Integral Transforms and Fourier Series

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Preface

The technique of integral transforms in solving boundary-value problems of mathematical physics finds its origin in Heaviside's works which initially contained solution of ordinary differential equations with constant coefficients. The technique developed by Heaviside, Bromwich and Carson, unified in the works of Doetsch, has found advantage over the classical methods of solving such problems and has been used in solving many intractable problems.

The present book is intended as a basic text for UG & PG students of Science, Computer Science and Engineering. It has been written with two pedagogical goals in mind to offer complete topics in one book and to keep the presentation student friendly to make it the best suited book for independent study as well.

Chapter 1 deals with the basic ideas, concepts, methods and related theorems of Laplace Transforms and their applications in finding the solutions of differential and integral equations with a large number of solved and unsolved problems of varied nature.

Chapter 2 covers the theory of Fourier series in all respect including Fourier series in case of change of intervals, even and odd functions and application of Drichlet's theorem to Fourier series with a variety of solved and unsolved problems of different categories.

Chapter 3 contains the basic concepts, techniques in all details to find the Fourier transform and Fourier Sine, Cosine transforms of various functions in different types of intervals with applications to boundary value problems.

Chapter 4 of the book is devoted to the definitions, basic ideas, properties and applications of Z-Transform to various types of problems. Z-Transform is of immense use to those pursuing engineering courses at various levels.

Chapter 5 provides basics of other important integral transforms such as Mellin, Hilbert, Hankel, Weierstrass and Abel transforms and detail study of Hankel transforms. Although they do not form the subject matter of UG syllabus, it will give an exposure to interested readers willing to pursue the branch at advanced level.

Since one learns by doing, a large number of carefully selected solved problems follow the theory in each of the sections, and, to build up confidence review exercises have been given in the end of each chapter. The novel approach presented in the book is an outcome of more than two decades of our teaching Integral Transform and Fourier series to undergraduate students. Over these years and also during preparation of the manuscript, we have been greatly helped and benefited by the excellent works of our senior authors. We sincerely acknowledge our indebtedness to all of them. Some of these valuable works appear in the Bibliography.

Our thanks are due to the publishers for their keen interest and cooperation in bringing out this book.

We look forward receiving constructive criticism and valuable suggestions which would help improving the future editions of the book.

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Laplace Transforms with Applications

1.1 INTRODUCTION

Of all the integral transforms, Laplace transform has been more widely used owing to its suitability for the problems in heat conduction and also in ordinary Differential Equations. This chapter exposes the readers towards its properties and mechanism to apply it.

1.2 DEFINITION

For a given function $f(t)$, defined $\forall\ t \geq 0$, Laplace transform $L\ \{f(t)\}$ is defined by

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\ dt, \tag{1.2.1}$$

and the value of integral in the right side is denoted by $F(s)$, where s is a parameter-real or Complex. Thus Laplace transform $L\ \{f(t)\}$ of a function $f(t)$ is $F(s)$, provided $F(s)$ exists.

The symbol L which transforms $f(t)$ into $F(s)$ is called the *Laplace transform operator.*

$$\because \qquad L\{f(t)\} = F(s),$$

We further define the *operator* L^{-1} as:

$$f(t) = L^{-1}\ \{F(s)\}$$

Thus L^{-1}, *the inverse Laplace transform operator*, transforms back $F(s)$ to $f(t)$.

1.3 BASIC INTEGRATION FORMULAS

Following basic formulas of integration will be required in evaluating Laplace transforms of different functions:

1. $\int x^n dx = \frac{x^{n+1}}{(n+1)}$ $(n \neq -1)$; $\int \frac{dx}{x} = \log_e^x$
2. $\int e^x dx = e^x$; $\int e^{kx}\, dx = \frac{1}{k} e^{kx}$ $(k \neq 0)$
3. $\int a^x dx = \frac{a^x}{\log_e^a}$
4. $\int \sin x\, dx = -\cos x$
5. $\int \cos x\, dx = \sin x$
6. $\int \tan x\, dx = \log \sec x$
7. $\int \cot x\, dx = \log \sin x$
8. $\int \sec x\, dx = \log(\sec x + \tan x)$
9. $\int \operatorname{cosec} x\, dx = \log(\operatorname{cosec} x - \cot x)$
10. $\int \sec^2 x\, dx = \tan x$
11. $\int \operatorname{cosec}^2 x\, dx = -\cot x$
12. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$
13. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \frac{a+x}{a-x}$
14. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x-a}{x+a}$
15. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$
16. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1} \left(\frac{x}{a}\right)$
17. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \left(\frac{x}{a}\right)$
18. $\int \cosh x\, dx = \sinh x$
19. $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$
20. $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$
21. $\int uv\, dx = u \int v\, dx - \int \left[\frac{d}{dx} u \int v\, dx\right] dx$

    (By parts rule)

1.4 ILLUSTRATIVE EXAMPLES ON § 1.2

We will illustrate definition 1.2 of Laplace transform in case of few simpler functions

Example 1.4.1 Find the Laplace transform of following functions:

(i) $f(t) = 1$, (ii) $f(t) = t^n$, (iii) $f(t) = e^{at}$,
(iv) $f(t) = \sin at$, (v) $f(t) = \cos at$, (vi) $f(t) = \sinh at$,
(vii) $f(t) = \cosh at$.

