THE ELEMENTARY FUNCTIONS

An Algorithmic Approach

G. Albert Higgins, Jr.

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G. Albert Higgins, Jr.

Formerly of
Kiewit Computation Center
Dartmouth College

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Preface

Recent years have seen changes in mathematics education at all levels. As a result, the need to revise methods for the learning and teaching of mathematics is continual. For example, the presentation of algebra, as well as geometry, from a deductive viewpoint insures a command of the notion of sets, logic, and the structure of the real number system which was absent a few years ago. Most recently the advent of computing in secondary and pre-secondary schools as an adjunct to course work, or as an extra-curricular vehicle, provides students with a significant extension of traditional methods of problem solving.

Presently, many texts are available which offer a wealth of programming suggestions and techniques. In general, these are suitable for reference with another text, or as texts for introductory computer science courses. It is the purpose of this text, however, to present a course in Elementary Functions of which the use of the computer is an intimate part. It assumes that the student will have:

(a) knowledge of BASIC or another simplified programming language; and,

(b) access to computing services on a regular basis.

The treatment of elementary functions is organized and presented in a fashion to reap the benefits of the computing ability of students as well as to give the student a sounder understanding of elementary functions. Recent CUPM recommendations for the pre-calculus course in mathematics have been considered in the choice of topics. It is suitable for use in first-year college courses in Elementary Functions or for advanced courses in secondary school where a computer is available.

An intuitive geometric point of view is adopted in developing the definitions of elementary functions whenever possible. In the introductory chapter on functions, for example, a special class of functions, which we call area functions, are introduced to help with the student's understanding of functions which are not defined by equations. Plane geometry and elementary algebra may be used to determine equations for many of these area functions. At the point where the natural logarithm is to be defined, then, not only does the student know how functions are defined by area, he already is familiar with their properties. This approach to functions should be helpful, not only for the student who goes on to calculus, but also to those headed for probability and statistics.

In the same manner, the study of wrapping functions on commensurable geometric figures presents problems which can be easily solved using geometry and algebra. It is only when the figure about which the wrapping takes place is incommensurable that the problems of measurement associated with the sine and cosine arise. But again, by that time the student knows quite a number of properties of wrapping functions which help with his understanding of the sine or cosine. Additionally, of course, the computer is available for making the necessary rational approximations.

In conjunction with the development of the algebra of functions, several fundamental ideas of calculus are presented intuitively in a *numerical* context: Routines are developed to calculate area, slope, maximums, and minimums, and to solve for irrational zeros. These routines are used judiciously throughout the text to provide the student with a "calculus" of his own so that the function may be studied in a sounder pedagogical order than is usually possible. A strong side effect of the approach is that when the student studies the calculus formally for the first time, he will be able to concentrate on the difficulties posed by its notation and formalism as he will have already developed an intuition for the concepts.

In addition to this dependence upon the numerical mathematics, the text departs from the conventional approach in another way. The student is provided with a substantial introduction to the historical origin and development of the functions. Further background is suggested through references. It has been the author's experience that students want to know more than a logically clean presentation and that their motivation is higher when more background is provided. Specifically, each chapter starts with a footnoted Historical Note, and many exercises have been included for amplification. Most must be solved by the computer, and all are highly recommended as they help illuminate the significant way in which the several functions studied are related or have been related from their inception. Historical exercises are marked with an H.

Programs used in the text are written in BASIC, although knowledge of any other programming language will enable students and teachers to understand the examples, due to the simplicity of BASIC. For those not familiar with programming, the Bibliography lists several introductory references. The Instructor's Commentary for the course will include flow charts and programs where appropriate. The computing exercises may be solved using any programming language and computer system, of course. All programs listed have been run at the Honeywell 635 System at Dartmouth. They also have been tested by Ann Waterhouse and her students on a Digital Equipment Corporation Time-Shared 8 System, as well as the DEC RSTS-11 System at Northfield Mount Hermon School.

