MATHEMATICS

concrete behavioral foundations

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To
Jeanne, Janie,
Joey, Julie

To the Instructor

The instructor who is entirely satisfied with the text he is presently using is not likely to adopt this one. If you fall into this category, you might just as well close the book now and save yourself the trouble of reading further. On the other hand, if you have been concerned that students who have studied your present text still don't understand what mathematics is all about (although they may know a fair amount of detail and/or new vocabulary), or cannot relate the mathematics they have learned to the real world, or have little feeling for the wide variety of behavior which knowing mathematics makes possible, then, under these conditions, it may pay you to look further.

This book has been designed as an intermediate level text on mathematics for elementary and middle school teachers, and may also be used in terminal courses for liberal arts students or as a supplement in courses for secondary school teachers. It would be appropriate as a first course in some colleges and universities, as it has been used at Pennsylvania, or as a first *year* course, or as a second semester (year) course in other colleges. Our experience with preliminary versions of the book shows that it is especially well suited for use by experienced teachers, and the book could very profitably be used either as a graduate text or for in-service teacher training courses.

The emphasis throughout the text is on building a cohesive and concrete mathematical edifice. Even where specific topics are being covered, a concerted attempt is made to show how these topics fit into the overall conception. Several techniques have been used to help accomplish this. First, basic mathematical ideas recur in new guises throughout the text. Sets, relations, operations, systems, and so on, are not only treated as important concepts in their own right, but are used throughout the text to help unify the treatment. Second, where possible, as in most of Part III, the chapters have been constructed in parallel fashion. It has been our experience that this makes it easier for the student to see the similarities as well as the differences which exist among number systems. In addition, where parallelisms seem to break down (or become cumbersome), I have tried to indicate why. Third, explicit attention has been given throughout to the relationship between mathematical ideas and reality. In fact, the book is unique among full-length texts in the degree of emphasis given to concrete embodiments of mathematical ideas. We hear much about relevance today, and students and teachers alike are no longer as enamored by "modern mathematics" as they were even a few years ago. If we are to keep them from "throwing out the baby with the bath water," we as mathematics educators are going to have to show them why modern mathematics is important. This text is designed to help make this possible. Fourth, although the emphasis is on mathematical content, the book is not limited to just that. Chapter 1 contains a unique analysis of the processes (e.g., detecting regularities, making inferences, etc.) which, although not normally considered to be part of mathematical content, are extremely crucial in *doing* mathematics. In many ways, this chapter sets the tone for much of the rest of the book. It illustrates the advantages, especially for teachers, of thinking about mathematics in terms of the kinds of behavior which knowing mathematics makes possible. This point of view recurs throughout the book and helps provide further insight into both the mathematical ideas themselves and how they might be taught to students. To take full advantage of these insights, methodological questions are included in most of the exercises.

The book requires very little in the way of prerequisite mathematics, but it does require that the student be able to read precise college level prose. Mathematical notation is used sparingly and only where necessary for clarity and precision of meaning. Many examples are given and the reader is asked questions at many points in the text. The latter are designed to elicit the student's involvement and to help ensure that he thinks about what he is reading and does not gloss over important details.

The text is divided into three parts. Part I (Chapter 1) on objectives and mathematical processes is the first concerted attempt to deal systematically at an elementary level with those basic mathematical competencies (processes) which are invariably involved in performing mathematical tasks but which are rarely made an object of explicit study.* This chapter has been singled out by many of our students as one which they have found most useful in their teaching. It has also been used with secondary school mathematics teachers with equal success. In addition, the point of view may be useful to the liberal arts student by providing him with a broader perspective on mathematical thinking.

Part II (Chapters 2–4) is designed to introduce the reader to many of the major ideas in contemporary mathematics. Chapter 2 deals with those general mathematical ideas which are basic to essentially all of mathematics (i.e., sets, relations, and operations) and emphasizes the similarities and differences both within and among them. The treatment is more complete than in most texts at this level. Chapter 3 deals with logic and deductive reasoning and shows informally how the set operations (i.e., Boolean algebra) provide a model for the statement logic. Chapter 4 introduces mathematical (algebraic) systems and embodiments, and theories, as the basic objects of mathematical study. The emphasis throughout this part is on the concrete reality of the mathematics in the lives of both the reader (elementary school teacher) and her students. The

* Among the other treatments presently available, the monograph *Mathematics in the Primary School* (Melbourne: Macmillan, 1966) by Z. P. Dienes is particularly recommended.

treatment has a number of other features which are not common at this level. These features include (1) the central role played by sets, relations, and operations in organizing content, (2) relationships between logical reasoning and the real world, (3) an introduction to relationships which may exist among different mathematical systems, and (4) a sharp distinction between semantics (systems) and syntax (properties of systems, or theories).

