

ADVANCED STUDIES IN PURE  
MATHEMATICS 57

Probabilistic Approach to Geometry



MATHEMATICAL SOCIETY OF JAPAN, TOKYO

# ADVANCED STUDIES IN PURE MATHEMATICS 57

Chief Editor of the Series: Sadayoshi Kojima (Tokyo Institute of Technology)

---

## Probabilistic Approach to Geometry

Edited by

Motoko Kotani (Tohoku University)

Masanori Hino (Kyoto University)

Takashi Kumagai (Kyoto University)



Mathematical Society of Japan

*This book was typeset by  $\text{\AA MS-}\text{\TeX}$  and  $\text{\AA MS-}\text{\LaTeX}$ , the  $\text{\TeX}$  macro systems of the American Mathematical Society, together with the style files `aspm.sty` and `aspmfm.sty` for  $\text{\AA MS-}\text{\TeX}$  written by Dr. Chiaki Tsukamoto and `aspmproc.sty` for  $\text{\AA MS-}\text{\LaTeX}$  written by Dr. Akihiro Munemasa and `aspm.cls` for  $\text{\AA MS-}\text{\LaTeX}$  provided by Livretech Co., Ltd., formerly named Tokyo Shoseki Printing Co., Ltd.*

*$\text{\TeX}$  is a trademark of the American Mathematical Society.*

©2010 by the Mathematical Society of Japan.

*All rights reserved. The Mathematical Society of Japan retains the copyright of the articles in the present volume except those indicated in the footnotes. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording or otherwise, without the prior permission of the copyright owner.*

*Edited by the Mathematical Society of Japan.*

*Published by the Mathematical Society of Japan.*

*Distributed by the Mathematical Society of Japan, the American Mathematical Society and the World Scientific Publishing Co., Ltd.*

*Distributed exclusively in North America by the American Mathematical Society.*

*Advanced Studies in Pure Mathematics 57*

*ISBN 978-4-931469-58-7*

PRINTED IN JAPAN  
by Livretech Co., Ltd.

*2000 Mathematics Subject Classification.*

*Primary 51-06, 60-06.*

*Secondary 34F05, 51F99, 58J65.*

## Advanced Studies in Pure Mathematics

Chief Editor

Sadayoshi Kojima (Tokyo Inst. of Tech.)

### Editorial Board of the Series

MARTIN GUEST (Tokyo Metro. Univ.)	SHIHOKO ISHII (Tokyo Inst. of Tech.)	TOSHITAKE KOHNO (Univ. of Tokyo)
SADAYOSHI KOJIMA (Tokyo Inst. of Tech.)	MOTOKO KOTANI (Tohoku Univ.)	KIMIO MIYAJIMA (Kagoshima Univ.)
MASATOSHI NOUMI (Kobe Univ.)	TAKAYUKI ODA (Univ. of Tokyo)	KEIJI OGUIISO (Osaka Univ.)
MASAYOSHI TAKEDA (Tohoku Univ.)	YOSHIHIRO TONEGAWA (Hokkaido Univ.)	AKIHIKO YUKIE (Tohoku Univ.)

## Probabilistic Approach to Geometry

# The Seasonal Institute of the Mathematical Society of Japan

*September 24, 2009*

Takashi TSUBOI, the President of MSJ

The aim of the conference series “Seasonal Institute” is to obtain an overview of the significant recent developments in a specific subject, and to present it especially to young researchers in Japan, as well as in Asian countries. For this purpose, we invite leading mathematicians of the field from all over the world, as well as promising young researchers from Asian countries to MSJ Seasonal Institutes.

The MSJ Seasonal Institute is held every year and its subject is chosen by the Scientific Committee of the Mathematical Society of Japan. This is a succession of our conference series MSJ-IRI (Mathematical Society of Japan, International Research Institute). We had 15 successful conferences in 14 years from 1993 to 2006, and our idea is to continue the conference in the same spirit, but also to contribute mathematical communities outside Japan.

We publish the outcome of each MSJ Seasonal Institute as a volume of Advanced Study in Pure Mathematics which will give a good reference of the developments in the chosen specific subject and contribute the developments of mathematics. This continues the tradition of MSJ-IRI.

This is the volume of Advanced Study in Pure Mathematics which contains the excellent outcome of the first Seasonal Institute of the Mathematical Society of Japan, “Probabilistic Approach to Geometry” held in Kyoto from the 28th July to the 8th August in 2008. We thank the contributors and the editors of the volume for their collaboration.