The author wishes to acknowledge his indebtedness to many people, particularly for the support of Thomas E. Kurtz, Director of Kiewit Computation Center, who has supported and encouraged him since 1966. Without the interest of Arthur Wester, Mathematics Editor at Prentice-Hall, Inc. and Dr. Laurence Binder of the National Science Foundation, the book would not have gained broad circulation. I am also indebted to John G. Kemeny and Donald L. Kreider and their colleagues in the Mathematics Department at Dartmouth College and the staff of Kiewit Computation Center for their many courtesies and suggestions during my two visits to their staff, in the summer of 1967 and during the academic year 1969–1970. Robert P. Weis, my colleague at Northfield Mount Hermon School, has made many helpful suggestions based upon two years of teaching the text in its Preliminary Edition, as have Ann Waterhouse of South Portland High School in Maine, Frank Geist of Vermont Academy, and Roland Young of Deerfield Academy. The comments of their students were helpful in improving this edition. Finally, without the support of Frederick G. Torrey, Arthur H. Kiendl, and Howard L. Jones, chief administrators at Northfield Mount Hermon School in the years since 1966, the work with Dartmouth College could not have taken place.

G.A.H.
Mount Hermon, Massachusetts

Foreword to Students

It is our task in the chapters that follow to help you develop an understanding and command of one of the most important mathematical concepts—the function. The function concept underlies almost every branch of mathematics. Understanding of this fundamental idea is necessary in advanced work and will give you additional insight into your previous mathematics courses. The examples of functions studied (such as the exponential, logarithmic, and circular functions) are called the elementary functions and are important in their own right in mathematics and its applications.

The theoretical approach of the text differs from that of conventional texts in two ways. First, we proceed to develop several numerical techniques which are used throughout the text as basic tools in solving problems and introducing concepts. The book presumes your ability to read and write simple computer programs, as the book is one concerned with functions primarily, rather than computer programming. We hope to use your past experience and ability as a base. Very few programs appear as examples in the text, but many of the exercises are to be solved by the computer. As you will have discovered, there are many correct ways of solving problems by a computer and these problems provide considerable latitude for you as to the complexity of your particular solution. Where the text includes complicated programs, you will have been introduced to the various components in exercises, and the final program appears in an Appendix. All programs are written in BASIC, but any programming language may be used to solve problems, and your own language should be sufficiently close to BASIC for you to read the few listed programs.

Second, an historical context has been provided so that you may be aware of some of the interesting details behind the development of the theory as we know it today. Many problems have been included to this end, amplifying the Historical Note which preceeds each chapter. The Notes include footnoted references so that you may readily read further should you be interested. It has been the author's experience that students are interested in more than mathematical competence and that appreciation of the conditions which stimulated the development of ideas adds to the enjoyment of mathematics. The computer can be of assistance here as well.

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1

Functions

HISTORICAL NOTE. The development of the function concept parallels that of modern mathematics. One of its earliest appearances is found in the work of Nichole Oresme (*ca.* 1360) who used graphic methods to anticipate Galileo's Law of Freely Falling Bodies (by two centuries).[1] More formal consideration of functions came with the development of calculus and its application to the problems of the late seventeenth-century science. Bell credits G. W. Leibnitz (1646–1716) as the first to deal with the function as a mathematical entity.[2] As was the case with the calculus of Newton and Leibnitz, when first formulated, the concept of function was not as rigorously presented as today but was refined to its present form throughout the period of analysis that continued into the nineteenth century.

Such evolution is characteristic of ideas in mathematics and science. First representations are usually crude, and often intuitive. Experience dictates change until finally rigorous criteria agree with empirical results. Indeed, mathematics is never static but always responding to the environment, fixed though it may seem to any one generation of students.

1.1. *The Function—Informal Definition*

A premature definition of a *function* is an unambiguous pairing of elements from two sets. To see what is meant by this description, consider the following examples of functions:

1. A list of the students in this classroom and their seat numbers.
2. A list of the students in this classroom and their weights.
3. A list of integers from 1 to 10 with their square roots.
4. A list of cars on sale at any nearby used car lot with the prices noted.
5. A set of assorted building blocks on which the volume of each piece is printed.

