Lecture Notes in Applied and Computational Mechanics 63

Ulrich Langer
Martin Schanz
Olaf Steinbach
Wolfgang L. Wendland
Editors

Fast Boundary Element Methods in Engineering and Industrial Applications

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Preface

This volume on mathematical aspects and applications of fast boundary element methods in engineering and industry contains eight contributions on the state of the art in this field. This book is strongly related to the annual Söllerhaus workshops on *Fast Boundary Element Methods in Industrial Applications*[1] where recent trends and new methodologies are discussed to solve todays challenging problems in almost all areas of applications. The stimulating atmosphere of the Söllerhaus workshops contributed a lot to new developments and new interdisciplinary cooperations which are also documented within this book. This spirit of strong cooperations between mathematicians and engineers, with direct applications in industry, follows an already long–ongoing history. To underline this, we just mention the volumes *Boundary Element Topics* (Springer 1997), *Boundary Element Analysis. Mathematical Aspects and Applications* (Springer 2007), and the special issue of *Computing and Visualization in Science* (Volume 8, 2005), which indicate the development of the mathematical foundations of boundary integral equation methods and the applications of fast boundary element methods.

Nowadays, fast boundary element methods are a powerful tool for the simulation of physical phenomena in different fields of applications. In particular, boundary integral equation techniques are well suited for the solution of partial differential equations in unbounded exterior domains, or for problems which are considered in complicated geometries, but with simple physical model assumptions. The latter also involves applications with nonlinear interface or transmission conditions, as they appear in multiphysics simulations. Several of these aspects are covered within this book. An efficient and accurate numerical simulation of time–dependent problems both in time and frequency domain belongs still to the most challenging problems. This book includes contributions on the mathematical analysis of boundary integral formulations, the numerical analysis of boundary element methods and the construction of robust and efficient preconditioning strategies, and the design and implementation of fast boundary element methods to solve challenging problems of interest.

[1] see http://www.numerik.math.tu-graz.ac.at/tagungen

The aim of this book is to present some of the current developments of fast boundary element methods and their applications. We are aware that such a book can not cover all aspects in the analysis and applications of fast boundary element methods. There are no contributions, e.g., for adaptive fast boundary element methods. Other missing topics include the use of fast boundary element methods for the simulation of complex multiphysics problems including the coupling with finite element methods, as well as related inverse and shape optimization problems. In fact, this book may serve to present some of the basic tools to handle the above mentioned problems. The ongoing work on the solution of these problems will be reported on future workshops and conferences, and the results will be documented in future publications as well.

We would like to thank all authors for their contributions to this volume. Moreover, we also thank all anonymmous referees for their work, their criticism, and their suggestions. These hints were very helpful to improve the contributions. Finally, we would like to thank Dr. T. Ditzinger of Springer Heidelberg for the continuing support and patience while preparing this volume.

Graz, Linz, Stuttgart
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Contents

Differential Forms and Boundary Integral Equations for Maxwell-Type Problems

Stefan Kurz and Bernhard Auchmann

Abstract. We present boundary-integral equations for Maxwell-type problems in a differential-form setting. Maxwell-type problems are governed by the differential equation $(\delta \mathrm{d} - k^2)\omega = 0$, where $k \in \mathbb{C}$ holds, subject to some restrictions. This problem class generalizes **curl curl**- and div **grad**-types of problems in three dimensions. The goal of the paper is threefold: 1) Establish the Sobolev-space framework in the full generality of differential-form calculus on a smooth manifold of arbitrary dimension and with Lipschitz boundary. 2) Introduce integral transformations and fundamental solutions, and derive a representation formula for Maxwell-type problems. 3) Leverage the power of differential-form calculus to gain insight into properties and inherent symmetries of boundary-integral equations of Maxwell-type.

1 Introduction

It is the goal of this paper to express the theory of boundary-integral equations for Maxwell-type problems in the language of differential-form calculus. Maxwell-type problems are governed by the differential equation

$$(\delta \mathrm{d} - k^2)\omega = 0,$$

where $k \in \mathbb{C}$ fulfills either $k = 0$ or $0 \leq \arg k < \pi, k \neq 0$ [27, eq. (9.13)]. The exterior derivative d and coderivative δ will be defined in Sect. 2.1. This problem class generalizes **curl curl**- and div **grad**-types of problems in three dimensions.

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It encompasses electro- and magnetostatics (potential problems), eddy-current and diffusion-type problems, as well as scattering problems.

