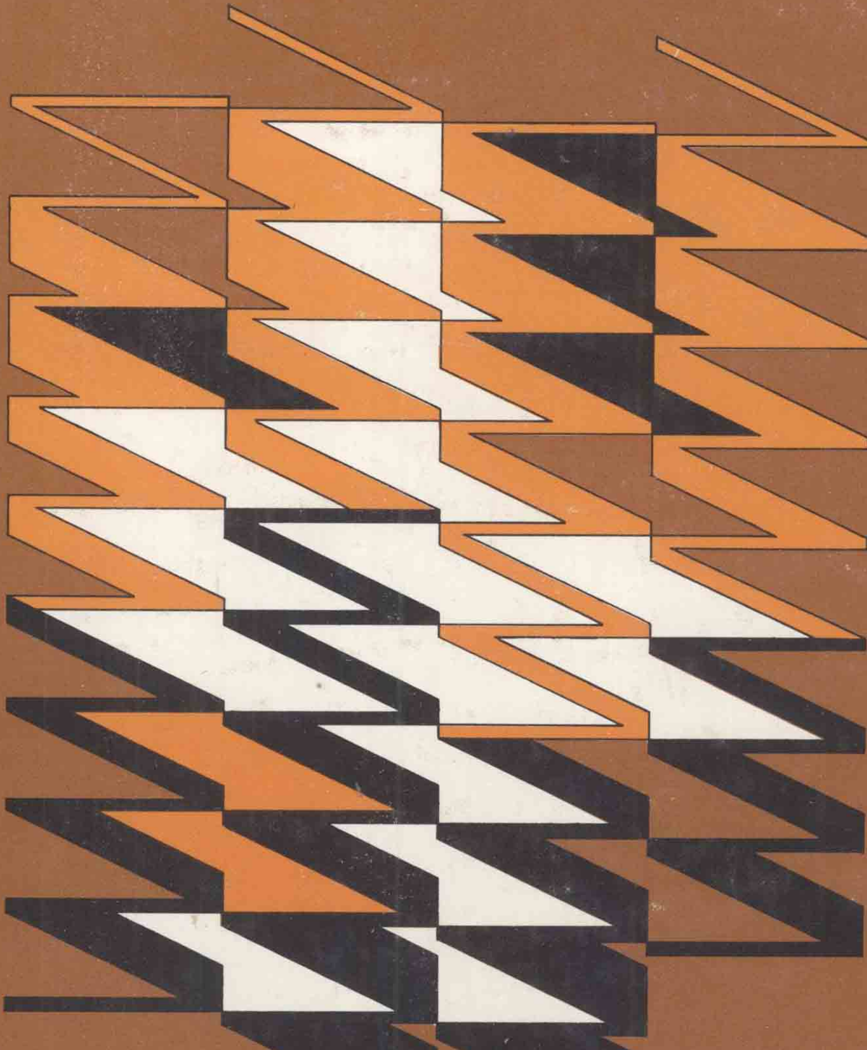
Linear Algebra
A Concrete Introduction

Dennis M. Schneider
Manfred Steeg
Frank H. Young



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A Concrete Introduction

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Preface

Linear algebra is deeply rooted in analytic geometry and the theory of systems of linear equations. Until the last twenty or thirty years, students were first introduced to some of the concepts of linear algebra in courses on analytic geometry and the theory of equations. When they were subsequently exposed to an abstract presentation of linear algebra they were equipped to deal with it because they were already familiar with some of the concrete problems that gave birth to the subject.

Recently, all of this has changed. Analytic geometry has been integrated into courses on the calculus. This has resulted in a substantial increase in the amount of material that needs to be covered in calculus courses. And if material needs to be omitted because of time constraints, it is usually the material on analytic geometry. Also courses on the theory of equations have long since vanished from the curriculum. Thus most students enter a course on linear algebra equipped only with some elementary facts about vectors in R^2 and R^3. Although vectors certainly provide some motivation for studying abstract vector spaces, it is not enough. Students do not see the need for such abstract concepts as linear independence, spanning, bases, and dimension arising simply out of the study of vectors in R^2 and R^3. However, they do see the need for these concepts arising out of concrete problems in analytic geometry and systems of linear equations. For these reasons we feel that most students will learn more abstract linear algebra from a concrete approach based on the theory of linear equations and analytic geometry than they would from the (now traditional) abstract approach.

