Basic Concepts of Mathematics

BUSH and OBREANU

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Preface

The flood during the last few years of elementary college mathematics texts with titles containing the words "basic," "fundamental," "modern," or "abstract" is a clear indication of the shift in emphasis in the undergraduate program in mathematics. These books serve various purposes and require various backgrounds. Since there is such a variety of content despite the similarity of titles, it is in order to give some explanation of the content of the present work.

The material presented has been developed from a course that was originally designed for first-year students in mathematics or physics who had taken "college" algebra, two-dimensional analytic geometry, and plane trigonometry in their last year of high school and that was then broadened to include a much greater variety of students. The course has also been taught in the summer school to a class consisting largely of teachers who were seeking a higher professional classification.

The purpose of the book is to lay a suitable foundation for later mathematics courses. Accordingly we have dealt with the basic concepts of a number of branches of mathematics rather than developing any one of them in great detail. While the scope is broad, the language of sets, relations, and functions provides a unifying thread.

We have departed from the current trend to start such a study with the propositional calculus and truth tables. We have chosen instead to give an informal discussion of the logical arguments used in mathematics. We believe that this is more meaningful and more important to the student at this level.

Although we introduce the concept of the quotient set, we do not make extensive use of it since experience has shown that this is a difficult idea for the student to handle. For this reason we do not discuss quotient groups or quotient rings. We also choose to introduce real numbers as infinite decimals, a procedure that has intuitive advantages for the student, although it may seem cumbersome to the expert.

The concept of a vector is introduced geometrically. We chose this approach rather than using ordered triples because it has more intuitive appeal, especially in the application of vectors to problems in geometry, even while we recognized the disadvantages from an axiomatic point of view since we cannot assume that the student has a proper axiomatic basis even for two-dimensional geometry.

In Part Four, we have emphasized the conceptual rather than the computational aspects of the calculus. This emphasis is in keeping with the rest of the material; a mathematics major would receive a course concentrating on the calculus and would need only a brief indication of the set theoretical background of the subject. Thus, Part Four is not an attempt to duplicate a calculus text.

In several places, particularly in examples, we have depended rather heavily on the reader's intuition. We have chosen this rather than a more rigorous presentation when we felt that it would aid the student's understanding of the material.

The book could be covered entirely in three semesters, or it offers a variety of two semester courses. We have used the material of Parts One. Two, and Three since our students were taking a concurrent calculus course, The first seven chapters are prerequisite to the later chapters (although they could be abbreviated). After Chapter 7 it is possible to begin at Chapters 8, 10, or 12. Some of the results in Parts Three and Four are stated in the language of groups and rings, but these require only the definition of these structures.

A variety of exercises is included. In many of these the emphasis is on the proof and so no short answer can be given. Answers to about one half of the exercises that have short answers are given under the heading "Selected Answers to Exercises."

We are indebted to a number of people for their help in writing this book. Professors E. Hewitt and B. W. Jones read the manuscript in detail and offered valuable criticisms. To them we owe our special thanks. Our colleagues who have taught from earlier versions of the work and our students who have studied from it have pointed out various places where improvements could be made; Mrs. E. M. Wight has patiently typed the various versions of the manuscript. Finally, we express our gratitude to our wives who have for many months tolerated our preoccupation with the preparation of this material.

Kingston, Canada
February, 1965

G.C.B.
P.E.O.

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Introduction The Language of Mathematics

As an introduction to our study of mathematics, we shall attempt an elementary analysis of the processes of mathematical reasoning. We shall do this by considering examples from high school mathematics and especially from geometry. Our study is far from complete. The whole problem of analyzing mathematical arguments belongs to the realm of mathematical (or symbolic) logic.

kingdom.

There are two different approaches in teaching mathematics. One, the traditional approach that is usually taught in high schools, attempts to start from definitions of the basic objects studied. The other, more modern, approach accepts certain objects as undefined elements, makes no attempt to define them, and uses them as the basis of a body of theorems and definitions. The second approach is more or less the one used in this book.

These two approaches are well illustrated in the study of Euclidean geometry. Euclid himself, followed by the authors of many high school texts up to the present, defined point and line as follows:

interpreted

"A point is that which has no parts." (1)

"A line is length without breadth." (2)

The student who encounters these definitions might well ask for definitions of part, length, and breadth. Such a quest for ultimate definition is endless.

