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MATHEMATICS

Models and Applications

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D. C. HEATH AND COMPANY Lexington, Massachusetts Toronto

Published simultaneously in Canada.

Printed in the United States of America.

International Standard Book Number: 0-669-01051-0

Library of Congress Catalog Card Number: 78-19918

To Chris and Joan

PREFACE

Students who are not required to take any specific college mathematics course quite often enroll in one of several introductory courses. One type of course might be called "mathematics appreciation"; it includes diverse topics such as sets, groups, number theory, and graphs, chosen to give an idea of what modern mathematics is all about. Another type is specifically aimed at elementary education majors. "Finite mathematics" is the label most often given to a third type of course. Although the content of our book substantially overlaps with material usually found in finite mathematics texts, the emphasis is different.

Designed for those students who will probably take only one college mathematics course, this text presupposes only one year of high school algebra. Our purpose is to provide a direct and explicit answer to the question, What good is mathematics? Since students are not satisfied by simply being told that the material they study will be needed in a later course, we have only included material that can be applied immediately to real problems. In particular the chapters on sets and logic that begin most finite mathematics texts will not be found here.

The great majority of exercises in this book are word problems. The ability to manipulate symbols is useless unless it can be applied to life in some way. The stage in the application of mathematics to real-world problems that comes *before* the manipulation of symbols is slighted throughout the education of most students; here it is emphasized.

The problems we have provided are tied to the real world, and the numbers involved are realistic. Sometimes students become so accustomed to working exercises that have been prearranged to come out in terms of pleasant, round numbers that they conclude they have made a mistake when an "unpleasant" number appears. In an age of computers and pocket calculators there is no excuse for this. For most calculations in this book three-digit accuracy is ample (except for finance problems, where each penny is considered sacred).

Pocket calculators today are relatively common in all fields and cannot be ignored, least of all in a mathematics course: A student can now buy a calculator that makes unnecessary most of the tables at the back of mathematics texts. We believe any student owning a calculator should be encouraged to use it in this course as much as possible. However, all appropriate tables *are* provided, and of course possession of a calculator is not at all necessary. Frequently we point out how a calculator can handle problems that would be impossible or excessively tedious with pencil and paper.

The metric system is clearly entering everyone's life. We feel that it is important for present-day students to take the metric system seriously and be able to convert between metric and British units as the need arises. To this end we have used both types of units throughout our examples and exercises and included many cases where conversion is necessary.

Many examples are given for each mathematical technique presented since at this level most learning is done by studying examples rather than by following and applying abstract mathematical explanations.

Chapter 1 begins with a review of how to translate word problems into algebraic expressions. This leads naturally to solving and graphing linear equations, graphing linear inequalities, and an introduction to linear programming.

The chapter on the computer is placed second for two reasons. Besides being of interest in its own right, computer knowledge allows those who have access to a computing facility an opportunity to have this powerful tool available for problems that arise in subsequent chapters. Many special exercises have been included that are too long and tedious to solve without the aid of a computer, or at least an electronic calculator. We have given enough explanation that students can actually learn to write simple programs in either BASIC or WATFIV (a variation of FORTRAN IV). We suggest that the students learn one language or the other but not both. The Appendix contains some additional information about each language, but, as with other chapters, our intention is to give a brief introduction rather than to be definitive in any sense. It is not necessary to have a computer facility available for students to get some insight into the computer from this chapter, nor is this chapter necessary for any of the subsequent chapters.

In Chapter 3 probability questions lead naturally to problems of counting. Many of our examples on probability use games and gambling because students seem to find these more interesting than pulling marbles from an urn. The use of mathematical expectation in decision-making is also emphasized.

The subject of statistics is introduced in Chapter 4 from the standpoint of trying to make sense in human terms of a set of measurements. The normal curve provides the main focus of the chapter.

Chapter 5 is aimed toward making sensible decisions as consumers. The consumer activities involving the most mathematics, borrowing, installment buying, investing, and annuities, get an especially thorough treatment here.

