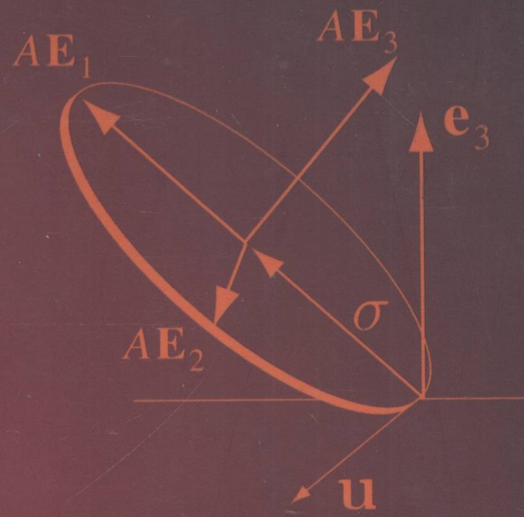

GEOMETRIC MECHANICS

Part II: Rotating, Translating and Rolling

DARRYL D HOLM

Imperial College Press

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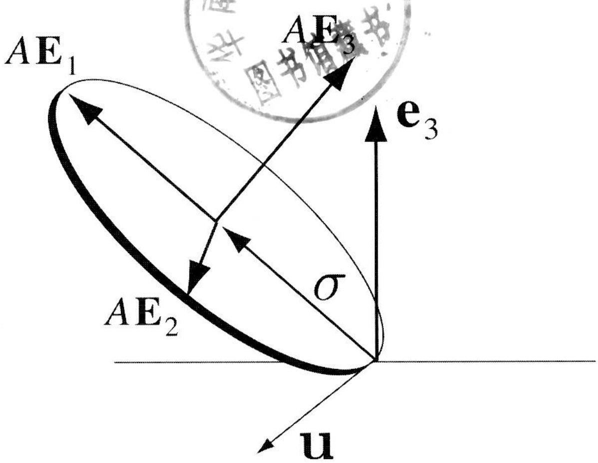

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GEOMETRIC MECHANICS

Part II: Rotating, Translating and Rolling

To Justine, for her love, kindness and patience.
Thanks for letting me think this was important.

Preface

Introduction

This text is based on a course of thirty three lectures in geometric mechanics, taught annually by the author to fourth year undergraduates in their last term in applied mathematics at Imperial College London. The text mimics the lectures, which attempt to provide an air of immediacy and flexibility in which students may achieve insight and proficiency in using one of the fundamental approaches for solving a variety of problems in geometric mechanics. This is the Euler-Poincaré approach, which uses Lie group invariance of Hamiltonian's principle to produce symmetry-reduced motion equations and reveal their geometrical meaning. It has been taught to students with various academic backgrounds from mathematics, physics and engineering.

Each chapter of the text is presented as a line of inquiry, often by asking sequences of related questions such as, What is angular velocity? What is kinetic energy? What is angular momentum? and so forth. In adopting such an inquiry-based approach, one focuses on a sequence of exemplary problems, each of whose solutions facilitates taking the next step. The present text takes those steps, forgoing any attempt at mathematical rigour. Readers more interested in a rigorous approach are invited to consult some of the many citations in the bibliography which treat the subject in that style.

Prerequisites

The prerequisites are standard for an advanced undergraduate student. Namely, the student should be familiar with linear algebra of vectors and matrices, ordinary differential equations, multivariable calculus and have some familiarity with variational principles and canonical Poisson brackets in classical mechanics at the level of a second or third year undergraduate in mathematics, physics, and engineering. An undergraduate background in physics is particularly helpful, because all the examples of rotating, spinning and rolling rigid bodies treated here from a geometric viewpoint are familiar from undergraduate physics classes.

How to read this book

Most of the book is meant to be read in sequential order from front to back. The 120 Exercises and 55 Worked Answers are indented and marked with ★ and ▲, respectively.

> Key theorems, results and remarks are placed into frames (like this one).

The three appendices provide supplementary material, such as condensed summaries of the essentials of manifolds (Appendix A) and Lie groups (Appendix B) for students who may wish to acquire a bit more mathematical background. In addition, the appendices provide material for supplementary lectures that extend the course material. Examples include variants of rotating motion that depend on more than one time variable, as well as rotations in complex space and in higher dimensions in Appendix C. The appendices also contain ideas for additional homework and exam problems that go beyond the many exercises and examples sprinkled throughout the text.