The second MSJ-Seasonal Institute “Arrangements of Hyperplanes” was held in Sapporo from the 1st to the 13th August in 2009 and the third MSJ-Seasonal Institute “Development of Galois-Teichmüller theory and anabelian geometry” will be held in Kyoto from 25th to 30th October 2010. The outcomes of these MSJ-Seasonal Institutes follow in the future volumes of Advanced Study in Pure Mathematics.

We are grateful to the precious efforts of the organizers of MSJ-Seasonal Institutes as well as to the warm hospitality of the host institutes. We are sure that publication of the outcomes of MSJ-Seasonal Institutes in Advanced Study in Pure Mathematics contributes the the developments of mathematics.

## Preface

This volume is the product of the first conference of the Seasonal Institute of the Mathematical Society of Japan titled,

“Probabilistic Approach to Geometry”  
28<sup>th</sup> July 2008 – 8<sup>th</sup> August 2008  
Kyoto University, Japan.

The conference was organized by the Society’s committee — Motoko Kotani (Tohoku University), Hitoshi Arai (University of Tokyo), Takashi Kumagai (Kyoto University) and Karl-Theodor Sturm (Universität Bonn) — in collaboration with the local organizing committee at Kyoto University — Takashi Kumagai, Masanori Hino, Tsuyoshi Kato and Shin-ichi Ohta.

The conference focused on the interaction between Geometry and Probability Theory, and the development of new directions in both research fields. To achieve this aim, survey lectures were given by leading researchers in one of the following four themes selected by the organizers:

Rough Path Analysis:

Shigeki Aida (Osaka University),  
Terry J. Lyons (University of Oxford)

Theory of Random Groups:

Yann Ollivier (École Normale Supérieure de Lyon)

Heat Kernel Analysis:

Laurent Saloff-Coste (Cornell University)

Mass Transport Problems:

Cédric Villani (École Normale Supérieure de Lyon)

Based on these themes, thirty-three invited talks and thirteen contributed talks were also presented along with active and fruitful discussions by the 180 participants, one third of whom were from abroad.

The editors and conference organizers thank the Mathematical Society of Japan, and the Japan Society for the Promotion of Science (JSPS) for financial support through the Grant for International Scientific Meetings in Japan and a Grant-in-Aid for Scientific Research. We would also like to express our gratitude to the staff of the Mathematical Society of Japan for the efficient handling of the conference and the publication of this volume. Last but not least, we extend our most sincere thanks to all

of the participants and speakers for making the conference so successful.

Editors:

Motoko Kotani (Tohoku University)

Masanori Hino (Kyoto University)

Takashi Kumagai (Kyoto University)

*All papers in this volume have been refereed and are in final form.  
No version of any of them will be submitted for publication elsewhere.*



## Program of the Conference

The first week (July 28th – August 1st)  
Kyoto University Clock Tower Centennial Hall

### July 28th (Mon.)

- 9:45–10:15 Registration
- 10:15–10:30 **President of the Mathematical Society of Japan**  
Opening remark
- 10:30–11:20 **Cédric Villani**, École Normale Supérieure de Lyon,  
Lecture 1  
Optimal transport in geometry
- 11:20–11:40 Tea
- 11:40–12:30 **Terry J. Lyons**, University of Oxford, Lecture 1  
Rough paths — A story in non-commutative analysis
- 12:30–14:00 Lunch
- 14:00–14:50 **Martin T. Barlow**, University of British Columbia  
Uniqueness of Brownian motion on the Sierpinski carpet
- 15:00–15:30 **Max von Renesse**, Technische Universität Berlin  
Entropic measure and Wasserstein diffusion
- 15:40–16:10 **Tsuyoshi Kato**, Kyoto University  
A dynamical pattern formation, tropical geometry and  
informative entropy
- 16:10–16:40 Tea
- 16:40–17:30 **Roland Friedrich**, Max-Planck-Institute für Mathe-  
matik  
The global geometry of stochastic Loewner evolutions

### July 29th (Tue.)

- 10:00–10:50 **Cédric Villani**, École Normale Supérieure de Lyon,  
Lecture 2  
Optimal transport in geometry
- 10:50–11:10 Tea
- 11:10–12:00 **Shigeki Aida**, Osaka University, Lecture 1  
Rough path analysis: An introduction
- 12:00–14:00 Lunch