Part III deals systematically with the more traditional topics on number systems, beginning with the natural numbers and building up through the positive rationals and integers to the rationals, reals, and further extensions. Although the main approach is constructive, the number line is introduced as an alternative in discussing the positive rationals and integers, and is used heavily in discussing the reals in Chapter 9. In addition to an emphasis on the concrete foundations of the subject, the various chapters (particularly Chapters 7 and 8) are explicitly constructed so as to parallel one another. This procedure provides increased insight into the relationships that exist both within and between the different number systems, and at the same time makes possible a more sophisticated approach than would otherwise be possible. In line with the difference between denoted entities and descriptions (introduced in Chapter 1 and continued in Part II), a sharp distinction is made throughout between numbers and number systems, on the one hand, and numerals and systems of numeration, on the other. Chapter 9 ends by describing a basic relationship between algebra (i.e., number systems) and geometry, and in the process introduces the idea of a limit as a prelude to analysis.

Although there is little geometry in the text itself, it is too important to leave out entirely, and Appendix A provides a short, self-instructional introduction to geometry which parallels a treatment developed by the School Mathematics Study Group for primary school teachers. Rather than using a text format, the material has been organized by listing the specific geometric tasks, which the reader is expected to perform, and then describing a rule for solving such tasks, together with examples.

Appendix B will also be useful to teachers and prospective teachers, because it is based on an analysis of ten elementary school textbook series in mathematics. This research was conducted under my direction by members of the Mathematics Education Research Group (MERG) under contract with Research for Better Schools, U.S. Regional Educational Laboratory, Philadelphia, Pennsylvania.

In order to facilitate use of the text, an answer book with many worked-out solutions is available on request from the publisher. (Answers to the odd-numbered exercises, together with many detailed solutions, are included at the end of this book.) In addition, a workbook has been designed to directly parallel the text. In this workbook, many of the key ideas have been identified and formulated as specific tasks to be performed. The reader may learn to solve these tasks either by reading (and interpreting) explicit rule statements for solving each kind of task or by inducing the underlying rule from the examples provided. Numerous exercises are also included for each kind of task. The approach used is based on a theory of mathematical knowledge,* which goes beyond the usual behavioral objectives point of view, and makes it possible to build transfer potential directly into the workbook. In particular, the workbook includes tasks which may be solved by combining in predetermined ways rules that have been learned earlier. This workbook has been evaluated empirically as part of a doctoral dissertation by Walter Ehrenpreis and the results unequivocally support the "higher order" form of analysis proposed in the aforementioned theory.

The selection of exercises has also been facilitated by keying them as follows: *Section* exercises are prefaced by an *S* and are based directly on the material in the particular section involved. *Extra* exercises are prefaced by an *E* and require information from other chapters or elsewhere. (This information may or may not be readily available to the student.) *Methods* exercises are prefaced by an *M* and deal with methodological issues involved in teaching the content in question. Exercises marked with an *asterisk* are more difficult than the others and require more ingenuity on the part of the student. *Brackets* indicate that the question deals with one or more of the ideas introduced in Chapter 1. Frequently these ideas do not affect the working of the problem, and the brackets are included merely to help make the student more aware of how the fundamental processes identified in Chapter 1 enter into *doing* mathematics. Finally, most sections have an additional exercise or two for *further thought and/or study*. In writing many of the exercises, a conscious effort has been made to supplement, as well as simply reflect, the text. I hope that the instructor will find these exercises, and the solutions provided, a valuable aid in promoting student learning as well as in testing.

The text may be used in a variety of ways depending on the instructor's interests and the needs of the students. In particular, the three parts of the book have been written so as to be largely independent of one another and the text can be entered equally well at Chapter 1, Chapter 2, or Chapter 5. In addition, Chapters 2 and 3, 3 and 4, and 7 and 8, respectively, may be interchanged (except that 2 should come before 4). It would be quite feasible, for example, to start with Chapter 5 of Part III, and then alternate between chapters in Parts I and II (Chapters 1–4) and those in Part III. Changes should not be made indiscriminately, however, and without good reason.

The book also lends itself to individualization in the sense that the development tends to parallel the order in which various topics are introduced in the elementary school. Thus, preschool and primary school teachers (N, K–2) may be allowed to concentrate on Chapter 5 and the initial parts of Chapters 6–8 (on the nature of numbers and numerals), together with the relevant parts of Parts I and II. Teachers in the upper elementary grades will correspondingly want to spend relatively more time on the algorithms (Chapter 6) and the body of Chapters 7, 8, and 9.