Description of contents

Galilean relativity and the idea of a uniformly moving reference frame are explained in Chapter 1. Rotating motion is then treated in

Chapters 2, 3 and 4, first by reviewing Newton's and Lagrange's approaches, then following Hamilton's approach via quaternions and Cayley-Klein parameters, not Euler angles.

Hamilton's rules for multiplication of quaternions introduced the adjoint and coadjoint actions that lie at the heart of geometric mechanics. For the rotations and translations in $\mathbb{R}^3$ studied in Chapters 5 and 6, the adjoint and coadjoint actions are both equivalent to the vector cross product. Poincaré [Po1901] opened the field of geometric mechanics by noticing that these actions define the motion generated by any Lie group.

When applied to Hamilton's principle defined on the tangent space of an arbitrary Lie group, the adjoint and coadjoint actions studied in Chapter 6 result in the Euler-Poincaré equations derived in Chapter 7. Legendre transforming the Lagrangian in Hamilton's principle summons the Lie-Poisson Hamiltonian formulation of dynamics on a Lie group. The Euler-Poincaré equations provide the framework for all of the applications treated in this text. These applications include finite dimensional dynamics of three-dimensional rotations and translations in the special Euclidean group $SE(3)$. The Euler-Poincaré problem on $SE(3)$ recovers Kirchhoff's classic treatment in modern form of the dynamics of an ellipsoidal body moving in an incompressible fluid flow without vorticity.

The Euler-Poincaré formulation of Kirchhoff's problem on $SE(3)$ in Chapter 7 couples rotations and translations, but it does not yet introduce potential energy. The semidirect-product structure of $SE(3)$, however, introduces the key idea for incorporating potential energy. Namely, the same semidirect-product structure is also invoked in passing from rotations of a free rigid body to rotations of a heavy top with a *fixed* point of support under gravity. Thereby, semidirect-production reduction becomes a central focus of the text.

The heavy top treated in Chapter 8 is a key example, because it introduces the dual representation of the action of a Lie algebra on a vector space. This is the diamond operation ($\diamond$), by which the forces and torques produced by potential energy gradients are represented in the Euler-Poincaré framework in Chapters 9, 10 and 11. The diamond operation ($\diamond$) is then found in Chapter 12 to lie at the heart of the standard (cotangent-lift) momentum map.

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Chapter 1

Galileo

1.1 Principle of Galilean relativity

Galileo Galilei

Principles of relativity address the problem of how events that occur in one place or state of motion are observed from another. And if events occurring in one place or state of motion look different from those in another, how should one determine the laws of motion?

Galileo approached this problem via a thought experiment which imagined observations of motion made inside a ship by people who could not see outside.

Galileo showed that the people isolated inside a uniformly moving ship are *unable to determine by measurements made inside it whether they are moving!*

> ... have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still.
> – Galileo Galilei, *Dialogue Concerning the Two Chief World Systems* [Ga1632]

Galileo's thought experiment showed that a man who is below decks on a ship cannot tell whether the ship is docked or is moving uniformly through the water at constant velocity. He may observe water dripping from a bottle, fish swimming in a tank, butterflies flying, etc. Their behaviour will be just the same, whether the ship is moving or not.

Definition 1.1.1 (Galilean transformations)
*Transformations of reference location, time, orientation or state of uniform translation at constant velocity are called **Galilean transformations**.*

Definition 1.1.2 (Uniform rectilinear motion)
*Coordinate systems related by Galilean transformations are said to be in **uniform rectilinear motion** relative to each other.*

Galileo's thought experiment led him to the following principle.

Definition 1.1.3 (Principle of Galilean relativity)
The laws of motion are independent of reference location, time, orientation, or state of uniform translation at constant velocity. Hence, these laws are invariant under Galilean transformations.