Solution (i) We have $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$. Choose $f(t) = 1$ to get

$$L\{1\} = \int_0^\infty e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s},$$

if $s > 0$. Thus $L\{1\} = \frac{1}{s} = F(s)$, when $s > 0$.

(ii) Choose $f(t) = t^n$ to get

$$L\{t^n\} = \int_0^\infty e^{-st}\, t^n\, dt$$

To evaluate the integral in right, put $st = p$, we get

$$L\{t^n\} = \int_0^\infty e^{-p} \left(\frac{p}{s}\right)^n \frac{dp}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-p}\, p^n\, dp$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-p}\, p^{(n+1)-1}\, dp$$

$$= \frac{\Gamma(n+1)}{s^{n+1}} \text{ if } n > -1$$

and $s > 0$, where

gamma function $\Gamma(n)$ has the integral $\int_0^\infty e^{-x} x^{n-1}\, dx$ $(n > 0)$, which reduces to $n!$ according to relation

$$\Gamma(n+1) = n!, \text{ when } n \in Z^+.$$

Thus $L\{t^n\} = \frac{n!}{s^{n+1}}$ where $s > 0$ and n is a + ve integer.

(iii) Choose $f(t) = e^{at}$ in (1.2) to get

$$L\{e^{at}\} = \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty = \frac{1}{s-a} \quad \text{if} \quad s > a$$

Likewise $L\{e^{-at}\} = \dfrac{1}{s+a}$ if $s > -a$

(iv) $$L\{\sin at\} = \int_0^\infty e^{-st} \sin at \, dt = \frac{a}{s^2 + a^2} \text{ if } s > 0,$$

by formula (19). Alternatively, $L\{\sin at\}$ = Imaginary part of $\int_0^\infty e^{-st} e^{ait} \, dt$

$$= \text{Im. part of } L\{e^{ait}\}$$

$$= \text{Im. part of } \frac{1}{s - ia}$$

$$= \frac{a}{s^2 + a^2}$$

(v) Proceeding the same way, as in (iv), we get

$$L\{\cos at\} = \int_0^\infty e^{-st} \cos at \, dt$$

$$= \frac{s}{s^2 + a^2} \text{ for } s > 0$$

(vi) $$L\{\sinh at\} = L\left\{ \frac{e^{at} - e^{-at}}{2} \right\}$$

$$= \frac{1}{2} [L\{e^{at}\} - L\{e^{-at}\}] \quad \text{(using 1.5.1)}$$

$$= \frac{a}{s^2 - a^2} \quad \text{(on simplification) for } s > |a|.$$

(vii) Likewise $L\{\cosh at\} = L\left\{ \dfrac{e^{at} + e^{-at}}{2} \right\} = \dfrac{s}{s^2 - a^2}$, $s > |a|$

1.5 PROPERTIES OF LAPLACE TRANSFORM

1.5.1 Linearity Property

Let f, g, h be the functions of t, then

$$L\{a f(t) + b g(t) + c h(t)\} = a L\{f(t)\} + b L\{g(t)\} + c L\{h(t)\},$$

where a, b, c are constants.

Proof: $$\text{LHS} = \int_0^\infty e^{-st} (a f(t) + b g(t) + c h(t)) \, dt$$

$$= a \int_0^\infty e^{-st} f(t) \, dt + \ldots + \ldots$$

$$= a\ L\{f(t)\} + b\ L\{g(t)\} + c\ L\{h(t)\}$$

$$= \text{Rhs.}$$

Note The property can be extended for any finite number of such combinations.

1.5.2 Illustrative Examples Based on 1.5.1

Example 1.5.2.1 Find the LT of following functions:

(i) $f(t) = e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$

(ii) $f(t) = \cos^2 kt$

(iii) $f(t) = 1 + 2\sqrt{t} + \dfrac{3}{\sqrt{t}}$

(iv) Find: $L\{\cos\sqrt{t}\}$

Solution (i) Taking LT on both the sides,

$$L\{f(t)\} = L\{e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t\}$$

$$= L\{e^{2t}\} + 4\ L\{t^3\} - 2\ L\{\sin 3t\} + 3\ L\{\cos 3t\} \quad \text{(using Linearity property)}$$

$$= \frac{1}{s-2} + \frac{24}{s^4} + \frac{3(s-2)}{s^2+9}.$$

(ii)

$$L\{\cos^2 kt\} = L\left\{\frac{1+\cos 2\,kt}{2}\right\}$$

$$= \frac{1}{2}L\{1\} + \frac{1}{2}L(\cos 2kt\}$$

$$= \frac{1}{2s} + \frac{1}{2}\left(\frac{s}{s^2+4k^2}\right)$$

$$= \frac{s^2+2k^2}{s(s^2+4k^2)}$$

(iii) Same as in (i) and (ii) above

(iv) By series formula of cost,

$$\cos\sqrt{t} = \sum_{n=0}^{\infty} \frac{(-1)^n (t^{1/2})^{2n}}{(2n)!},$$

so that $L\{\cos\sqrt{t}\}$

$$= L\left\{\sum_{n=0}^{\infty} \frac{(-1)^n (t^{1/2})^{2n}}{(2n)!}\right\}$$

$$= \sum_{n=0}^{\infty} L\left\{\frac{(-1)^n\, t^n}{(2n)!}\right\} \qquad \text{(by 1.5.1)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\,(n)!}{(2n)!\; s^{n+1}}$$

1.5.3 First Shifting Property

If $\qquad L\{f(t)\} = F(s)$, then $L\{e^{at} f(t)\} = F(s-a)$.

Proof: RHS

$$= \int_0^{\infty} e^{-(s-a)t} f(t)\, dt$$

$$= \int_0^{\infty} e^{-st}\left(e^{at} f(t)\right) dt$$

$$= L\{e^{at} f(t)\}$$

1.5.4 Illustrative Examples on 1.5.3

Example 1.5.4.1 Find LT of following functions: (i) $e^{at}\sin bt$ (ii) $e^{at}\cos h\; bt$

Solution (i) In $e^{at}\sin bt$, let $f(t) = \sin bt$, then

$$L\{f(t)\} = L\{\sin bt\} = \frac{b}{s^2+b^2} = F(s), \text{ say}$$

so that $\qquad L\{e^{at}\sin bt\} = L\{e^{at} f(t)\} = F(s-a)$

$$= \frac{b}{(s-a)^2+b^2}$$

Thus $\qquad L\{e^{at}\sin bt\} = \dfrac{b}{(s-a)^2+b^2}$

(ii) Likewise $\quad L\{\cosh bt\} = \dfrac{s^2}{s^2-b^2} = F(s)$, then

$$L\{e^{at}\cosh bt\} = \frac{(s-a)}{(s-a)^2-b^2}.$$

Example 1.5.4.2 Find the LT of $f(t)$ if

(i) $f(t) = e^{3t}\, t^{5/2}$ (ii) $f(t) = e^{-3t}\,(2\cos 5t - 3\sin 5t)$ (iii) $f(t) = \sinh at \sin at$

Solution (i) Let $f(t) = t^{5/2} \Rightarrow L\{f(t)\} = L\{t^{5/2}\}$

$$= \frac{\Gamma\left(\frac{5}{2}+1\right)}{s^{5/2+1}} = \frac{15\sqrt{\pi}}{8\, s^{7/2}} = F(s), \text{ say}$$

Thus $$L\{e^{3t}\, t^{5/2}\} = F(s-3) = \frac{15\sqrt{\pi}}{8(s-3)^{7/2}}$$

(ii) Let $f(t) = 2\cos 5t - 3\sin 5t$

Then, $$L\{f(t)\} = \frac{2s}{s^2+5^2} - \frac{15}{s^2+5^2} = F(s), \text{ say}$$

so that $$L\{e^{-3t}\,(2\cos 5t - 3\sin 5t)\} = \frac{2s-9}{s^2+6s+34}$$

(iii) $\sinh at \sin at = \frac{1}{2}\,(e^{at} - e^{-at}) \sin at$

Choose $$f(t) = \sin at \text{ and let } L\{f(t)\} = L\{\sin at\}$$

$$= \frac{a}{s^2+a^2} = F(s), \text{ say}$$

Then $$L\{e^{at}\, f(t)\} = F(s-a) = \frac{a}{(s-a)^2+a^2}$$

and $$L\{e^{-at}\, f(t)\} = F(s+a)$$

$$= \frac{a}{(s+a)^2+a^2}$$

so that, finally

$$L\{\sinh at \sin at\} = \frac{2a^2 s}{s^4+4a^4}.$$

1.5.5 Change of Scale Property

If $$L\{f(t)\} = F(s), \text{ then}$$

$$L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

Proof: $$L\,\{f(at)\} = \int_0^\infty e^{-st} f(at)\, dt$$

$$= \frac{1}{a}\int_0^\infty e^{\frac{-su}{a}} f(u)\, du$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right), \text{ putting } u = at$$

