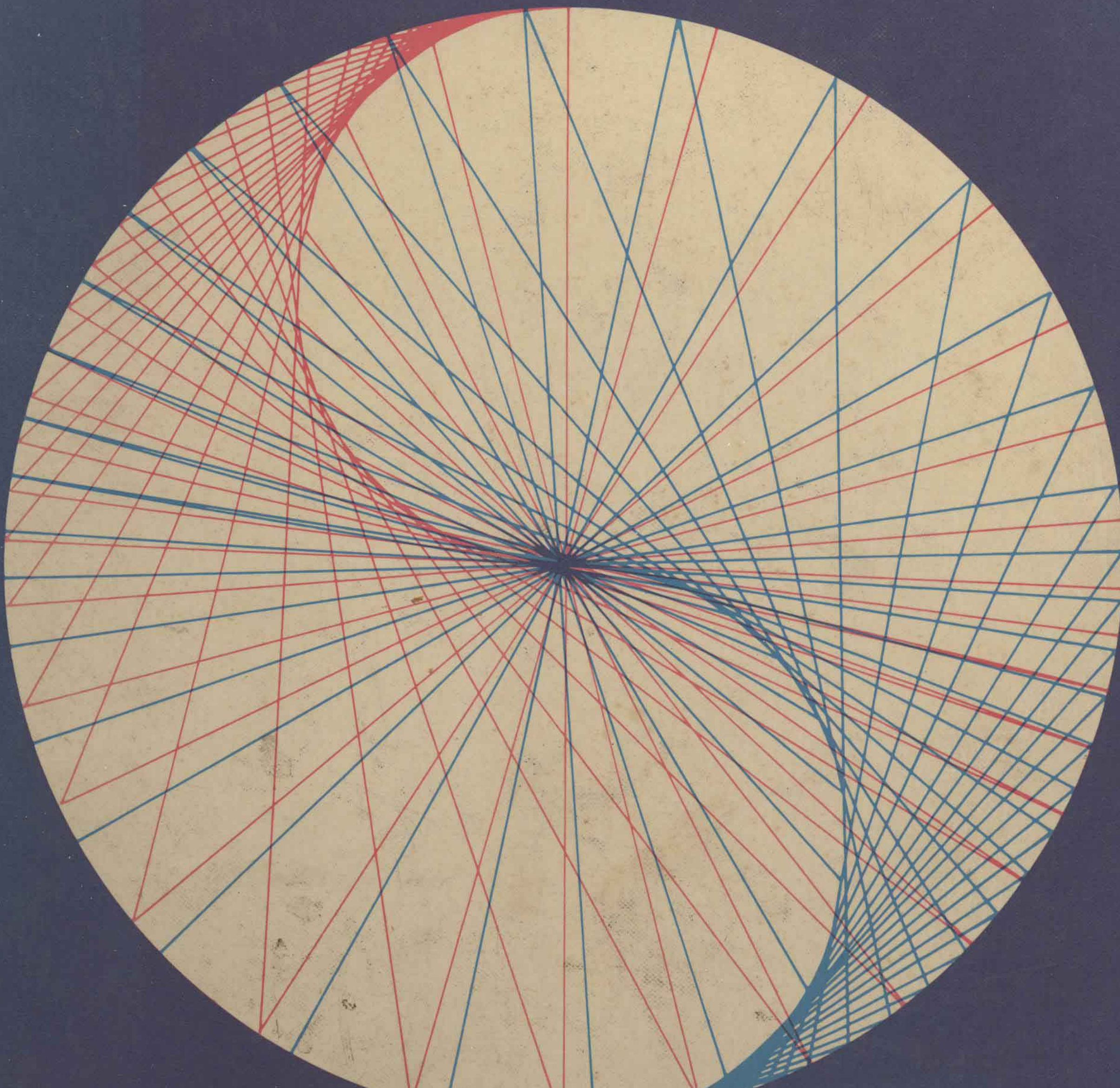
# FINITE MATHEMATICS

HUGH G. CAMPBELL
and ROBERT E. SPENCER



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# Finite Mathematics

**Under the Editorship of Carl B. Allendoerfer**

**To Linnie, Dorothy, and Dorothy Lynn**

# Preface

One distinguishing feature of this book is its preoccupation with motivation—a vital part of the learning process. The authors have devoted considerable energy toward the goal of furnishing the reader with reasons for studying the mathematics included in this book. Almost every section in the book includes a subsection entitled "Applications"; these subsections, however, are not designed as comprehensive lists of real-world applications, nor is any claim made concerning the realism of many examples. Rather, these "indications of usefulness" are designed to point out in a limited and elementary way that the material is relevant to a broad spectrum of disciplines and also that it has considerable potential for future applications. Many of the applications are documented to permit further investigation by the reader.