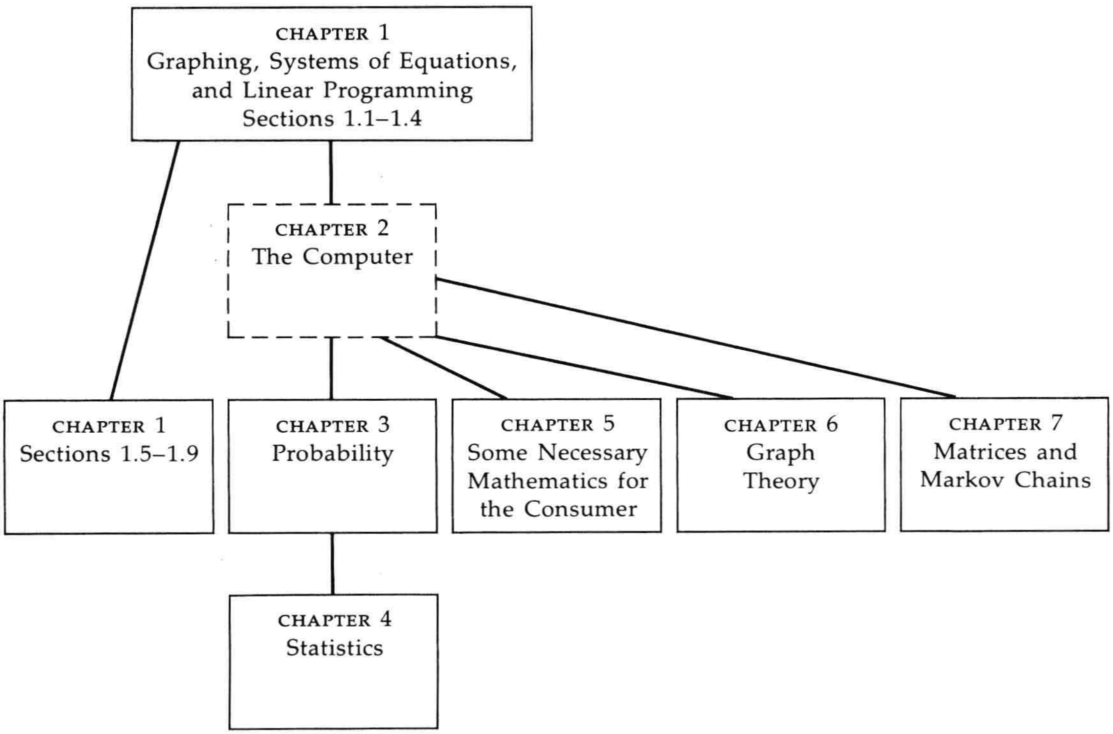
One of the rapidly expanding and increasingly important areas of mathematics furnishes the material for Chapter 6. Barely 200 years old, graph theory is finding its way into the solution of an ever-increasing variety of real-world problems. Our main intent is to get students to recognize when a graph is an appropriate model for a situation rather than to present technical theorems of graph theory. The chapter ends with the PERT method for scheduling complex projects.

The final chapter introduces matrices, one of the most practical tools of modern mathematics. We give applications to inventory matrices, solving systems of linear equations, and Markov chains.

Traditional mathematics is not neglected, for example, graphing (in linear programming and statistics), exponents (in probability and interest problems), and solving equations (in linear programming and Markov chains). Yet the emphasis is never mathematics for the sake of mathematics, but always solving some problem of the outside world. For example, a straight line is first graphed not to picture an arbitrarily presented equation but to show how a couple's federal income tax depends on their taxable income.

We have found that there is ample material in this book for a one-semester course. Thus the instructor may wish to skip some chapters or even rearrange them. The chapters are for the most part independent, as the chart indicates.

Chapter 2 is optional. However, if it *is* covered, it is preferable to



do so directly after Chapter 1 so that the computer may be used as a tool in later chapters.

Throughout this book the mathematics background assumed is kept to a minimal level; Chapters 2 and 6 in particular might form a good introduction to the course.

It is a paradox that at a time when the influence of mathematics in all phases of life has grown and is growing enormously, the understanding of how mathematics is *applied* seems to have decreased, even among college-educated people. We hope that this book can play a part in remedying this situation.

L. C. E.
C. L. V. E.

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GRAPHING, SYSTEMS OF EQUATIONS, AND LINEAR PROGRAMMING

CHAPTER ONE

1.1 MATHEMATICAL STATEMENTS

We live in a mathematical world. If this is not generally recognized, it is because many people are not sensitive to the mathematical relationships around them.

A large newspaper may easily contain as many quantitative statements as a mathematics text. Consider the first two paragraphs of an article by Gene Salorio, headlined "Pumpkin Prices Climb," appearing in the *New York Times* of November 11, 1973*:

> Like turkeys, the pumpkin business reaches its peak at this time of year, with millions of Thanksgiving and Christmas turkeys followed by pumpkin pies. Those pies will, however, be a bit more expensive this year.
>
> Pumpkin farmers are receiving about $12 a ton, up 20 percent from last year's price and 50 percent from that of 1971. During the preceding two decades the price had remained stable at $8 per ton.