In view of these remarks, we chose as our point of departure the theory of systems of linear equations. In Chapter 1 we completely develop the Gaussian elimination process and thereby teach the student how to solve any system of linear equations. This chapter is also the natural place to introduce vectors and matrices in order to represent systems of equations. (*The material in this chapter should be covered as rapidly as possible.*) In Chapter 2 we use concrete problems concerning linear systems to motivate the abstract concepts of linear algebra (in n-dimensional Euclidean space only). We also stress the interplay between these concepts and the geometry of two- and three-dimensional space. In Chapter 3 we again return to the theory of linear systems, but this time use inconsistent systems to motivate the need to extend the concepts of length and angle to higher-dimensional spaces. Chapter 4 (which requires calculus and may be omitted) extends the concepts of Chapters 2 and 3 to the function space setting. Chapter 5 introduces the notion of a linear transformation and discusses the relationship between these transformations and matrices.

It is not until Chapter 6 (which may be omitted) that we finally give an abstract definition of a vector space and an inner product space. The student should at this point be prepared to appreciate how these definitions provide a single conceptual framework for dealing with problems in linear algebra.

In Chapter 7 we briefly discuss determinants and their relationship to the geometry of space. Eigenvalues and eigenvectors are discussed in Chapter 8. Our discussion leads naturally to the problem of diagonalizing a matrix and the spectral theorem for symmetric matrices. In Sections 8.7 through 8.10 (which require calculus and may be omitted) we apply the theory of linear algebra to systems of differential equations. Finally, in Chapter 9 we discuss some numerical techniques that are useful for solving problems in linear algebra with a computer.

The instructor should note that the text deals exclusively with real vector spaces. Except for a brief remark in Chapter 8, all matters concerning complex numbers are left to the appendix. In the appendix, we define complex numbers and develop their arithmetic. We then point out (via examples and exercises) that all of the material in Chapters 1 and 2 extends to C^n with no change. After motivating a definition of an inner product on C^n we show (again via examples and exercises) that the material in Chapter 3 extends to C^n with no change. The extension of the material in Chapter 8 to C^n (where it belongs) now follows immediately.

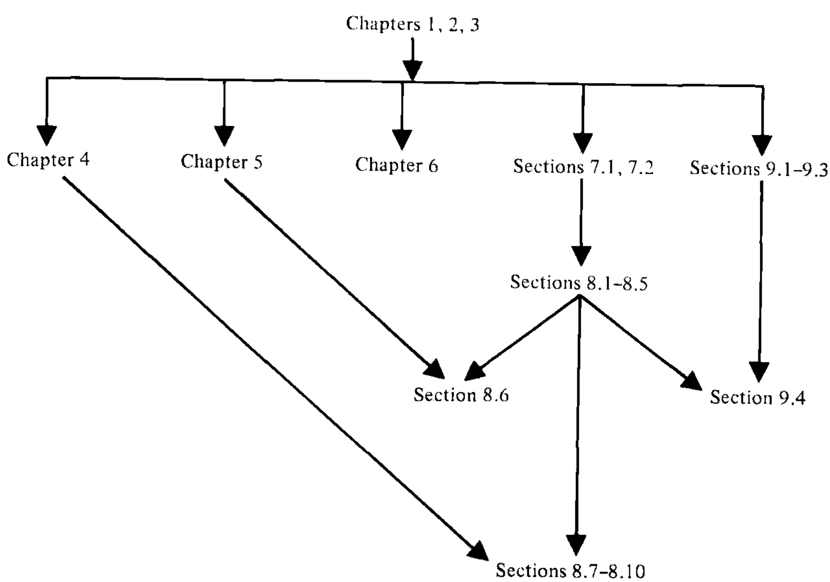
The applications that are presented are an important part of the text. They provide the student with a sense of the vast scope and rich nature of the subject. We believe that the theory and applications of linear algebra illuminate each other. Not only does a knowledge of the theory help one to understand the applications, but a knowledge of the applications helps one to understand the theory. The applications that we have chosen are real, not artificial. They are taken primarily from biology, economics, sociology, circuit theory, and data analysis. Each example is carefully motivated, explained, and developed.