A second distinguishing feature of this book is its organizational form. A conscious effort was made to organize the book in such a way that the instructor can alter the emphasis or length of a course to suit his needs. Many of the sections are independent of other sections and are so advertised at the outset of each chapter; all the applications are optional; and the format of the text is such that the theory can be emphasized or avoided as desired. The chart shown in Table A is given to summarize the flexibility of topic selection. Using Table A, instructors may construct alternative courses by varying the emphasis placed on each of the seven chapters; for example, one course can be constructed that gives major emphasis to Chapters 4, 5, 6, and 7 with as few as 6 lessons on Chapters 1, 2, and 3, whereas another course might give major emphasis to Chapters 1, 2, 3, and 4 with minimum or moderate emphasis on Chapters 5, 6, and 7.

Ordinarily, the material found in Chapter 1 is relegated to a less prominent position in the study of finite mathematics. The authors have found, however, that this material is ideal for an introduction to finite mathematics and certainly facilitates the study of logic. This is probably due in part to the fact that a great many diagrams are involved which allow the student to obtain a visual conception of some of the basic ideas. The material in the first chapter also offers the opportunity to present, very early in the course, diverse applications of interest.

Above all, the guiding principle in the construction of this book has been consideration for the needs of the student. The sincerity of the authors in this regard is illustrated by the attempt to motivate through the subsections on applications, the list of new vocabulary at the end of each chapter, the numerous exercises, and the many examples illustrating the concepts.

The authors express sincere appreciation to the many people who have assisted us in the production of this book. Special gratitude is expressed

**Table A**

| | SUGGESTED SECTIONS FOR MINIMUM EMPHASIS | SUGGESTED SECTIONS FOR MODERATE EMPHASIS | SUGGESTED SECTIONS FOR MAXIMUM EMPHASIS |
|---|---|---|---|
| CHAPTER 1 (Systems) | 1.1, 1.2 (1 or 2 lessons) | 1.1–1.3 (3 lessons) | Entire chapter (4 lessons) |
| CHAPTER 2 (Logic) | 2.1, 2.2 (2 lessons) | 2.1–2.3 (3 lessons) | Entire chapter (5 lessons) |
| CHAPTER 3 (Sets) | 3.1, 3.2, 3.5 (3 lessons) | 3.1–3.3, 3.5, 3.6 (5 lessons) | Entire chapter (6 lessons) |
| CHAPTER 4 (Probability) | 4.1–4.4, 4.10 (5 lessons) | 4.1–4.6, 4.8, 4.10 (8 lessons) | Entire chapter (10 lessons) |
| CHAPTER 5 (Matrices) | 5.1–5.3, 5.5 (4 lessons) | 5.1–5.8 (8 lessons) | Entire chapter (10 lessons) |
| CHAPTER 6 (Linear Programming) | 6.1, 6.2 (2 lessons) | 6.1–6.4 (4 lessons) | Entire chapter (6 lessons) |
| CHAPTER 7 (Game Theory) | 7.1, 7.2 (2 lessons) | 7.1–7.3 (3 lessons) | Entire chapter (4 lessons) |
| TOTAL | 19–20 lessons | 34 lessons | 45 lessons |

to Dr. Carl Allendoerfer, Mr. Everett Smethurst, and Dr. H. Earl Spencer for their time and efforts in behalf of this project. Our respective wives, Allen and Carole, also deserve considerable thanks for their encouragement during the development of the book.

HUGH G. CAMPBELL
ROBERT E. SPENCER

*Blacksburg, Virginia*

# Contents

# 0 An Orientation

Over the years the development of mathematics has closely paralleled the need for quantification of natural and other phenomena. From the time man invented symbols that correspond to our numerals 1, 2, . . . , there has been a relentless march toward the use of mathematics in ever more diverse ways. Certainly, the rate of this advance has not been constant and often has been related to the intellectual and political climates of the age as well as the maturity and influence of sister disciplines. Early interest in astronomy, for example, provided considerable impetus for the development of related mathematical concepts and techniques. Over a period of years, many of these concepts and techniques have led to a comparatively sophisticated development of the engineering sciences; the development of the computer and space travel give ample evidence of this fact. The computer and other engineering developments, in turn, are now having a tremendous effect upon modern mathematics, as well as having the effect of opening vast new areas for quantification in the social, management, and life sciences.