The purpose of such definitions is to help the student to an intuitive grasp of the subject. He is encouraged to draw from the real objects around him some abstract concept. This is a valid and useful approach. Intuition is required to understand mathematical facts and to discover new ones.

= direct

The more modern approach to mathematics starts from undefined terms and thus avoids two pitfalls. The first is the endless regression from one definition to another, the second is the so-called vicious circle in which two words are defined in terms of each other. It uses intuition and abstrac-

return to

tion from real objects only as motivation and excludes them from the formal statement of theorems. In keeping with this modern approach, we shall make no attempt to define such entities as line, point, or set. We consider such terms *primitive concepts* and give exist. certain statements called *axioms* or *postulates* that express relations among these concepts. One example of such an axion, chosen from Euclidean geometry, is

> "There is exactly one line which passes through two distinct points." (3)

There has been until quite recently a traditional distinction between axioms and postulates. According to this tradition, which grew out of the teaching of Euclid's *Elements*, an axiom was a "self-evident truth" without which thought would be impossible. A postulate was not considered self-evident but was believed to be a fundamental truth about the subject matter being studied, a truth that was beyond question.

By the nineteenth century leading mathematicians abandoned this distinction. They realized that the essence of mathematics is not the pursuit of an absolute truth; in fact the idea of truth based on overwhelming evidence is quite foreign to mathematics. In the new view a mathematical theory is concerned only with proofs of the validity of statements of the form: "If P holds, then Q holds." In constructing such a system we start from a few propositions which we call axioms, but we are no longer concerned about whether these axioms express some truth that is beyond all possible doubt. We shall sometimes state that "A is true," but we interpret this to mean only that A comes at the end of a chain of arguments of the type "If P holds, then Q holds," a chain that begins with one or more of our axioms. This view frees us from the need for a priori evidence for the truth of the axioms. It is no longer claimed that axioms represent the behavior of the physical world.

The present view of axioms has come about as the result of study in various branches of mathematics. The concern about the *parallel postulate* of Euclid (the fifth postulate in the *Elements*) is typical of these studies. In the form given by the Scottish mathematician J. Playfair (1748–1819), this postulate is:

> "Through a given point outside a given line one and only one line can be drawn parallel to the given line." (4)

To the followers of Euclid this postulate seemed less "self-evident" than the others he had stated, and they attempted to find a proof for it. This search for a proof ended when the Russian mathematician N. I. Lobatschefski (1793–1856) constructed a geometry in which the parallel postulate does not hold. This new geometry is one of the class of geometries referred to as *non-Euclidean*. Other geometries of this class have been constructed and provide a model of the physical world that for some purposes is more convenient than the geometry of Euclid.

We shall not spend more time on the history of the axiomatic method, which now permeates all of mathematics. The more familiar we become with the method, the less we feel the need for explicit definitions of the primitive concepts and the more we realize that all we need to know about them is contained in the system of axioms. This book does not completely adopt the axiomatic approach, but Chapters 8 and 9 come very close to it.

We introduce new objects in any mathematical system by means of statements that we call *definitions*. In these definitions we use words that we assume to be already known. Examples of such definitions are the following from geometry.

"The line A is said to be parallel to the line B if A and B have no common point." (5)

"By a segment ab we mean the set of all points of the line passing through a and b and which are between a and b." (6)

The next example is drawn from arithmetic.

"An integer p is called a prime number if its only positive divisors are 1 and p." (7)

If we look at definition (6), we see that it contains undefined words such as "point," "line," and "between." If we are prepared to assume a knowledge of these words as our undefined concepts which have been introduced in the axioms, then (6) is a satisfactory definition of a segment.

Similarly, (7) is a satisfactory definition of a prime number only if we understand the meaning of "integer," "positive," and "divisor."

A definition may be thought of as a description of how to construct a certain new object from known objects and relations. A definition may also be considered to describe a certain object and to assign a name to it. It is natural to ask whether the construction can be carried out or whether an object with the specified properties exists.

We have already seen that, in a certain system where the parallel postulate does not hold, the construction required for definition (5) could not be carried out, because in this system there are no parallel lines.

We shall not pursue the nature of the definition further but shall emphasize that it is possible to give a definition of an object that does not exist. We must be careful to avoid such a mistake. Failure to recognize this point can lead to the development of a theory that is vacuous, in that there are no objects to which it applies.