Every linear algebra text must contain exercises. They are vital because it is through working the exercises that the student actually confronts the subject and completes the learning process. Exercises will be found in the textual material to encourage the student to read the book in an active rather than passive manner. The problems at the end of each section begin with some fairly routine exercises. More substantial problems follow these. To assist the student, answers for computational exercises and for those problems marked with an asterisk are provided at the end of the book. All problems involving calculus are so indicated.

Since we have found that at this stage most students find set theoretical notation and the sigma notation more of a hinderence than a help, we have avoided their use in the text. We have also avoided using mathematical induction.

The book is organized so that it may be used for both a calculus-based and a non-calculus-based course. Only a very basic understanding of calculus is required for a calculus-based course. If calculus is not required, Chapter 4 and Sections 8.7 through 8.10 must be omitted. In the remainder of the book, the few problems and examples that require calculus are clearly marked.

Although we believe that there are strong pedagogical reasons for covering the topics in the book in the order in which they are presented, the instructor does have many options. For example, after covering Chapters 1, 2, and 3 there are essentially five options available: Chapter 4; Chapter 5; Chapter 6; Chapters 7 and 8; Chapter 9. We have noted the exact dependencies in the following chart.



To our students at Knox College and to the students at Monmouth College who have suffered through using Xerox copies of this book in the various stages of its development, we wish to express our sincere thanks. Special thanks are due to those students, colleagues (especially George Converse and Lyle Welch at Monmouth College), and the reviewers who have made many valuable suggestions and criticisms. We also wish to express our gratitude to Betsy Kelly, Mavis Meadows, and Jonathan Young for their excellent typing of the manuscript. Finally, we wish to thank the entire staff at Macmillan for their effort.

D. M. S.
M. S.
F. H. Y.

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Chapter 1

Systems of Linear Equations and Gaussian Elimination

Linear algebra is a subject of crucial importance to mathematicians and users of mathematics. Applications of linear algebra are found in subjects as diverse as economics, physics, sociology, and engineering. Workers in these fields, as well as mathematicians, statisticians, computer scientists, and management consultants, use linear algebra to express ideas, solve problems, and model real activities.

Linear algebra has its beginnings in the study and solution of systems of linear equations. Before studying linear algebra as an abstract mathematical subject, it is necessary for the student to have some understanding and appreciation of the concrete origins of the subject. We therefore devote this introductory chapter to the theory of systems of linear equations, introducing necessary terminology and notation as well as describing applications and techniques for computation. *We recommend that the material in this chapter be covered as rapidly as possible. Some or all of the applications in Section 1.8 may be postponed until Chapter 8.*

1.1 SYSTEMS OF LINEAR EQUATIONS

The equation $y = mx + b$ is familiar to mathematics students as an equation that represents a nonvertical straight line. It is an example of what we call a linear equation. A linear equation in the two variables x_1 and x_2 is an equation that can be written in the form

$$a_1x_1 + a_2x_2 = b,$$

where a_1, a_2, and b are numbers. In general, a **linear equation in the n variables** $x_1, x_2, \dots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_n x_n = b,$$

where the **coefficients** $a_1, a_2, \dots, a_n$ and the **constant term** b are numbers. We adopt the usual notation of using subscripts because this makes it easier to understand and manipulate equations involving several variables (and we do not have to worry about running out of letters). The following are examples of linear equations:

$$x_1 + 7x_2 = 3, \qquad x_1 - 3x_2 + x_4 = \frac{5}{2},$$
$$0.5x_1 = 3x_2 - 7, \qquad x_1 + x_2 + \cdots + x_n = 4.$$

Some examples of equations that are not linear are:

$$x_1^2 + x_1x_3 = 5, \qquad \frac{1}{x_1} + x_2 + x_3 = 7,$$
$$e^{(x_1)} + x_2 = \frac{1}{2}, \qquad \frac{x_1 + x_2}{x_3 + x_4} = x_5 + 7.$$

Any equation that contains a power of a variable (x_i^r, where $r \neq 1$) or a product of two or more variables (e.g., x_ix_j) is not a linear equation.

A **solution** of a linear equation is a collection of values for the variables such that when these values are substituted for the variables, the equation is true. For example, a solution of $x_1 + x_2 = 0$ is $x_1 = 0$, $x_2 = 0$. Another solution of this equation is $x_1 = 1$, $x_2 = -1$. Solving a linear equation involves finding values (numbers) for the variables that make the equation true. Since these values are initially unknown, we often refer to the variables as unknown quantities or **unknowns.** "Variable" and "unknown" are interchangeable terms.