In this book we have chosen a few mathematical topics that have shown considerable promise in the social, management, and life sciences. Extensive quantification in these areas is a relatively new thing, and there is not yet universal acceptance of the usefulness of the mathematical topics in these disciplines. Although many criticisms are probably valid and useful, we should not lose sight of the fact that similar battles were fought over the usefulness of calculus in the natural sciences back in the 17th and 18th centuries. It should be emphasized that the art of application of the topics of this book to the behavioral, management, and life sciences is in its infancy, and while there is considerable reason for hope, much work remains to be done.

One of the predominant characteristics of modern mathematics is the stress placed on the structure of mathematics; this approach is very important as far as applications are concerned because it lends itself readily to the applied concept of quantitative models. Although the development of structure will be emphasized, it is not the purpose of this book to give an airtight, logical development of the topics selected. The purpose is to give the reader an idea of the nature of mathematical structure and modern mathematical thought, along with material that may be useful in quantitative applications in his discipline. In constructing a mathematical structure we begin with a foundation consisting of certain undefined and defined terms, together with a set of assumptions. Beyond this foundation we make conjectures and more definitions. Those conjectures that are justified according to certain rules of logic are called theorems and become part of the superstructure, so to speak. Applications then follow naturally from the structure. In other words, mathematical techniques can be thought of not as a collection of unrelated magical practices, but as natural operations performed within a unifying structure.

To illustrate these and other important ideas, a treatment of the customary topics of sets and logic is given by constructing in each of the first three chapters a specific mathematical structure which has elements or building blocks consisting of entities other than numbers. In the third chapter the material is further unified by showing that each of the first three structures is really a special case of a single abstract structure known as a Boolean Algebra.

# 1 Systems and the Algebra of Components

*This chapter has a twofold purpose. (1) The reader is introduced to the idea of using a mathematical structure in which elements other than numbers are combined by various operations. Ideally, this will considerably broaden the reader's perspective of both the structure and uses of mathematics. (2) The study of logic, which is to be studied in Chapter 2, is made easier by first studying the analogous and easier material of this chapter. The diagrams of this chapter provide a visual conception of some of the basic ideas. Furthermore, this material is fundamental in the design of various electrical and electronic devices such as the digital computer. Although it is beyond the scope of this text to investigate the construction of these devices, some of the applications are intended to make clearer their nature, design, and use. Also, for those readers who have access to a computer terminal that is capable of accepting the APL computer language, an appendix to this chapter is provided for instruction in its use in making evaluations.*

*Prerequisites: high school algebra*
*Suggested Sections for Moderate Emphasis: 1.1–1.3*
*Suggested Sections for Minimum Emphasis: 1.1, 1.2 (For those who prefer to reserve more time for Chapters 4, 5, 6, and 7, a one- or two-lesson survey of 1.1 and 1.2 will satisfy the minimum prerequisites for Chapters 2 and 3.)*

## 1.1 Components and Connectives

The word "system" is quite common in our current language. We frequently hear about such diverse and important systems as computer systems, digestive systems, business systems, spacecraft systems, political systems, transportation systems, and many others. It is important in various disciplines that participants have some idea of what is meant by a system and the various ways that mathematics can serve in analyzing, modifying, or improving identifiable systems; often this service can be achieved by means of constructing mathematical models of physical, natural, or social systems. This procedure will be illustrated later.

Webster defines the word **system** to be "a set or arrangement of things (components) so related or connected as to form a unity or organic whole." In this chapter we shall be concerned with those systems whose components are connected with parallel and series connections. In order to give the reader specific illustrations of the abstract concepts that we are trying to teach, we shall make our presentation of the algebra of components by giving the word **component** a specific interpretation. Consider a set of "on–off" switches similar to a common light switch found near the door of most rooms or a set of switches found in the interior of a digital computer. We shall use the letters $p$, $q$, $r$, $s$, $t$, and so on, to represent these switches. Switches can be connected in parallel as shown in Figure 1.1, or they can be connected in series as shown in Figure 1.2.