1.5.6 Illustrative Examples on 1.5.5

Example 1.5.6.1 If $L\{f(t)\} = \frac{e^{-1/s}}{s}$, find $L\{e^{-t} f(2t)\}$

Solution $$L\{f(t)\} = \frac{e^{-1/s}}{s} = F(s),$$

Making use of property 1.5.5 with $a = 2$,

we get $$L\{f(2t)\} = \frac{1}{2}\,F\left(\frac{s}{2}\right) = \frac{1}{2}\,\frac{e^{-2/s}}{s/2} = \frac{e^{-2/s}}{s}$$

which on applying first shifting property

$$L\{e^{-t}\,f(2t)\} = \frac{e^{-2/(s+1)}}{s+1}$$

1.5.7 Laplace Transform of Derivatives

If $L\{f(t)\} = F(s)$, and $f'(t)$ be continuous then

(a) $$L\{f'(t)\} = sF(s) - f(0)$$

Proof: $$L\{f'(t)\} = \int_0^\infty e^{-st}\,f'(t)\,dt,$$

Integrating by parts,

$$L\{f'(t)\} = \left[e^{-st}\,f(t)\right]_0^\infty - \int_0^\infty (-s)\,e^{-st}\,f(t)\,dt$$

Assuming that $\lim_{t\to\infty} e^{-st}\,f(t) = 0$, i.e., $f(t)$ is of exponential order s, then

$$L\{f'(t)\} = -f(0) + s\int_0^\infty e^{-st}\,f(t)\,dt$$

$$= -f(0) + sF(s)$$

(b) Higher order derivatives

If $f'(t)$ and its first $(n-1)$ derivatives be continuous then

$$L\{f^n(t)\} = s^n\,F(s) - s^{n-1}\,f(0) - s^{n-2}\,f'(0)\,\ldots - f^{n-1}_{(0)}$$

Proof: $$L\{f^n(t)\} = \int_0^\infty e^{-st}\,f^n(t)\,dt,$$

Which, on using the general rule of integration by parts, gives us

$$\left[e^{-st}f^{n-1}(t) - (-s)\,e^{-st}\,f^{n-2}(t) + (-s)^2\,e^{-st}\,f^{n-3}(t) + \ldots + (-1)^{n-1}(-s)^{n-1}e^{-st}f(t)\right]_0^\infty$$

$$+ (-1)^n\,(-s)^n\int_0^\infty e^{-st}\,f(t)\,dt$$

Assuming that $\lim_{t\to\infty} e^{-st}\,f^p(t) = 0$ for

$p = 0, 1, 2, \ldots\, n-1$, we get the RHS.

1.5.8 Illustrative Examples on 1.5.7

Example 1.5.8.1 Find the LT of following functions using transform of derivatives

(i) $f(t) = t^2$

(ii) $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}\, e^{-1/4s}$, Given $L\{\sin\sqrt{t}\} = \sqrt{\frac{\pi}{s}} \cdot \frac{1}{2s} \cdot e^{-1/4s}$

Solution (i) Let $f(t) = t^2$, then $f(0), f'(0) = 0\ f''(0) = 2$, to give us

$$L\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$

$$L\{2\} = s^2 L\{f(t)\} - 0; \text{ thus}$$

$$L\{f(t)\} = \frac{L(2)}{s^2} = 2/s^3$$

$$\therefore \qquad L\{t^2\} = 2/s^3.$$

(ii) Taking $f(t) = \cos\frac{\sqrt{t}}{\sqrt{t}}$ which is derivative of $\sin\sqrt{t}$, let $g(t) = \sin\sqrt{(t)}$, giving $g'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$, $g(0) = 0$

$$L\{g'(t)\} = s\, L\{g(t)\} - g(0)$$

$$L\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = s\, L\{\sin\sqrt{t}\} - 0$$

$$\Rightarrow \qquad \frac{1}{2} L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = s\sqrt{\frac{\pi}{s}} \cdot \frac{1}{2s} \cdot e^{-1/4s}$$

$$\Rightarrow \qquad L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \left(\frac{\pi}{s}\right)^{1/2} \cdot e^{-1/4s}.$$

1.5.9 Laplace Transform of Integrals

If $L\{f(t)\} = F(s)$, then $L\left\{\int_0^t f(u)\, du\right\} = \frac{F(s)}{s}$

Proof: Let $g(t) = \int_0^t f(u)\, du$ then $g'(t) = f(t)$ and $g(0) = 0$

$$L\{g'(t)\} = s\, L\{g(t)\} - g(0)$$

$$L\{f(t)\} = s\, L\left\{\int_0^t f(u)\, du\right\}$$

$$\Rightarrow \qquad L\left\{\int_0^t f(u)\, du\right\} = \frac{L\{f(t)\}}{s} = \frac{F(s)}{s}.$$

1.5.10 The Function $t^n f(t)$

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s), \text{ where } n = 1, 2, \ldots$$