In teaching undergraduate elementary school teachers in training, we have

* J. M. Scandura, *A Theory of Mathematical Knowledge: Can Rules Account for Creative Behavior?* Mathematics Education Research Group, Report No. 52, Structural Learning Series, University of Pennsylvania, Philadelphia, Pa. 19104 (also in *Journal for Research in Mathematics Education*, in press).

found that it makes an important difference whether the material is presented before, during, or after practice teaching. If presented before, or particularly after, the indicated order is just as good as any other. For students who have never taught or observed, however, it is sometimes useful to postpone the introduction of Chapter 1 until after they have covered Part II (or Part III). If the course is taken at the same time as practice teaching, it is frequently useful to get into Part III, particularly Chapters 5–7, as soon as possible. The students are frequently fearful and concerned more with what they perceive as immediate matters—for example, the more traditional topics on arithmetic which they see in the children's texts—than with topics which the instructor may deem more crucial and of long-range benefit. Under these circumstances, it is better to let the students get their feet wet in practice teaching, and thereby get a feeling for what is important in the school setting, before getting into Parts I and II. Nonetheless, the ideas inherent in these chapters can be profitable and should be introduced implicitly in discussing Part III. For example, the question "How would you teach this?" immediately raises the question of whether to use examples (i.e., teach by discovery) or exposition. Students who have had to face up to this kind of question find the discussion in Chapter 1 particularly relevant and useful.

In graduate courses for experienced teachers, just the reverse has been true. Teachers take very naturally to the ideas discussed in Parts I and II. This, in turn, tends to provide them with a useful background for increasing their depth of understanding of the more traditional topics discussed in Part III. Parts of Chapters 3 and 4 are somewhat more advanced than the rest (e.g., the discussion of homomorphisms) and can be made optional. However, many teachers who do succeed in mastering these ideas have been most anxious to try them out with their pupils. (We have been pleased with the many success stories we have heard from teachers who have tried this.)

Chapters 5 and 6 tend to go rather smoothly. Special care should be taken in introducing Chapters 7 and 8, as the approach is generally unfamiliar to most teachers. The basic arithmetical operations in these chapters are defined first and then their concrete realizations are discussed. Although contrary to the order preferred by many instructors, this tended to facilitate the discussion as well as make it more precise and efficient. Furthermore, there is some research by Suppes and Dienes which suggests that this order of presentation may in the long run be the most efficient. The instructor, of course, is free to use whichever order he deems best for his classes.

It is also worth noting that the parallel construction of Chapter 7 and Chapter 8 tends to make things get progressively easier for the student as he goes along, even where he may have been confused at the beginning. The student should be made aware of this and provided with encouragement where needed. Chapter 9 is more compact than the others and is included largely as icing on the cake for the better student. The instructor should feel free to pick and choose from this chapter as he sees fit. The introduction and summary sections are particularly recommended, as these provide the student with a useful overview of the different number systems. The treatment of the rationals, and particularly the reals, is more thorough than in other books written for the elementary school teacher.

We have found that both pre- and in-service elementary school teachers are well able to study Appendixes A and B independently. With preservice teachers, however, it will generally be useful to discuss Appendix B in class (in order to supplement the views expressed there with your own).

In a terminal course in mathematics for liberal arts and junior college students, the text may be used as is, although the instructor may wish to postpone the introduction of Chapter 1 until after Part II has been completed. In addition, of course, he may want to ignore the material pertaining to methodology. This can be accomplished by simply eliminating those exercises marked with an *M*. It might be possible to cover most of the book in one semester in some colleges and universities, but in most, and certainly in junior colleges, there is more than enough material for a full year. Parts I and II, together with sections from Part III, constitute a particularly appropriate curriculum for a one semester course for liberal arts students.

With secondary school teachers, the text is probably best used as either a supplement or as one of two texts. Chapter 1 should be covered thoroughly and the students required to make up examples of their own for each of the basic processing skills identified. The remainder of the book can be covered more rapidly than with elementary school teachers, but we have been continually appalled by how much of the material even our better students do not know. Coverage of this content is particularly important for the junior high school teacher (who frequently has to be an expert in remedial mathematics). Mathematics majors can generally cover most of the text (including Chapter 9) in about half a semester, leaving the remainder for other purposes.

Although the text is entirely self-contained, it has been our experience that the major ideas in the text can be made available to even the weakest student by carefully coordinating the use of the text with the workbook (description above) which has been prepared for this purpose. This workbook has been designed to parallel the text closely and can be used either to introduce the key topics before reading the text or as highly directed practice which can be accomplished after the student has read the text, possibly in lieu of the text exercises. The instructor may want to suggest both alternatives to the students and let them select the way which seems to best fit their own learning style.