In each of the examples above you will note there are two specific sets involved —e.g., in (1) we have names paired with seat numbers and in (2) names paired with weights. That the pairing must be unambiguous implies that for each name on the class list there can be only one seat assigned [or one weight as in (2)]. Beyond this each name must have a seat (or weight). Since in all functions we have a pairing between two sets, we give names to these sets. The first set, called the *domain*, identifies the objects with which our function is concerned. The second, the *range*, gives specific information concerning the corresponding elements in the domain. The domain and range of the functions (1)–(5) are as follows:

[1] Carl B. Boyer (1968), p. 290.
[2] E. T. Bell (1937), p. 98. (c) 1937 by E. T. Bell. Permission of Simon and Schuster.

	Domain	Range
1.	{Names of class}	{Seat numbers}
2.	{Names of class}	{Student weights}
3.	{1, 2, 3, . . , 10}	{1, 1.414 . . . , 1.732 . . . , . . . , 3.162 . . .}
4.	{All cars in lot}	{Prices}
5.	{Blocks}	{Volumes}

Functions such as these are somewhat hard to describe except as we have paired them. As the domain and range are listed, you will note that braces have been used to set off the elements of each set. This practice is customary when specifying sets. Clearer specification is difficult. It is important to recognize and describe the domain sets accurately. In many, if not most cases, they will be sets of numbers that may be listed or represented using an algebraic statement.

If for some reason you are not sure that the expression under consideration is a function, critical examination of the domain and range is essential in resolving the identity.

Because a function is considered as a pairing, rather than list the domain and range separately an alternate, clearer listing of a function can be made by recording the actual pairings of domain and range elements. Thus such a set of pairs for (3) above would be

$$\{(1, 1), (2, 1.414\ldots), (3, 1.732\ldots), \ldots, (10, 3.162\ldots)\}$$

where each element of the domain is paired with its square root approximation.

Exercises 1-1A

1. The domain of a function is the set $\{a, e, i, o, u\}$. Find the range if each range element is the order number in the alphabet—e.g., for a, we have order number 1; for b, 2; for c, 3; etc.
2. Pair with each integer from 1 to 10 its cube. List in pairs. List the domain and range.
3. Make a table of names of your friends in this class and give their ages.
4. Make a list that pairs each of the ages {16, 17, 18, 19, 20} with the names in Exercise 3 with that age. Is this a function? Why?
5. A function (although not always so classified) from plane geometry is the distance function. It gives the distance between any pair of points in the plane if the coordinates of the points are known. If P has coordi-

nates (x_1, y_1) and Q has coordinates (x_2, y_2), the distance d_{PQ} is given by

$$d_{PQ} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Complete the function table for the points listed here:

Domain (pairs of points)	Range (distances)
(3, 2), (5, 9)	$\sqrt{(3-5)^2 + (2-9)^2} = \sqrt{4 + 49} = \sqrt{53}$
(−1, 3), (2, 0)	
(5, 3), (−2, 1)	
(4, 7), (7, 7)	

What is the entire domain? The range?

6. A useful function in computer science is called SIGNUM. Its domain is the set of real numbers and its range is the set $\{-1, 0, 1\}$. As there are but three elements in the range there is a natural division of the domain into three subsets that are paired to the elements -1, 0, and 1. Make a conjecture (a reasoned guess) as to the appropriate definition of SIGNUM.
7. Consider the common objects below. Are they examples of functions? If not, could they be modified to be functions?
   (a) Telephone book
   (b) Photograph
   (c) Computer

The uniqueness (or the unambiguity) of the pairing is the essential quality of functions that leads to their importance in mathematics where we are most concerned with the avoidance of ambiguity. Where a pairing of objects from two sets is ambiguous—i.e., where for each domain element there may be more than one range element—such a pairing is called a *relation*. Relations are fairly common in mathematics and students sometimes confuse relations with functions. The following are examples of relations:

1. Names of students with high schools attended.
2. The pairs of numbers (x, y) satisfying the equation $x = y^2$. For example, (4, 2) and (4, −2) both satisfy the equation. Find two other pairs that illustrate ambiguity.
3. The points of the xy plane satisfying the equation $x^2 + y^2 = 4$. Can you identify the domain and range of this relation? Find at least four pairs (x, y) that do satisfy this relation. Can you identify the graph represented?