EXERCISE 1 Verify that the indicated values are solutions of the given linear equations.

(a) $3x_1 + 2x_2 = 7, \qquad x_1 = 1, x_2 = 2$
(b) $3x_1 + 2x_2 = 7, \qquad x_1 = -3, x_2 = 8$
(c) $x_1 - x_2 = 5, \qquad x_1 = 2, x_2 = -3$

Frequently, we have more than one equation involving the same variables. For example,

$$\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ 3x_1 - 2x_2 + 2x_3 &= 6 \end{aligned} \tag{1}$$

is a **system** of linear equations in the three variables x_1, x_2, and x_3. This system has two equations and three unknowns. In general, a **system of linear equations** (also called a **linear system**) in the variables $x_1, x_2, \ldots, x_n$ consists of a finite number of linear equations in these variables. The general form of a system of m equations in n unknowns is

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & b_m. \end{array}$$

We will call such a system an $m \times n$ (m by n) linear system. The double subscripting of the coefficients has been arranged so that the first subscript refers to the equation and the second subscript refers to the variable. In other words, a_{ij} is the coefficient of x_j in the ith equation. In (1), $a_{12} = 1$ and $a_{21} = 3$. Similarly, b_i is the constant term in the ith equation. In (1), $b_1 = 4$ and $b_2 = 6$. Note also in (1) that $m = 2$ and $n = 3$. We can determine m and n (once we are given the system of equations) by counting equations and variables, respectively.

Exercise 2 Given the linear system

$$\begin{array}{rcrcrcrcr} 4x_1 & + & 2x_2 & - & 3x_3 & + & 4x_4 & = & 2 \\ 5x_1 & - & 3x_2 & + & 4x_3 & - & 2x_4 & = & -7 \\ x_1 & + & x_2 & - & x_3 & + & 3x_4 & = & 2. \end{array}$$

What is the value of a_{31}? of a_{21}? of a_{12}? of a_{24}? of b_3? This is an $m \times n$ system. What do m and n equal?

It should come as no surprise that a solution of a system of linear equations is a collection of values for the variables which makes *all* the equations true. When these values are substituted for the variables, every single one of the equations is a true statement. System (1) has $x_1 = 2$, $x_2 = 1$, $x_3 = 1$ as a solution because $2 + 1 + 1 = 4$ *and* $3 \cdot 2 - 2 \cdot 1 + 2 \cdot 1 = 6$. (The student who is actively reading this book should have just asked if there are any other solutions. Are there?) However, $x_1 = 4$, $x_2 = 0$, $x_3 = 0$ is not a solution of system (1) despite the fact that it is a solution of the first equation of the system. Since $3 \cdot 4 - 2 \cdot 0 + 2 \cdot 0 = 12 \neq 6$, these values do not satisfy the second equation and thus cannot be a solution of the system.

Exercise 3 Which of the following are solutions of system (1)?

(a) $x_1 = 1$, $x_2 = 1$, $x_3 = 1$
(b) $x_1 = -2$, $x_2 = 0$, $x_3 = 6$
(c) $x_1 = 1$, $x_2 = 1$, $x_3 = 2$

A system of linear equations may not have any solutions. For example, consider the system

$$\begin{aligned} x_1 + 3x_2 &= 17 \\ x_1 + 3x_2 &= 24. \end{aligned}$$

If this system were to have a solution, 17 would have to equal 24, since both are equal to $x_1 + 3x_2$. This equality is clearly impossible. A linear system that does not have any solutions is called **inconsistent.** A linear system that does have solutions is called **consistent.**

EXERCISE 4 Find a value for b such that the following system is inconsistent.

$$\begin{aligned} 2x_1 + 3x_2 &= 4 \\ 4x_1 + 6x_2 &= b \end{aligned}$$

In the next section we discuss applications of linear systems in several disciplines.