- 14:00–14:50 **Bálint Virág**, University of Toronto  
Random matrices, probability, and geometry
- 15:00–15:30 **Tatsuya Tate**, Nagoya University  
Bernstein measures on convex polytopes
- 15:40–16:10 **Chang-Wan Kim**, Korea Institute for Advanced Study  
Ricci and flag curvatures in Finsler geometry
- 16:10–16:40 Tea
- 16:40–17:30 **Shin-ichi Ohta**, Kyoto University  
Optimal transport and Ricci curvature in Finsler geometry

### July 30th (Wed.)

- 10:00–10:50 **Shun-ichi Amari**, Riken  
Information geometry, its applications and related mathematical problems
- 10:50–11:10 Tea
- 11:10–12:00 **Shigeki Aida**, Osaka University, Lecture 2  
Rough path analysis: An introduction
- 12:00–14:00 Lunch
- 14:00–14:50 **Sumio Watanabe**, Tokyo Institute of Technology  
What we can estimate about a singularity from random samples
- 15:00–15:30 **Hiroshi Matsuzoe**, Nagoya Institute of Technology  
Statistical manifolds and affine differential geometry
- 15:40–16:10 **Kazuhiro Kuwae**, Kumamoto University  
On discrete harmonic maps into  $CAT(k)$ -spaces via Markov chains
- 16:10–16:40 Tea
- [Poster] **Hyun Yoo**, Hankyong National University  
Projections in the reproducing kernel Hilbert spaces and the conditional probabilities of determinantal point processes in discrete spaces
- [Poster] **Wen-Haw Chen**, Tunghai University  
On topological obstructions of compact Riemannian and combinatorial positively Ricci curved manifolds

- 16:40–17:30 **Shinto Eguchi**, The Institute of Statistical Mathematics  
Information divergence geometry and its application to machine learning
- 18:30–20:30 Buffet-style party (Kyodai-Kaikan)

### July 31st (Thu.)

- 10:00–10:50 **Cédric Villani**, École Normale Supérieure de Lyon,  
Lecture 3  
Optimal transport in geometry
- 10:50–11:10 Tea
- 11:10–12:00 **Shigeki Aida**, Osaka University, Lecture 3  
Rough path analysis: An introduction
- 12:00–14:00 Lunch
- 14:00–14:50 **Takashi Shioya**, Tohoku University  
Geometric analysis on Alexandrov spaces
- 15:00–15:30 **Yuzuru Inahama**, Tokyo Institute of Technology  
A stochastic Taylor-like expansion in the rough path theory
- 15:40–16:10 **Kazumasa Kuwada**, Ochanomizu University  
Characterization of maximal Markovian couplings for diffusion processes
- 16:10–16:40 Tea
- 16:40–17:30 **Keisuke Hara**, Ritsumeikan University  
Rough path condition for smooth paths

### August 1st (Fri.)

- 10:00–10:50 **Cédric Villani**, École Normale Supérieure de Lyon,  
Lecture 4  
Optimal transport in geometry
- 10:50–11:10 Tea
- 11:10–12:00 **Vladimir Pestov**, University of Ottawa  
Urysohn's universal, or random, metric space, its group of isometries, and other related structures
- 12:00–14:00 Lunch

- 14:00–14:50 **Robert J. McCann**, University of Toronto  
Curvature, continuity and uniqueness of optimal transportation maps
- 15:00–16:10 Contributed Talks
- Kouji Yano**, Kobe University  
Excursions away from a regular point for one-dimensional symmetric Lévy processes without Gaussian part
- Hiroshi Kawabi**, Okayama University  
Riesz transforms on a path space with Gibbs measures
- AbdulRahman Al-Hussein**, Qassim University  
Time-dependent backward stochastic evolution equations
- 16:10–16:40 Tea
- 16:40–17:30 **Yukio Otsu**, Kyushu University  
Statistical mechanics of 1-particle ideal gas and deformation of Alexandrov spaces

## August 2nd (Sat.)

Excursion 13:30–17:30

## The second week (August 4th – August 8th) Shiran Kaikan

### August 4th (Mon.)

- 10:00–10:50 **Laurent Saloff-Coste**, Cornell University, Lecture 1  
Heat kernel estimates
- 10:50–11:10 Tea
- 11:10–12:00 **Yann Ollivier**, École Normale Supérieure de Lyon,  
Lecture 1  
Survey on random groups
- 12:00–14:00 Lunch
- 14:00–14:50 **John Lott**, University of Michigan  
Optimal transport and Perelman's reduced volume