The distinction, then, between functions and relations is that for each element in the domain of a function there corresponds *exactly* one range element, whereas for a relation there may be one or more range elements corresponding to each domain element. Graphically, if we represent a function on the xy-plane, there can be at most one intersection of the curve with any vertical line drawn. See Fig. 1-1. The *graph of a function* is the set of all points whose

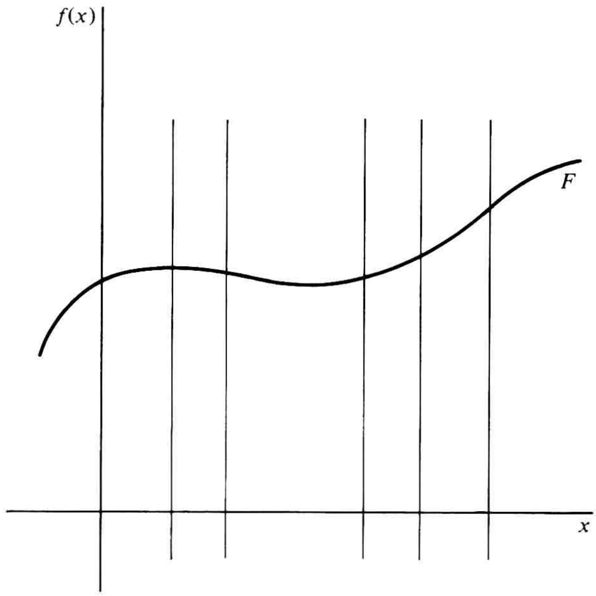

Figure 1-1

coordinates satisfy the equation. We usually let the horizontal axis be the domain axis, with the variable x representing the domain. The range is plotted vertically as y or $f(x)$, or other suitably chosen labels.

It should be observed that it is not a contradiction of the definition that the same range element is paired with more than one domain element. Thus if a horizontal line were drawn in Fig. 1-1, do you see that it is possible to find values of x having the same y value? Do you recall examples of functions having this property from your earlier mathematics courses?

The relation, when graphed, will have *at least one* multiple intersection with some vertical line, as represented in Fig. 1-2.

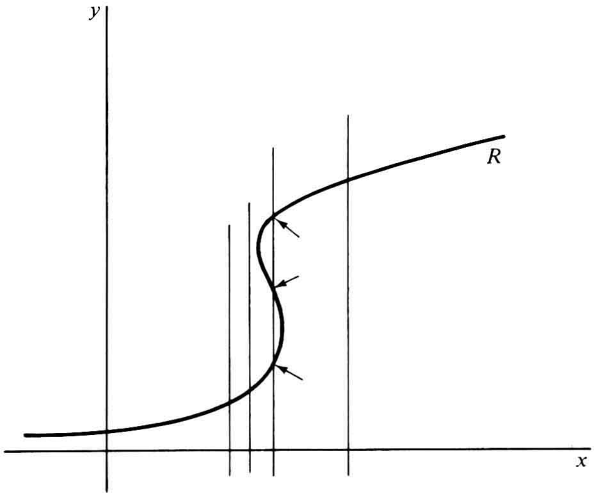

Figure 1-2

Our last consideration is the means by which the domain and range elements are associated. Usually we have a *rule*, which may be a simple phrase such as "to each boy associate his age," but more commonly it is an equation relating two variables. We can list a table or *roster* of the pairings, such as

$$\{(2, 3), (3, 4), (5, 6), (7, 9), (11, 12), \ldots\}$$

or we can write the rule in sentence or equation form. For example, the function F, which to each integer pairs its square, could be represented as

$$\mathrm{F} = \{\ldots, (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9), \ldots\}$$

Three dots at each end of the roster indicate elements of similar form precede and follow those listed. Except for functions with a few pairings this method is unwieldy. A more succinct form of the same function is

$$\mathrm{F} = \{(x, y)\colon y = x^2, x \text{ any integer}\}$$

or

$$\mathrm{F} = \{(x, y)\colon y = x^2, x \in I\}$$

In the latter form of F, "$x \in I$" is read as "x belongs to the set of all integers."