In the authors' view, differential-form calculus features a range of advantages over classical vector analysis, that are particularly interesting in the field of boundary-integral equations. We give four examples: (i) Being independent of dimension, operators of the same class act upon fields on the domain and on the boundary. (ii) For a comprehensive treatment of the subject, only two families of functional spaces are required on the domain and on the boundary, respectively. The two families are related via Hodge duality. (iii) Involved computations with cross-products of normal vectors and tangent vectors are replaced by more elegant tools. (iv) A discretization of the functional spaces in terms of discrete differential forms is readily available and, in fact, an integral part of the differential-form setting. In this context HIPTMAIR writes in [17, p. 239ff.]: "Suitable finite elements for electromagnetic fields should be introduced and understood as discrete differential forms. ... Finite elements that lack an interpretation as discrete differential forms have to be used with great care.". For establishing spaces of discrete differential forms on two-dimensional surfaces we also point to [7, Sect. 4.1.].

The reader will find that, in many ways, the theory and proofs outlined in this paper are reminiscent of vector-analysis literature. This is not surprising, since a major part of our work consisted in translating classical proofs to the more general differential-form setting. In other places, presumably well-known subjects may look strangely unfamiliar. Study of the theory from the viewpoint of differential-form calculus reveals structural layers that are often hidden or obscured by the nature of vector analysis. For examples we point to the definition of generalized integral transforms, the image spaces of Sobolev spaces under the Hodge operator, or the symmetry of Calderón projectors under dual transformations. We hope that, with this work, we can help to spark the curiosity for differential-form calculus in the community, and do our share to lay the groundwork for future progress in the field. After all, ROTA wrote [31, p. 46], "Exterior algebra is not meant to prove old facts, it is meant to disclose a new world."

From a historical perspective, the idea to generalize Maxwell's equations, using p-forms in n-dimensional Euclidean space, was first put forward in a seminal paper by WEYL in 1952 [38]. Comparable work for the static case, that is, for potential problems, was accomplished by KRESS in 1972 [23]. Related work about higher dimensional electromagnetic scattering on Lipschitz domains in $\mathbb{R}^n$ was published by JAWERTH and MITREA in 1995 [21]. Recently, PAULY has published a series of papers, where the low frequency asymptotics for generalized Maxwell equations have been examined under rather general assumptions [29].

In Sect. 2 we give a concise summary of relevant topics of differential-form calculus. This summary is intended mainly for reference purposes. Readers who are not familiar with the formalism might want to consult [20], [16, Chap. A] or [2, Sect. 2]. Sect. 2 also includes contributions on topics such as integral transformations, and fundamental solutions of Helmholtz-type equations. So-called translation isomorphisms are introduced, that carry the differential-form setting in three-dimensional Euclidean space over to the classical vector-analysis setting. Sect. 3 presents a differential-form based Sobolev-space framework that sets the scene for the discussion of Maxwell-type problems, their solutions, and boundary data. The section builds upon a 2004 work by WECK [36]. Translation isomorphisms are used to establish the link with Sobolev spaces in classical calculus. Sect. 4 is devoted entirely to the representation formula for Maxwell-type problems. The results generalize the Kirchhoff and Stratton-Chu formulae. In Sect. 5 we introduce boundary-integral operators and establish some of the properties that are required to prove the well-posedness of boundary-value problems. Finally, Sect. 6 studies properties of the Calderón projector and reveals a powerful symmetry with respect to dual transformations.

In our notation, we seek to strike a balance between readability on the one hand, and the addition of information that helps to interpret the compact differential-form notation on the other hand. If in doubt, we tend to favor the former over the latter, assuming that the generality and elegance of differential-form calculus best serve the readers' interest. For example, operators in Sect. 2 are defined for forms of arbitrary degrees, and on (Riemannian) manifolds of arbitrary dimension. We therefore do not generally distinguish in our notation between, for example, the Hodge operators acting on forms of various degrees on a domain Ω, and the Hodge operators acting upon the traces of said forms on the boundary Γ. The metric tensor which applies in the definition of each operator is clear from the context. A generalization that we did not adopt is to introduce graded Sobolev spaces on the entire exterior algebra of differential forms. We have opted for spaces of homogeneous degree and highlight the degree in the notation. All along the text, the relationship to results of classical vector analysis is established in framed paragraphs, to keep the paradigms separate in the main body of the paper.

2 Differential Forms – Preliminaries

In this section, we intend to summarize important results of differential-form calculus. Throughout the paper, n denotes the dimension of the problem domain; the degree of forms is frequently denoted by p and q, which are always related by $q = n - p$.