- 15:00–15:30 **Yoshikata Kida**, Tohoku University  
Orbit equivalence rigidity for some groups acting on trees
- 15:40–16:10 **Atsushi Atsuji**, Keio University  
Estimates on the number of omitted values of meromorphic functions
- 16:10–16:40 Tea
- 16:40–17:30 **Vadim Kaimanovich**, Jacobs University Bremen  
Random graphs and equivalence relations

### August 5th (Tue.)

- 10:00–10:50 **Laurent Saloff-Coste**, Cornell University, Lecture 2  
Heat kernel estimates
- 10:50–11:10 Tea
- 11:10–12:00 **Yann Ollivier**, École Normale Supérieure de Lyon, Lecture 2  
Discrete positive curvature, Markov chains and concentration of measure
- 12:00–14:00 Lunch
- 14:00–14:50 **Terry J. Lyons**, University of Oxford, Lecture 2  
Rough paths — A story in non-commutative analysis
- 15:00–16:10 Contributed Talks
- Masayoshi Watanabe**, Tohoku University  
Concentration of measure via approximated Brunn-Minkowski inequalities
- Kei Funano**, Tohoku University  
Concentration of 1-Lipschitz maps and group action
- Asuka Takatsu**, Tohoku University  
On Wasserstein geometry of the space of Gaussian measures
- 16:10–16:40 Tea
- 16:40–17:30 **Hiroyasu Izeki**, Tohoku University  
A fixed-point property of discrete groups and an energy of equivariant maps

## August 6th (Wed.)

- 10:00–10:50 **Laurent Saloff-Coste**, Cornell University, Lecture 3  
Heat kernel estimates
- 10:50–11:10 Tea
- 11:10–12:00 **Yann Ollivier**, École Normale Supérieure de Lyon,  
Lecture 3  
Discrete positive curvature, Markov chains and concentration of measure
- 12:00–14:00 Lunch
- 14:00–14:50 **Anton Thalmaier**, Université du Luxembourg  
Li-Yau type inequalities and a priori estimates for heat equations by stochastic analysis
- 15:00–15:30 **Takefumi Kondo**, Kyoto University  
Fixed-point property of random groups
- 15:40–16:10 **Jun Kigami**, Kyoto University  
Measurable Riemannian geometry on the Sierpinski gasket
- 16:10–16:30 Tea
- 16:30–17:10 Contributed Talks
- Naotaka Kajino**, Kyoto University  
Weyl type spectral asymptotics for the Laplacian on Sierpinski carpets
- Ryoki Fukushima**, Kyoto University  
Brownian survival among perturbed lattice traps
- 19:00–21:00 Banquet (Ganko Takasegawa Nijoen)

## August 7th (Thu.)

- 10:00–10:50 **Laurent Saloff-Coste**, Cornell University, Lecture 4  
Heat kernel estimates
- 10:50–11:10 Tea
- 11:10–12:00 **Yann Ollivier**, École Normale Supérieure de Lyon,  
Lecture 4  
Discrete positive curvature, Markov chains and concentration of measure
- 12:00–14:00 Lunch

- 14:00–14:50 **Dominique Bakry**, Université Paul Sabatier  
Gradient bounds for some hypo-elliptic heat equations
- 15:00–16:10 Contributed Talks
- Juillet Nicolas**, University of Bonn  
Synthetic Ricci curvature bounds in the Heisenberg group
- Takumi Yokota**, University of Tsukuba  
Perelman's reduced volume and gap theorem for the Ricci flow
- Shinichiroh Matsuo**, University of Tokyo  
The Runege theorem for instantons
- 16:10–16:40 Tea
- 16:40–17:30 **Atsushi Kasue**, Kanazawa University  
Functions of finite Dirichlet sum and compactifications of infinite graphs

### **August 8th (Fri.)**

- 10:00–10:50 **Ichiro Shigekawa**, Kyoto University  
Non symmetric diffusions on a Riemannian manifold
- 10:50–11:10 Tea
- 11:10–12:00 **Terry J. Lyons**, University of Oxford, Lecture 3  
Rough paths — A story in non-commutative analysis
- 12:00–14:00 Lunch
- 14:00–14:50 **Andrzej Zuk**, Université Paris 7  
Automata groups
- 15:00–15:50 **Kenneth David Elworthy**, University of Warwick  
Stochastic flows and geometric analysis on path spaces