A number of precise mathematical statements are made in these two paragraphs, but because words rather than symbols are used to express them, the reader may not realize this. In the first paragraph, for example, we learn that pumpkin pies will cost more in 1973 than previously. Let us denote by C_3 the average cost of a pumpkin pie in 1973, and let C_2 denote the corresponding cost in 1972. Then we could express the fact of rising pie cost by

SUBSCRIPTS
The 3 in C_3 is a *subscript,* and C_3 is read "C-sub-three," or, more briefly, "C-three."

$$C_3 > C_2.$$

Here $>$ is the usual mathematical symbol for "is greater than."

In the same way we translate "Pumpkin farmers are receiving about $12 a ton, up 20 percent from last year's prices and 50 percent from that of 1971" into mathematical symbolism by letting P_3, P_2, and P_1 be the price per ton of pumpkins to farmers in 1973, 1972, and 1971, respectively, and writing

PERCENT
"Percent" means "hundredths," so 20% of the 1972 pumpkin price P_2 is $\frac{20}{100}$ of P_2, or $0.20P_2$. If prices are *up* 20% from 1972, the new price must be $P_2 + 0.20P_2$.

$$P_3 = \$12,$$

$$P_3 = P_2 + 0.20P_2,$$

and

$$P_3 = P_1 + 0.50P_1.$$

We do not claim, of course, that our symbolic formulations are better than the ones we started with. It depends on how they are to be

*An article by Gene Salorio headlined "Pumpkin Prices Climb," appearing in the *New York Times* of November 11, 1973. (© 1973 By the New York Times Company. Reprinted by permission.)

used. For a newspaper article the first paragraph of Mr. Salorio's account is certainly better than

> PUMPKIN PRICES CLIMB
> Let C_2 and C_3 denote the average price of a pumpkin pie in 1972 and 1973, respectively. Then $C_2 < C_3$.

For the purpose of further mathematical analysis, however, the latter formulation may be better.

There are various advantages mathematical language may have over plain English. One is that the *conciseness* of mathematics allows the relationships expressed to be understood at a glance. A second is that mathematics has a *precision* that ordinary language often lacks. A third is that mathematical expressions can be *manipulated* (by algebra, for example) to derive facts not at first obvious.

We need to be able to translate ordinary sentences into mathematical symbolism. Let us continue to do this with the pumpkin article.

The statement that pumpkin prices to farmers held steady at $8 per ton during the two decades preceding 1971 might be expressed as follows. Let $P(n)$ be the price per ton farmers got for pumpkins in year n. Then $P(n) = \$8$ for $n = 1951, 1952, \ldots, 1970$.

The article continues:

> "All my expenses are up, and tonnage per acre is down to 10 or 12 where it usually runs between 15 and 18," declared Robert Robson, a pumpkin farmer in Geneva, N.Y., a major growing area.

Let Y be the 1973 pumpkin yield of Robert Robson, in tons per acre, and let U be his usual yield. Then

$$10 \leqslant Y \leqslant 12$$

and

$$15 \leqslant U \leqslant 18.$$

> Despite higher prices, growers are finding a brisk market. The pumpkins, harvested during the past month, go principally to bakeries and canneries, although Mr. Robson estimated that 10 percent of his crop became Halloween jack-o'-lanterns.

Let B be the part of the 1973 pumpkin crop going to bakeries and canneries, let J be the part becoming jack-o'-lanterns, and let T be the total crop. Then

$$B > \tfrac{1}{2}T,$$

and

$$J = 0.10T.$$

It should be noted that not all the mathematical statements we found above were equations. Although the idea is common that mathematics consists chiefly in manipulating equations, this is not the case. Other symbols that find frequent use are $<$ ("is less than"), $\leqslant$ ("is less than or equal to"), $>$ ("is greater than"), and $\geqslant$ ("is greater than or equal to"). For example, the following statements are true:

$$3 < 5,$$

$$3 \leqslant 5,$$

$$3 \leqslant 3;$$

while the following are false:

$$3 < 3,$$

$$5 < 3,$$

$$5 \leqslant 3.$$

INEQUALITIES IN PLAIN ENGLISH

There are many ways of expressing an inequality in the English language. For example, each of the following statements amounts to $A < B$:

A is less than *B*
A is smaller than *B*
B exceeds *A*
B is more than *A*
B is in excess of *A*
B is greater than *A*

Likewise $A \leqslant B$ might be expressed as

A is less than or equal to *B*
A is at least as small as *B*
A is not greater than *B*
A does not exceed *B*
A is not more than *B*
B is greater than or equal to *A*
B is not less than *A*
B is not smaller than *A*

A statement like

$$J = 0.10T$$

is meaningless in itself, that is, without any explanation of what J and T represent. In our examples we tried to choose letters suggesting the quantities they represented, for example, T for the total pumpkin crop and J for the part of the crop going for jack-o'-lanterns. Although this is not necessary, it is often helpful for remembering the meaning of the symbols.

