which stands for the set which has no members, called the *empty set*. This set often arises in combining perfectly well-defined sets. We shall denote the empty set by

$$\phi. \tag{12}$$

It is always a true statement that whatever x may be,

$$x \notin \phi.$$

It is also the case that we may always write

$$\phi \subset S$$

(which includes the possibility that S is ϕ!). The above inclusion is always valid, because it is impossible to exhibit an element in ϕ which is not in S. In fact, it is impossible to exhibit an element in ϕ at all!

SYNOPSIS

There are two methods of describing a set: listing all the items and describing the set in terms of a defining property. The notations appropriate for these two methods are

$$\{\cdots\} \quad \text{and} \quad \{x \mid x \text{ satisfies } \cdots\}.$$

The set membership symbol indicates that an item x belongs to a set S. This is written

$$x \in S.$$

If x is not in S, we write

$$x \notin S.$$

The equality of two sets,

$$S = T,$$

means that S and T consist of the same members. The inclusion symbol $\subset$ as in

$$T \subset S$$

means that each element in T is also in S. The empty set ϕ is the set which has no members.

QUIZ

Answer *true* or *false:*

1. If E is the set of even whole numbers, then $2\frac{1}{2} \in E$.
2. If M is the set of all whole numbers and E is the set of all even whole numbers, then $E \subset M$.
3. If A and B are subsets of a set S, then the set of all items which belong to either A or B (possibly to both A and B) is a subset of S.
4. If A and B are subsets of a set S, then the set of all items which belong to both A and B is a subset of S.
5. If X is a set, then $X \subset X$.
6. If X is a set, then $X \subset \phi$.
7. If X is a set, then $\phi \subset X$.
8. It is never possible that $\phi \in X$ for any set X.
9. The set which consists of the empty set, i.e., $\{\phi\}$, has no elements in it.
10. $\phi \in \{\phi\}$.

EXERCISES

1. List all subsets of the following set: $S = \{0, 1, 2\}$.
2. List all the subsets of the set of letters appearing in the word "banana."
3. Show by example that an element of a set can also be a subset of the set.
4. Let N be the set of all *natural numbers*, 1, 2, 3, ... Let

$$S = \{x \mid x = n^2, n \in N\}.$$

That is, S is the set of numbers which are squares of whole numbers. Recall that n^2 is just the product of n with itself. For each of the following numbers indicate whether $x \in S$ or $x \notin S$:

$$x = 4,\ x = 0,\ x = \tfrac{1}{4},$$
$$x = 10,\ x = 25,\ x = 1{,}000,$$
$$x = 1{,}000{,}000.$$

5. Let M be the set of counting numbers 0, 1, 2, 3, . . . , and let

$$E = \{x \mid x = 2m,\ m \in M\}, \qquad T = \{x \mid x = 3m,\ m \in M\},$$
$$S = \{x \mid x = m^2,\ m \in M\}, \quad \text{and} \quad R = \{x \mid x = s + 1,\ s \in S\}.$$

Insert in each blank space of the following table either YES or NO according as the number on the left is or is not an element of the set above the blank space.

$\in$	M	E	T	S	R	ϕ
9						
5						
0						
6						
26						
-1						

6. Let M, E, T, S, R be the sets defined in Exercise 5. Insert in each blank space in the following table either YES or NO according as the set indicated on the left of the blank space is or is not a subset of the set above it.

$\subset$	M	E	T	S	R	ϕ
M						
E						
T						
S						
R						
ϕ						

7. How many subsets are there of a set which consists of five elements? Can you answer this question without listing all the subsets?
8. Let M be the set of nonnegative whole numbers, i.e., M is the set of ordinary *counting numbers*, 0, 1, 2, ... Let $S = \{x \mid x = 2m + 3n, m \in M, n \in M\}$; that is, S is the set of all whole numbers which can be written in the form $2m + 3n$, where m and n are nonnegative whole numbers. If E is the set of even nonnegative whole numbers and O is the set of odd nonnegative whole numbers, show that $E \subset S$ but that O is not a subset of S. Show that $S \subset M$. Is it true that $S = M$? If not, exhibit an element of M which is not in S.

1.2 Combining Sets

There are several fundamental operations that can be performed on subsets of a set S to produce other subsets of S. The simplest of these operations is *complementation*. Thus, if S is a set and $X \subset S$, we denote by

$$X' \tag{1}$$

the set of all elements in S which are *not* in X. The subset X' is called the *complement* of X with respect to S. The set S is assumed to be known even though it is not specified in our notation. We can write down the definition of X' as follows:

$$X' = \{x \mid x \in S, x \notin X\}.$$

In other words, X' consists of those items in S not in X. For example, if $S = M$ is the set of all nonnegative whole numbers, i.e., M consists of the numbers 0, 1, 2, 3, . . . , and if E is the set of even nonnegative whole numbers, and if O is the set of all odd nonnegative whole numbers, then

$$E' = O$$

and

$$O' = E.$$

Also observe that

$$(E')' = E.$$

For $(E')'$, which will henceforth be written E'', is the complement of $E' = O$, and therefore consists of all those whole numbers which are not in O, i.e., $E'' = E$. It is true, of course, that whatever the subset X of a set S may be,

$$X'' = X. \tag{2}$$

To verify (2), just remember that X'' is the complement of the set of objects not in X. In other words, $x \in X''$ means that it is not true that $x \notin X$, i.e., it is true that $x \in X$. We can visualize this situation by means of a simple picture known as a *Venn diagram* (Figure 1.1).

(3)

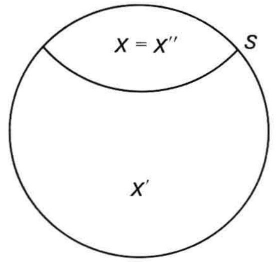

FIGURE 1.1

Another elementary method for producing new subsets of a set S is the formation of the *intersection* of two subsets. If X and Y are subsets of S, then the intersection of X and Y, denoted by

(4) $$X \cap Y,$$

is the subset of S which consists of precisely those elements which belong to both X and Y. In our set notation we can write

(5) $$X \cap Y = \{z \mid z \in X \quad \text{and} \quad z \in Y\}.$$

For example, if $S = M$, the set of nonnegative whole numbers, $X = \{0, 1, 2, 3\}$, $Y = \{2, 3, 4, 5\}$, then $X \cap Y = \{2, 3\}$. As another example using M, if we take X to be E, the set of even nonnegative whole numbers, and Y to be O, the set of odd nonnegative whole numbers, it is clearly true that $E \cap O = \phi$, since no whole number is both even and odd. This is a special case of the following general statement: If $X \subset S$, then

(6) $$X \cap X' = \phi.$$

We next discuss the operation of forming the *union* of two subsets X and Y of a set S. The union of X and Y, written

$$X \cup Y,$$

is the set of all items which belong to X or Y. We use the word "or" in the nonexclusive sense, i.e., the elements of $X \cap Y$ are also in $X \cup Y$. As an example, if