Remark 1.1.4 (Two tenets of Galilean relativity)
Galilean relativity sets out two important tenets:
(1) It is impossible to determine who is actually at rest; and
(2) Objects continue in uniform motion unless acted upon.
*The second tenet is known as **Galileo's Law of Inertia**.*
*It is also the basis for **Newton's First Law of Motion**.*

1.2 Galilean transformations

Definition 1.2.1 (Galilean transformations)
Galilean transformations of a coordinate frame consist of space-time translations, rotations and reflections of spatial coordinates, as well as Galilean "boosts" into uniform rectilinear motion.

In three dimensions, the Galilean transformations depend smoothly on ten real parameters, as follows:

- **Space-time translations,**

$$g_1(\mathbf{r}, t) = (\mathbf{r} + \mathbf{r}_0, t + t_0) .$$

These possess four real parameters: $(\mathbf{r}_0, t_0) \in \mathbb{R}^3 \times \mathbb{R}$, *for the three dimensions of space, plus time.*

- **Spatial rotations and reflections,**

$$g_2(\mathbf{r}, t) = (O\mathbf{r}, t) ,$$

for any linear orthogonal transformation $O : \mathbb{R}^{3N} \mapsto \mathbb{R}^{3N}$ *with* $O^T = O^{-1}$. *These have three real parameters, for the three axes of rotation and reflection.*

- **Galilean boosts** *into uniform rectilinear motion,*

$$g_3(\mathbf{r}, t) = (\mathbf{r} + \mathbf{v}_0 t, t) .$$

These have three real parameters: $\mathbf{v}_0 \in \mathbb{R}^3$, *for the three directions and rates of motion.*

Definition 1.2.2 (Group)
A ***group*** G *is a set of elements that possesses a binary product (multiplication),* $G \times G \to G$, *such that the following properties hold:*

1. *The product* gh *of* g *and* h *is associative, that is,* $(gh)k = g(hk)$.

2. *An identity element exists,* e: $eg = g$ *and* $ge = g$, *for all* $g \in G$.

3. *Inverse operation* $G \to G$, *so that* $gg^{-1} = g^{-1}g = e$.

Definition 1.2.3 (Lie group)
A ***Lie group*** *is a group that depends smoothly on a set of parameters. That is, a Lie group is both a group and a smooth manifold, for which the group operations are smooth functions.*

Proposition 1.2.4 (Lie group property)
Except for reflections, Galilean transformations form a Lie group.

Proof. Any Galilean transformation $g \in G(3) : \mathbb{R}^{3N} \times \mathbb{R} \mapsto \mathbb{R}^{3N} \times \mathbb{R}$ may be expressed uniquely as a composition of the three basic transformations $\{g_1, g_2, g_3\} \in G(3)$. Consequently, the set of elements comprising the transformations $\{g_1, g_2, g_3\} \in G(3)$ closes under the binary operation of composition. The Galilean transformations also possess an identity element $e : eg_i = g_i = g_i e$, $i = 1, 2, 3$, and each element g possesses a unique inverse g^{-1}, so that $gg^{-1} = e = g^{-1}g$. These are the defining relations of a group. The smooth dependence of the group of Galilean transformations on its ten parameters means the ***the Galilean group*** $G(3)$ ***is a Lie group*** (except for the reflections, which are discrete, not smooth). ■

Remark 1.2.5
Compositions of Galilean boosts and translations commute. That is,

$$g_1 g_3 = g_3 g_1 \,.$$

However, the order of composition does matter in the composition of Galilean transformations when rotations and reflections are involved. For example, the action of the Galilean group composition $g_1 g_3 g_2$ *on* $(\mathbf{r}, t)$ *from the left is given by*

$$g(\mathbf{r}, t) \quad = \quad (O\mathbf{r} + t\mathbf{v}_0 + \mathbf{r}_0, t + t_0)\,,$$

for

$$g = g_1(\mathbf{r}_0, t_0) g_3(\mathbf{v}_0) g_2(O) =: g_1 g_3 g_2 \,.$$

> **Exercise.** Write the corresponding transformations for $g_1 g_2 g_3$, $g_2 g_1 g_3$ and $g_3 g_2 g_1$ as well, showing how they depend on the order in which the rotations, boosts and translations are composed. Write the inverse transformation for each of these compositions of left actions.
>
> ★

Answer. For translations, $g_1(\mathbf{r}_0, t_0)$, rotations $g_2(O)$ and boosts