PROBLEMS 1.1

*1. Which of the following equations are linear?
(a) $3x_1 - 2x_2 + x_3 = 0$ (b) $2x_1 + 3x_2 + x_4 = x_3 - 8$
(c) $x_1x_2 + x_3 = 2$ (d) $2x_1 + x_2^2 - x_3 = x_4$
(e) $\dfrac{x_1 - x_2}{x_3} = x_4$ (f) $x_1 + x_2 - x_5 = 4$

2. Identify the linear equations.
(a) $3x = 5y - z$ (b) $5x_1 = 3x_2 - 1$
(c) $x_1 + x_7 = 0$ (d) $\log(x_1) = 4$
(e) $\dfrac{1}{x_1} + x_2 = -52$ (f) $\dfrac{x_2}{x_4} = \dfrac{x_3}{x_1}$

*3. Which of the following are solutions of the equation $2x_1 - 5x_2 + x_3 = 3$?
(a) $x_1 = 1, x_2 = -1, x_3 = -3$ (b) $x_1 = 10, x_2 = 2.5, x_3 = -4.5$
(c) $x_1 = 1, x_2 = -1, x_3 = -4$ (d) $x_1 = \dfrac{2}{9}, x_2 = -\dfrac{5}{9}, x_3 = -\dfrac{2}{9}$

4. Which of the following are solutions of the equation $3x_1 - 2x_2 + 4x_3 + x_4 = 0$?
(a) $x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 4$
(b) $x_1 = 1, x_2 = 2, x_3 = 1, x_4 = 3$
(c) $x_1 = 2, x_2 = 1, x_3 = -1, x_4 = 0$
(d) $x_1 = 2, x_2 = 1, x_3 = -1, x_4 = 1$

*5. Determine which of the following are solutions of the system

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 3 \\ -x_1 + x_2 - 5x_3 &= 0 \\ 3x_1 + 3x_2 + x_3 &= 6. \end{aligned}$$

*Asterisks indicate problems which have answers in the answers section.

(a) $x_1 = 1, x_2 = 1, x_3 = 0$ (b) $x_1 = -7, x_2 = 8, x_3 = 3$
(c) $x_1 = 9, x_2 = 0, x_3 = 3$ (d) $x_1 = 3, x_2 = 0, x_3 = 0$
(e) $x_1 = 9, x_2 = -6, x_3 = -3$ (f) $x_1 = 1, x_2 = 2, x_3 = 2$

6. Determine which of the following are solutions of the system

$$\begin{aligned} 3x_1 - 2x_2 - 3x_3 - 2x_4 &= -1 \\ x_1 + x_2 - x_3 + x_4 &= 8 \\ 2x_1 + 3x_2 + x_3 \qquad &= 21. \end{aligned}$$

(a) $x_1 = 1, x_2 = 1, x_3 = 1, x_4 = -1$
(b) $x_1 = 5, x_2 = 3, x_3 = 2, x_4 = 2$
(c) $x_1 = 2, x_2 = 6, x_3 = -1, x_4 = -1$
(d) $x_1 = 0, x_2 = -2, x_3 = 1, x_4 = 1$

7. Explain why the system

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 1 \\ 3x_1 + 3x_2 + 6x_3 &= 2 \end{aligned}$$

cannot have any solutions.

***8.** Find three different values for b that will make the following system inconsistent.

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 1 \\ 3x_1 + 3x_2 + 6x_3 &= b \end{aligned}$$

9. For which values of b will the following system have solutions?

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ 2x_1 + 4x_2 - 2x_3 &= b \end{aligned}$$

***10.** Given the system

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 4 \\ x_1 - x_2 + 3x_3 &= 2, \end{aligned}$$

find an equation with the property that when it is included in the system the resulting system of three equations is inconsistent.

11. Verify that an infinite number of solutions of the system

$$\begin{aligned} 3x_1 - 2x_2 - 3x_3 - 2x_4 &= -1 \\ x_1 + x_2 - x_3 + x_4 &= 8 \\ 2x_1 + 3x_2 + x_3 \qquad &= 21 \end{aligned}$$

is given by $x_1 = 3 + t$, $x_2 = 5 - t$, $x_3 = x_4 = t$, where t is any number.

12. Verify that an infinite number of solutions of the system

$$\begin{aligned} x_1 \qquad + x_3 + x_4 &= 7 \\ x_1 + x_2 \qquad - x_4 &= 4 \\ x_2 - x_3 - 2x_4 &= -3 \end{aligned}$$

is given by $x_1 = 2 + t - s$, $x_2 = 3 - t + 2s$, $x_3 = 4 - t$, $x_4 = 1 + s$, where s and t are arbitrary numbers.