Powers of minus one followed by operators, as in $(-1)^{pq}\mathrm{op}_1\,\mathrm{op}_2\,\phi$, are to be read as follows: The degree p refers to the differential form that the sequence of operators acts upon from the left. In this example, p is the degree of the form ϕ. n is always the dimension of the problem domain, even if operators and forms on the domain boundary are considered; and $q = n - p$ following the above rule.

2.1 Basic Definitions

We introduce differential forms on a smooth, orientable Riemannian manifold (M, g) of finite dimension n, where g denotes the metric tensor. We have $\mathbb{R}^3$, or a subset thereof, with Euclidean metric in mind. Throughout this section V denotes a vector space over a field $\mathbb{F}$, where $\mathbb{F}$ may be either $\mathbb{R}$ or $\mathbb{C}$.

A *simple p-vector* may be thought of as an ordered p-tuple of vectors that belong to a vector space V. The p-tuple is interpreted as a p-parallelepiped with oriented volume. An elementary permutation in the tuple changes the orientation. A change of orientation is indicated by a change of sign of the simple p-vector. More precisely, a simple p-vector is an equivalence class of ordered p-tuples of vectors that (i) span the same subspace of V; (ii) span p-parallelepipeds of identical oriented volume. *p-vectors* are linear combinations of simple p-vectors. They form a vector space $\Lambda^p V$ of dimension $\binom{n}{p}$, $0 \le p \le n$. Up to dimension $n = 3$ all p-vectors are simple p-vectors. We find $\Lambda^1 V = V$, $\Lambda^p V = \emptyset$ for $p > n$ and for $p < 0$, and we set $\Lambda^0 V = \mathbb{F}$.

Alternatively, p-vectors are defined in [15] via an isomorphism that identifies $\Lambda^p V$ with the vector space of skew-symmetric tensors of rank p over V.

Let $(\mathbf{e}_i \,|\, 1 \le i \le n)$ denote an ordered basis of V. We pick an *ordered basis of* $\Lambda^p V$

$$(\mathbf{e}_J \,|\, J \in \mathscr{J}_p^n),$$

where $J = j_1 j_2 \dots j_p$ is a multiindex,

$$\mathscr{J}_p^n = \{J = j_1 j_2 \dots j_p \,|\, 1 \le j_1 < j_2 < \dots < j_p \le n\},$$

and $\mathbf{e}_J$ is the equivalence class that contains the p-tupel $(\mathbf{e}_{j_1}, \mathbf{e}_{j_2}, \dots, \mathbf{e}_{j_p})$.

The *exterior product*, or wedge product, is a bilinear mapping

$$\wedge : \Lambda^k V \times \Lambda^\ell V \to \Lambda^{k+\ell} V : (\mathbf{v}, \mathbf{w}) \mapsto \mathbf{v} \wedge \mathbf{w},$$

defined by the following properties:

(i) $\wedge$ is associative, $(\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w} = \mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w})$, $\quad \mathbf{u} \in \Lambda^j V$;
(ii) $\wedge$ is graded anticommutative, $\mathbf{v} \wedge \mathbf{w} = (-1)^{k\ell} \mathbf{w} \wedge \mathbf{v}$ for $\mathbf{v} \in \Lambda^k V$ and $\mathbf{w} \in \Lambda^\ell V$;
(iii) $1 \wedge \mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in \Lambda^k V$.

To compute the exterior product we first relate the basis vectors of V to those of $\Lambda^p V$. Let $K = k_1 k_2 \dots k_p$ be an arbitrary p-index, and $\sigma(K)$ a permutation of K. Then we define

$$\mathbf{e}_{k_1} \wedge \dots \wedge \mathbf{e}_{k_p} = \begin{cases} +\mathbf{e}_{\sigma(K)} & \sigma(K) \in \mathscr{J}_p^n, \quad \sigma \text{ even}, \\ -\mathbf{e}_{\sigma(K)} & \sigma(K) \in \mathscr{J}_p^n, \quad \sigma \text{ odd}, \\ 0 & \text{otherwise}. \end{cases} \tag{1}$$