## CONTENTS

Shigeki AIDA — Rough path analysis: An introduction	1
Marc ARNAUDON and Anton THALMAIER — Li-Yau type gradient estimates and Harnack inequalities by stochastic analysis	29
Atsushi ATSUJI — Estimates on the number of the omitted values by meromorphic functions	49
K. David ELWORTHY — Stochastic flows and geometric analysis on path spaces	61
Roland FRIEDRICH — The global geometry of stochastic Lœwner evolutions	79
Ryoki FUKUSHIMA, Atsushi TANIDA and Kouji YANO — Non-Markov property of certain eigenvalue processes analogous to Dyson's model	119
Keisuke HARA — Rough path conditions for smooth paths	129
Tae HATTORI and Atsushi KASUE — Functions of finite Dirichlet sums and compactifications of infinite graphs	141
Yuzuru INAHAMA — A note on rough differential equations with unbounded coefficients	155
Hiroyasu IZEKI — A fixed-point property of finitely generated groups and an energy of equivariant maps	171
Nicolas JUILLET — On a method to disprove generalized Brunn-Minkowski inequalities	189
Vadim A. KAIMANOVICH and Florian SOBIECZKY — Stochastic homogenization of horospheric tree products	199
Naotaka KAJINO — Remarks on non-diagonality conditions for Sierpinski carpets	231
Tsuyoshi KATO — Pattern formation from projectively dynamical systems and iterations by families of maps	243
Takefumi KONDO — Fixed-point theorems for random groups	263
Kazumasa KUWADA — Couplings of the Brownian motion via discrete approximation under lower Ricci curvature bounds	273



Kazuhiro KUWAE and Takashi SHIOYA — Infinitesimal Bishop-Gromov condition for Alexandrov spaces	293
Hiroshi MATSUZOE — Statistical manifolds and affine differential geometry	303
Shin-ichi OHTA — Optimal transport and Ricci curvature in Finsler geometry	323
Yann OLLIVIER — A survey of Ricci curvature for metric spaces and Markov chains	343
Vladimir PESTOV — Concentration of measure and whirly actions of Polish groups	383
Laurent SALOFF-COSTE — The heat kernel and its estimates	405
Ichiro SHIGEKAWA — Non-symmetric diffusions on a Riemannian manifold	437
Asuka TAKATSU — On Wasserstein geometry of Gaussian measures	463
Sumio WATANABE — A limit theorem in singular regression problem	473
Kouji YANO — Two kinds of conditionings for stable Lévy processes	493
Takumi YOKOTA — A gap theorem for ancient solutions to the Ricci flow	505

## Rough path analysis: An introduction

Shigeki Aida

### Abstract.

We explain basic results in rough path analysis and their applications in stochastic analysis.

### §1. Introduction

Let  $x : [0, T] \rightarrow \mathbb{R}^d$  be a  $C^1$ -path with  $x_0 = 0$ .  
 Let  $f$  be a  $C^\infty$ -map from  $\mathbb{R}^d$  to  $L(\mathbb{R}^d \rightarrow \mathbb{R}^m)$ . Then the limit

(1.1)

$$\begin{aligned} I_{0,T}(x) &= \int_0^T f(x_u) dx_u \\ &= \lim_{|D| \rightarrow 0} \sum_{i=1}^N f(x_{s_{i-1}}) (x_{t_i} - x_{t_{i-1}}) \in \mathbb{R}^m, \quad t_{i-1} \leq s_{i-1} \leq t_i, \end{aligned}$$

exists. Here  $D = \{0 = t_0 < \cdots < t_N = T\}$  and  $|D| = \max_{1 \leq i \leq N} (t_i - t_{i-1})$ . Moreover the functional  $x \mapsto I_{0,T}(x)$  is continuous in the topology of  $C^1$ . Let  $p \geq 1$  and define the  $p$ -variation norm of  $x$  by

$$\|x\|_p := \left\{ \sup_D \sum_{i=1}^N |x_{t_i} - x_{t_{i-1}}|^p \right\}^{1/p}.$$

We denote by  $B_{p,T}(\mathbb{R}^d)$  the Banach space which consists of continuous paths starting at 0 with finite  $p$ -variation norm  $\| \cdot \|_p$ . The 1-variation norm of  $x$  is the same as the total variation of  $x$ . The  $p$ -variation norm defines a weaker topology of the  $C^1$ -path space with  $x_0 = 0$ . Actually it is proved that  $I_{0,T}(x)$  is a continuous functional of  $x$  in the  $p$ -variation

---

Received January 15, 2009.

Revised April 22, 2009.

2000 *Mathematics Subject Classification.* 34F05, 60H05, 60H10, 93E03.

*Key words and phrases.* Rough path, stochastic differential equation.