Exercises 1.1

Express symbolically the quantitative information given in each problem. In the first eight problems use the letters given to denote the various quantities. In the remaining problems invent (and explain) your own notation.

1. "At Filasky's [Brookville, L.I.] roadside stand pumpkins are selling for 12 cents a pound, the same price as last year." [From the same article.] Let F_3 be Filasky's 1973 price per pound for pumpkins, and let F_2 be their corresponding 1972 price.

2. In 1960 hurricane Donna swept the entire U.S. Atlantic coast. High wind speeds of 121 and 130 miles per hour were recorded in Ft. Myers, Florida, and Block Island, R.I., respectively. The storm caused 50 deaths in the United States and an estimated $150 million damage.

Let Y be the year hurricane Donna hit. Let FMW and BIW denote the high wind speeds recorded from it in Ft. Myers and Block Island, and let d and D represent the number of deaths and value of damage due to the hurricane in the United States. (Here FMW and BIW are to be considered as single symbols, not as F times M times W, for example. Although such multiletter symbols are not common in traditional mathematics (except as subscripts), they are often used in computer programming.)

3. In July 1967 there were 97,945,000 men and 101,173,000 women in the United States. The median age of the men was 26.4, of the women, 29.0.

Let M and W be the number of men and women in the United States in July 1967 and let A_M and A_W be the median ages of these two groups.

4. In England and Wales there are over 800,000 births every year, of which 500,000 occur in hospitals. In the United States, where an even greater proportion of births take place in hospitals, there are nearly 4 million births a year, and more than 100,000 women in maternity wards on any given day.

Let B and I denote the number of births and births in hospitals in England and Wales, and let B' and I' denote the corresponding numbers in the United States. Let n denote the number of women in maternity wards in the United States on an arbitrary day.

5. The diameter of the planet Jupiter is 11 times the diameter of Earth, while the mass of Jupiter is more than 300 times Earth's. Let D_J and D_E be the diameters of Jupiter and Earth and let M_J and M_E be the masses of these planets.

6. In 1974 the suicide rate in the United States (per 100,000) was 12.2. This was more than twice that of Italy but less than one-third the rate for Hungary. Let the suicide rates for the United States, Italy, and Hungary be denoted by SUS, SI, and SH.

7. In 1969 the U.S. county with more than 50,000 population with the highest median family income was Montgomery County in Maryland. The median family income there, \$16,710, was more than \$1000 more than that of the runner-up, Fairfax County, Virginia. Let M and F be the 1969 median family incomes in Montgomery and Fairfax Counties.

8. The total number of people in the regular military forces of the United Kingdom is more than four times the figure for Canada, which is 80,000. The forces of the United Kingdom and Canada combined are still less than those of France, 502,100. Let UK, C, and F represent the number of people in the forces of the United Kingdom, Canada, and France, respectively.

9. The largest city and capital of Paraguay is Asunción, with a population of 305,160, according to a 1962 census. The population of the whole country was estimated to be 2,161,000 in 1967.

10. Roger Bannister of Britain became the first man to run the mile in less than 4 minutes in 1954, when he ran it in 3:59.4. The previous record was 4:01.4, run by Gunder Haegg of Sweden in 1945. The record time in 1864 was 4:56, held by Charles Lawes of Britain.

11. The average (adult) human eyeball weighs about $\frac{1}{4}$ ounce and is about 1 inch in diameter. The average male eye is about $\frac{1}{50}$ inch larger than the average female eye. About $\frac{1}{4000}$ of the weight of an adult is eyeball; in a baby the ratio is $\frac{1}{400}$.

12. In September 1969 production workers in manufacturing in the United States earned an average of \$3.24 per hour. The average price of a pound of American Cheddar cheese at that time was 95.4¢, for which an average worker would have to work about 18 minutes. In about 11 minutes he could earn enough to buy a package of 48 tea bags.