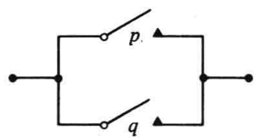


**Figure 1.1.** *Parallel connection of switches.*


**Figure 1.2.** *Series connection of switches.*

We shall designate a parallel connection of switches by the symbol $+$, and hence the arrangement of switches in **parallel** as shown in Figure 1.1 can be expressed symbolically as $p + q$. The symbol "$+$" can be read "is connected in parallel with." A connection of switches in **series** as shown in Figure 1.2 can be expressed as $r \times s$. The symbol "$\times$" can be read "is connected in series with." When two or more single switches are connected in parallel or series (or both), we have a system of switches. For reasons that will become apparent later, we choose to call such a system a **compound switch.** Large and complicated compound switches can be built, and, in fact, one of the purposes of this study is to point the way toward the analysis of compound switches with the view of simplifying them. Such simplification can be very useful in certain applications. Any single switch or compound switch that makes up part or all of a compound switch can be thought of as a **component** of a **system of switches.** The compound switch consisting of components $p$, $q$, and $r$ and arranged according to Figure 1.3 can be expressed as $(p \times q) + (p \times r)$. We shall prove later that the compound switch $(p \times q) + (p \times r)$ will function in an identical manner as the simpler compound switch $p \times (q + r)$, which is shown in Figure 1.4.

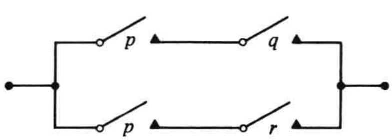


**Figure 1.3**

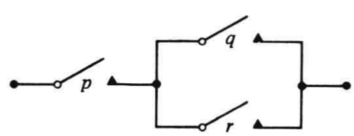


**Figure 1.4**

In Figure 1.3, notice that switch $p$ appears twice. Two or more switches can be constructed in such a way that they are "on" at the same time and "off" at the same time; when this is the case, the same letter is used to represent the switches that behave in an identical manner.

**Example 1**
Sketch the diagram of the compound switch $[q + (r \times q)] \times p$. Start from within, namely $(r \times q)$, and sketch that as shown in Figure 1.5. Then connect compound switch $(r \times q)$ with $q$ in parallel as shown in Figure 1.6. Finally, connect compound switch $[q + (r \times q)]$ in series with $p$ as shown in Figure 1.7. ▮

**Figure 1.5**

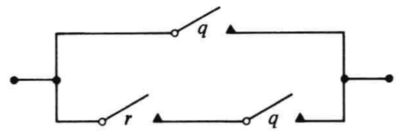

**Figure 1.6**

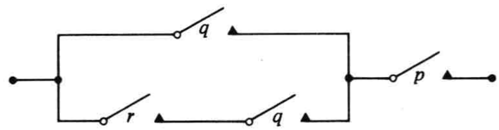

**Figure 1.7**

In this section we have begun to lay the foundation for a new mathematical structure which we will call the Algebra of Switches (components). The elements or building blocks of this structure are not numbers but are single or compound switches (components) $p$, $q$, $r$, $s$, $t$, and so on. Over these elements, the operations presented so far are "connection in parallel," denoted by the symbol $+$, and "connection in series," denoted by the symbol $\times$. Our construction is continued in the remaining sections of the chapter. Hereafter when the word "switch" ("component") is used, it means a single or compound switch (component).

## APPLICATIONS

The applications given at the end of each section are certainly not meant to be an exhaustive listing of the applications of that particular section, nor is any claim made concerning the realism of many of the examples. Rather, these "evidences of usefulness" are supplied primarily in the interest of motivating the reader and with the hope that occasionally one will provide a spark that may arouse a much deeper, productive study in some areas of interest. All applications are optional in that they are not prerequisite to later material in the main part of the text.

**Example 2**
Consider a specific system having $n$ parts or components that are interrelated in a certain way. A "system" may be given a variety of inter-

pretations, such as a spacecraft, a football team, a management team, a computer, a living organism, and so forth. If the components of a system are interrelated in such a way that the entire system fails when any one of the components fails, then the system is called a **series system.** On the other hand a **parallel system** is one that will fail only if all of its components fail. Naturally we can have systems with both parallel and series systems as components. For example, consider a space flight in which we consider as the primary components the spacecraft, the pilot, and the copilot. Such a system can be diagrammed as shown in Figure 1.8.