Next we define for basis vectors $\mathbf{e}_I \in \Lambda^k V$, $\mathbf{e}_J \in \Lambda^\ell V$

$$\begin{aligned} \mathbf{e}_I \wedge \mathbf{e}_J &= (\mathbf{e}_{i_1} \wedge \dots \wedge \mathbf{e}_{i_k}) \wedge (\mathbf{e}_{j_1} \wedge \dots \wedge \mathbf{e}_{j_\ell}) \\ &= \mathbf{e}_{i_1} \wedge \dots \wedge \mathbf{e}_{i_k} \wedge \mathbf{e}_{j_1} \wedge \dots \wedge \mathbf{e}_{j_\ell}, \end{aligned}$$

where $I \in \mathscr{J}_k^n$, $J \in \mathscr{J}_\ell^n$. The right hand side is defined by (1), or zero for $k+\ell > n$, respectively. Finally, the exterior product extends by linearity to the entire spaces. We say that $\Lambda^p V$ is the p-th exterior power of V. The direct sum $\Lambda V = \Lambda^0 V \oplus \Lambda^1 V \oplus \cdots \oplus \Lambda^{n-1} V \oplus \Lambda^n V$ is again a vector space. The pair $(\Lambda V, \wedge)$ has the structure of a graded algebra. It is called the *exterior algebra* over V.

Recall that a *tangent vector* on a manifold can be defined as a directional-derivative operator. Coordinates $(x^1, ..., x^n)$ on a patch $U \subset M$ induce a canonical coordinate basis $(\partial_{x^1}, ..., \partial_{x^n})$ of the tangent space $T_X M$. Here TM is the tangent bundle over M, and $T_X M$ is a fibre in a point $X \in U$. *Cotangent vectors*, on the other hand, are elements of the dual space $T_X^* M$, the cotangent space. In particular, the differential of a scalar function λ on M taken at a point X is a cotangent vector. The action of a cotangent vector $(\mathrm{d}\lambda)_X \in T_X^* M$ on a vector $\mathbf{v} \in T_X M$ equals the directional derivative of λ in direction of $\mathbf{v}$, $(\mathrm{d}\lambda)_X(\mathbf{v}) = \mathbf{v}(\lambda)$. The canonical coordinate basis of the cotangent space in a point $X \in U$ reads $(\mathrm{d}x^1, ..., \mathrm{d}x^n)$, and we see that $\mathrm{d}x^i(\partial_{x^j}) = \partial_{x^j}(x^i) = \delta_j^i$, with δ_j^i the Kronecker delta.

A vector field $\mathbf{v}$ is a section of TM. The space of smooth vector fields (component functions are C^∞) is denoted $\mathscr{X}_1(M)$. The space of smooth differential 1-forms is denoted $\mathscr{F}^1(M)$; its elements are smooth sections of the cotangent bundle. Note that the definitions of $\mathscr{X}_1(M)$ and $\mathscr{F}^1(M)$ require smoothness of the manifold M.

A p-vector field is a section of the p-th exterior power of the tangent bundle $\Lambda^p TM$, whose fibres are the spaces $\Lambda^p T_X M$. The space of *smooth p-vector fields* is denoted $\mathscr{X}_p(M)$, and the space of *smooth differential p-forms* $\mathscr{F}^p(M)$.

Coordinate bases of $\mathscr{X}_p(M)$ and $\mathscr{F}^p(M)$ are given in $U \subset M$ by $(\partial_{x^J} \,|\, J \in \mathscr{J}_p^n)$ and $(\mathrm{d}x^J \,|\, J \in \mathscr{J}_p^n)$, respectively. Hence the *basis representation*

$$\omega = \sum_{J \in \mathscr{J}_p^n} \omega_J \, \mathrm{d}x^J \in \mathscr{F}^p(M),$$

where $\omega_J \in C^\infty(M)$ are the component functions.

In the sequel, we will encounter function spaces of differential forms. A generic function space of p-forms defined on M will be denoted $\mathscr{H}\Lambda^p(M)$. For instance, $\mathscr{F}^p(M) = C^\infty \Lambda^p(M)$.

We denote by

$$\langle \cdot | \cdot \rangle_X : \Lambda^p T_X^* M \times \Lambda^p T_X M \to \mathbb{F}$$

the algebraic *duality product* at a point $X \in M$.

The exterior product above extends naturally to p-forms and p-vector fields. An alternative notation is given by

$$\mathrm{j} : \mathscr{F}^\ell(M) \times \mathscr{F}^k(M) \to \mathscr{F}^{\ell+k}(M) : (\omega, \eta) \mapsto \mathrm{j}_\eta \omega = \eta \wedge \omega,$$

and, analogously,

$$\mathrm{j} : \mathscr{X}_\ell(M) \times \mathscr{X}_k(M) \to \mathscr{X}_{\ell+k}(M) : (\mathbf{v}, \mathbf{w}) \mapsto \mathrm{j}_\mathbf{w} \mathbf{v} = \mathbf{w} \wedge \mathbf{v}.$$