It is an open secret that no book is the product of simply one man. This book is no exception. In writing the book, I have had more than a little assistance from a number of members of the Mathematics Education Research Group at the University of Pennsylvania. Joanna Burris, Walter Ehrenpreis, and Judy Gera assisted me when I introduced many of the ideas for the first time in a class for elementary school teachers. Later during a sabbatical year, John Durnin assisted me with the book in any number of ways. His help ranged from pulling together needed materials, to administrative assistant, to proofreader par excellence. Furthermore, I incorporated his comments, criticisms, and suggestions on more than one occasion and the book is a far better product for it. This help came at a most appropriate time, as I was then also engaged in writing my research monograph on *Mathematics and Structural Learning* (Englewood Cliffs, N.J.: Prentice-Hall) and a number of research articles. Upon my return I received the ready assistance of a number of my students in fine-tuning the text and in writing exercises and answers. George Lowerre, Louis Ackler, Julia Gatter, Sister Jeannine Grammick, Julia Hirsch, George Luger, and Christopher Toy all played important roles in this process. I must single out George Lowerre, however, for the special role he played in collating, criticizing, and editing both the exercises and the comments on the text itself. Both Durnin and Lowerre also assisted me later in going over the edited manuscript and in reading galleys and page proofs. It is for these reasons that I have singled them out by acknowledging their help on the title page: "with the assistance of." In addition, Walter Ehrenpreis provided substantial assistance with Appendix A, Gerald Satlow with Appendix B, and Linda Hunsicker with the index.

In spite of all the help I have received, however, I must bear full responsibility for the contents. I planned and wrote the book, I decided what kinds of exercises to include, and I decided what changes if any were to be made and how. I readily acknowledge that whatever limitations still remain in the text may well be due to the many suggestions or comments made that I chose to ignore.

In addition, I want to thank those of the Harper & Row staff who have been most helpful during the final phases of the work. Blake Vance, the editor, and Tony Asch, his field assistant, played no small part in my decision to publish with Harper & Row. I only hope that their trust in me and the purposes of this book prove justified. Karen Judd and Susan Emry of the Harper production staff, who labored so long over the manuscript itself, made my job much easier than it would otherwise have been.

Credit is also due the typists who worked so long and so hard. Mrs. Mary Tye did such a fine job for me in California that I still frequently send her things to do all the way from Philadelphia. Katherine Whipple and Lee Carvalho did an equally competent job in making the final corrections.

Last, but not least, I would like to express my deep gratitude to my parents and to my wife, Alice, and my children for their support while I was writing the book. During this period, as well as throughout much of my professional career, my wife has borne the burden of most of the day-to-day problems of raising a family of four children and has left for me most of the joys.

I shall feel more than repaid for the effort that went into this book if mathematics teaching in some small way may benefit as a result. Since my children would be among the beneficiaries of such improvement, it is to them that I dedicate this book.

JOSEPH M. SCANDURA

To the Student

The book you are about to study is different from most college mathematics texts. To be sure, it has its so-called formulas and symbols but it has fewer of them than is common in most books which cover the same content and in the same depth.

Although much of the book is written in ordinary prose, you must not be deceived by this. As with any mathematics book it is not possible to skip sentences, phrases, or even words without taking the chance that you will overlook something important.

Even more important than reading the material carefully is that you *think* about what you read. After you read through a paragraph or short section, or even a particularly meaning-filled sentence, ask yourself what it means before going on. Try to visualize the idea. Ask yourself questions about the material. Or, rephrase the material in your own words. These are all useful techniques for making sure that you know exactly what is being said. To help you in this regard, numerous examples are included and questions are interspersed throughout the text. It is important that you check these examples and answer the questions.

In studying the text, you should also have a pencil and paper in hand so that you can work things out on paper where necessary. Remember, even professional mathematicians do much of their hard thinking on paper. Don't try to do it all in your head. It is simply not possible. (Of course, many questions will be easy enough to answer in this way—and, if so, you will know it.)

One final point on the text: The exercises are *not* optional. Full understanding is rarely achieved simply through reading. It is absolutely necessary to also work enough of the exercises to get a feeling for how the various ideas apply, how they interact, and when to use them and why.

If you have the workbook as well as the text, there are several ways in which you might use them. Try different alternatives until you find out which one best fits your own learning style. If you are are a good reader, for example, you might want to first read the text somewhat more rapidly than you otherwise would. Then, go through the workbook and exercises. Finally, you might pick out and work those text exercises that still seem novel to you. (You can check the answers to the odd-numbered exercises at the end of the text. Many solutions are worked out in detail for your benefit.) If you are an average reader and feel your mathematics background to be particularly weak, on the other hand, you might want to go through each section in the workbook to firm up the key ideas before going to the text to see how they all fit together. As in the other technique, you would then work those text exercises which still appear novel. (In using the workbook, some students find it best after reading each task to go directly to the examples rather than to first read the corresponding rule.)

Before you begin, you may want to browse through the table of contents to get an overview of the course. For more details, you might even want to peek at the preface for your instructor.

Good luck on your journey. I hope you will enjoy studying this book. Even more, I hope you will profit from it.

JOSEPH M. SCANDURA

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* indicates optional section