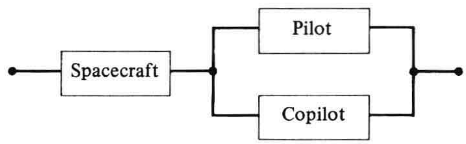


**Figure 1.8**

Let the **reliability** of a system be defined as the probability that the system will perform satisfactorily for a specified time interval under specified environmental conditions. In Chapter 4 after a study of probability is made, we can verify that the reliability $R_s$ of a series system consisting of $n$ independent components is the product of the reliabilities $R_1, R_2, \ldots, R_n$ of the respective components. That is, $R_s = (R_1)(R_2)\cdots(R_n)$. For example, if a series system had 3 components with reliabilities of $\frac{1}{3}$, $\frac{3}{4}$, and $\frac{5}{6}$, then the reliability $R_s$ of the system would be

$$R_s = (\tfrac{1}{3})(\tfrac{3}{4})(\tfrac{5}{6}) = \tfrac{15}{72} \approx 21 \text{ per cent.}$$

We can also verify later that the reliability $R_p$ of a parallel system with $n$ independent components is given by the formula

$$R_p = 1 - [(1 - R_1)(1 - R_2)\cdots(1 - R_n)].$$

For example, if a parallel system had components with reliabilities of $\frac{1}{3}$, $\frac{3}{4}$, and $\frac{5}{6}$, then the reliability of the system would be

$$\begin{aligned} R_p &= 1 - [(1 - \tfrac{1}{3})(1 - \tfrac{3}{4})(1 - \tfrac{5}{6})] \\ &= 1 - [(\tfrac{2}{3})(\tfrac{1}{4})(\tfrac{1}{6})] \\ &= 1 - \tfrac{1}{36} \\ &= \tfrac{35}{36} \approx 97 \text{ per cent.} \end{aligned}$$

The reliability of a system can be improved by the appropriate inclusion of parallel components. In space travel such parallel subsystem components are often referred to as "backup systems." Further discussion of reliability may be found in Example 6 of Section 4.4, and in Miller [35], pp. 362–364. ▮

## EXERCISES

Suggested minimum assignment: Exercises 1, 4, 5, 8, and 9.

In Exercises 1–4 sketch the diagrams of the following switches:

1. $(p + q) \times (r + s)$.
2. $(p \times q) + (r \times s)$.
3. $[p + (p \times q)] + (p \times r)$.
4. $[p \times (p + q)] \times (p + r)$.

5–8. For Exercises 5–8 write the symbolic expressions for the compound switches shown in Figures 1.9–1.12.

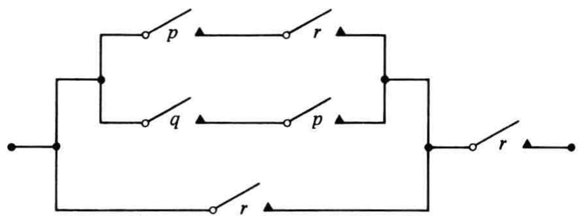


**Figure 1.9**

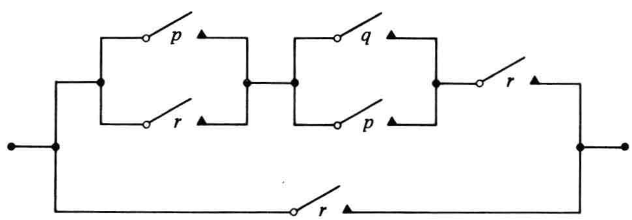


**Figure 1.10**

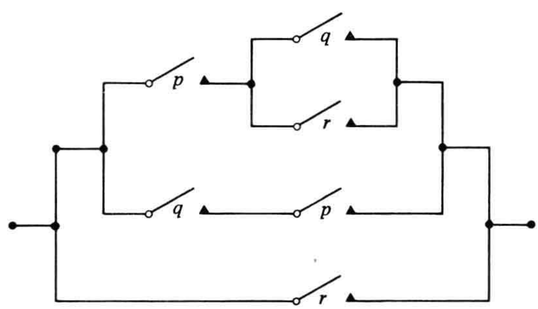


**Figure 1.11**

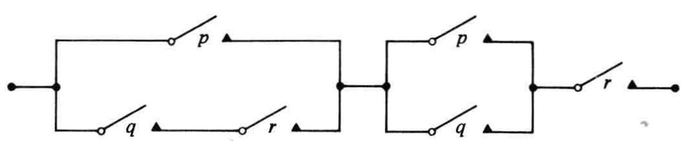


**Figure 1.12**