Perhaps the chief characteristic of mathematics is the fact that, after a list of axioms and definitions have been accepted, there follows a body of other statements called *theorems*. In order for a statement to be a theorem, a *proof* is required. Our confidence in the statement rests upon the proof, not upon the fact that the statement has been discovered to be true whenever it has been tested. Our confidence in the fact that $x^2 + y^2 = z^2$, where x, y, z are the lengths of sides of a right triangle, rests upon the proof of the theorem of Pythagoras, not upon the number of right triangles for which we have measured the sides. In the absence of a proof we may have strong evidence for our belief in a certain statement but we cannot state that it is true. Many theorems have been discovered by observing their truth in special cases, but until a proof is provided they remain only statements that may be true or false. It is the proof alone that elevates them to the rank of theorems.

No one can give an easy means by which to discover or prove theorems. Euclid said long ago to the King of Egypt that there is no royal road to geometry. This is still true, not only of geometry but of all branches of mathematics. Anyone who wants to discover and prove theorems must have as his tools intuition, imagination, and mathematical skill. He must also be prepared for hard work. The same could be said for an athlete—there is no easy way to success. In athletics there are certain rules of good form that are essential. These do not make a good athlete, but their absence can be ruinous. Similarly, in mathematics good form and a sound grasp of fundamentals do not guarantee success, but their absence may cause failure.

With these things in mind we shall examine some of the essential principles of logic that a mathematician or a student of mathematics must have at his command.

I.1 Implication

Many mathematical statements that we shall encounter are of the form "If . . . , then . . . ," where each set of dots represents a statement. For example:

If a and b are two distinct points, *then* there is exactly one line passing through a and b.

If a triangle has two equal angles, *then* it also has two equal sides.

If a prime number p divides the product of two integers, *then* p divides at least one of the integers.

In each case we have started with two statements, say A and B, and have combined them into a new statement: "If A then B." This new statement is called an *implication.* We sometimes say "A implies B" and use the notation $A \Rightarrow B$. This same relationship can be expressed as "A is a sufficient condition for B" or "B is a necessary condition for A." For example:

It is sufficient that a triangle have two equal angles in order that the triangle have two equal sides.

It is necessary that a triangle have two equal sides if the triangle has two equal angles.

The distinction between necessary and sufficient conditions must be clearly understood. We give a nonmathematical example to emphasize that this is a logical, not a mathematical concept.

Since every horse is a quadruped but not every quadruped is a horse, we say that being a quadruped is a necessary condition for being a horse, but it is not a sufficient condition.

We shall use the words "proposition" or "statement" without attempting to define them. For our purposes these are primitive or undefined terms. We expect that the reader will have an adequate intuitive grasp of this concept.

Many statements we shall consider have the form ". . . has the property . . . ," where the first blank is to be filled with the name of some object and the second with a description of some property. We shall also encounter statements such as "Every . . . has the property . . . " and "There exists a . . . which has the property . . . ." We shall not enter into a discussion of truth and falsehood but merely point out that, as the student already knows, these propositions may be either true or false depending on what is written in the blanks.

The composition of two propositions, which we call implication $A \Rightarrow B$, is a new proposition that states that, whenever A is true, then B is also true. Therefore if we know that the proposition $A \Rightarrow B$ is true and the proposition A is true, then the proposition B is also true. Such reasoning is perhaps the most important type in mathematical proofs. It is referred to as the *rule of detachment* or *modus ponendo ponens*. This phrase can be translated approximately as "the rule of establishing by establishing," an appropriate name because, by establishing A, we establish B, provided of course that we have established "If A, then B."

The implication $A \Rightarrow B$ is true if, whenever A is true, then B is also true. It does not put any restriction on the truth or falsehood of B in the case when A is false. We sometimes express this by saying that a false statement implies any statement. Students frequently have difficulty at this point. We shall give two examples in an attempt to clarify the idea. The first is nonmathematical, the second mathematical.

Suppose that a person says, "If an event A takes place, then I shall perform the act B." If A does not take place we have no basis for charging the person with breaking a promise whether or not he performs B.

Consider the implication, "If 5 divides 27 then this polygon is regular." This is a true statement because in every case for which 5 divides 27 (which is never) the polygon is regular. There is no case in which A is true and therefore there is nothing to check about the truth or falsehood of B when A is true.

I.2 Converse of an implication

Consider an implication $A \Rightarrow B$. By the *converse* of this we mean the proposition $B \Rightarrow A$. For example, we have the implication: "If $\triangle$ *DEF* is equilateral, then $\triangle$ *DEF* is equiangular."