13. Verify that a solution of

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

is given by

$$x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{12}a_{21}}$$

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

provided that $a_{11}a_{22} - a_{12}a_{21} \neq 0$.

1.2 SOME EXAMPLES OF SYSTEMS OF LINEAR EQUATIONS

Most students of high school algebra have been subjected to problems such as this:

> Find three numbers whose sum is 20 and such that (1) the first plus twice the second plus three times the third equals 44 and (2) twice the sum of the first and second minus four times the third equals −14. [This problem is found (together with 19 others involving digits, water tanks, boats, work, and freight trains) in H. B. Fine, *A College Algebra*, Ginn & Co. 1904, pp. 150–152.]

This problem is equivalent to finding the solution of the following system of linear equations (where x_1, x_2, and x_3 are the three numbers we are trying to find).

$$\begin{aligned} x_1 + x_2 + x_3 &= 20 \\ x_1 + 2x_2 + 3x_3 &= 44 \\ 2x_1 + 2x_2 - 4x_3 &= -14 \end{aligned}$$

Problems such as this, although of interest to professional (or habitual) problem solvers, are not important applications of linear equations. They give practice in translating English into the language of mathematics as well as practice in computation but do not give the student adequate motivation for studying the mathematics that is being used. In this section we give several practical and important examples where systems of linear equations arise naturally.

Electric Circuits

Most people think of electricity as something that "flows" through wires. Indeed, it is usually convenient to think of electricity as electrons flowing through wires. When we think of something flowing we naturally think of

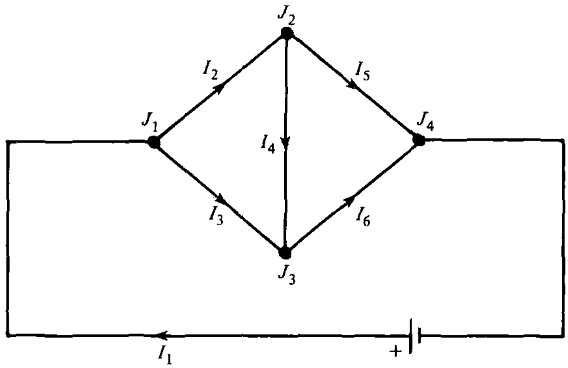

Figure 1.1

the "pressure" behind the flow and the "quantity" of substance flowing. For electrical circuits, the "pressure" behind the electrons is measured in volts and the "quantity" of electrons flowing, called the current, is measured in amperes or amps. For the sake of simplicity we will consider only direct-current (dc) circuits, circuits in which the electricity travels in one direction in each wire.

Let us consider the electric circuit given by Figure 1.1. This circuit has four junctions, places where many wires come together (J_1, J_2, J_3, and J_4). There are also six branches with currents $I_1, I_2, \ldots, I_6$ and one source of electricity in the branch from J_4 to J_1. Each branch of this circuit has been (arbitrarily) assigned an arrow indicating a direction of flow. The actual direction of flow will be given by the sign of the current in that branch. A positive current will mean a flow in the direction of the arrow; a negative current will flow in the opposite direction.

We can use an ammeter to measure both the direction and the amount of current flowing in each wire of this circuit. If the currents in all the wires that come together at a junction are added, it is found that the sum is zero. This is not too surprising—it expresses the fact that the substance (the electrons) which is flowing is not being created or destroyed at the junction. In brief, what goes in must equal what comes out. This is one of two basic laws regarding electric circuits which were first formulated by G. R. Kirchhoff in 1845.

At J_1 we have I_1 amps flowing in and $I_2 + I_3$ amps flowing out. By Kirchhoff's law, $I_1 - I_2 - I_3 = 0$. This is a linear equation with currents as the variables. There will be one equation for each junction. Looking at all four junctions we get the following four equations.

$$\begin{array}{rrrrrrll} I_1 & -I_2 & -I_3 & & & & = 0 & \text{(junction } J_1\text{)} \\ & I_2 & & -I_4 & -I_5 & & = 0 & \text{(junction } J_2\text{)} \\ & & I_3 & +I_4 & & -I_6 & = 0 & \text{(junction } J_3\text{)} \\ -I_1 & & & & +I_5 & +I_6 & = 0 & \text{(junction } J_4\text{)} \end{array}$$