APPLIED AND NUMERICAL HARMONIC ANALYSIS

Sampling, Wavelets, and Tomography

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Birkhäuser

Sampling, Wavelets, and Tomography

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Birkhäuser
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Library of Congress Cataloging-in-Publication Data

Sampling, wavelets, and tomography / John J. Benedetto, Ahmed I. Zayed, editors.
p. cm. – (Applied and numerical harmonic analysis)
Includes bibliographical references and index.
ISBN 0-8176-4304-4 (acid-free paper) – ISBN 3-7643-4304-4 (acid-free paper)
1. Harmonic analysis. 2. Wavelets (Mathematics) 3. Fourier analysis. 4. Sampling
5. Tomography. I. Benedetto, John. II. Zayed, Ahmed I. III. Series.

QA403.S24 2003
515'.2433–dc22 2003061722
CIP

AMS Subject Classifications: 42C40, 44A12, 92C55, 94A20

ISBN 0-8176-4304-4 Printed on acid-free paper.

Birkhäuser

Printed in the United States of America. (SB)

9 8 7 6 5 4 3 2 1 SPIN 10885525

Birkhäuser is part of *Springer Science+Business Media*

www.birkhauser.com

Applied and Numerical Harmonic Analysis

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Series Preface

The *Applied and Numerical Harmonic Analysis (ANHA)* book series aims to provide the engineering, mathematical, and scientific communities with significant developments in harmonic analysis, ranging from abstract harmonic analysis to basic applications. The title of the series reflects the importance of applications and numerical implementation, but richness and relevance of applications and implementation depend fundamentally on the structure and depth of theoretical underpinnings. Thus, from our point of view, the interleaving of theory and applications and their creative symbiotic evolution is axiomatic.

Harmonic analysis is a wellspring of ideas and applicability that has flourished, developed, and deepened over time within many disciplines and by means of creative cross-fertilization with diverse areas. The intricate and fundamental relationship between harmonic analysis and fields such as signal processing, partial differential equations (PDEs), and image processing is reflected in our state of the art *ANHA* series.

Our vision of modern harmonic analysis includes mathematical areas such as wavelet theory, Banach algebras, classical Fourier analysis, time-frequency analysis, and fractal geometry, as well as the diverse topics that impinge on them.

For example, wavelet theory can be considered an appropriate tool to deal with some basic problems in digital signal processing, speech and image processing, geophysics, pattern recognition, biomedical engineering, and turbulence. These areas implement the latest technology from sampling methods on surfaces to fast algorithms and computer vision methods. The underlying mathematics of wavelet theory depends not only on classical Fourier analysis, but also on ideas from abstract harmonic analysis, including von Neumann algebras and the affine group. This leads to a study of the Heisenberg group and its relationship to Gabor systems, and of the metaplectic group for a meaningful interaction of signal decomposition methods. The unifying influence of wavelet theory in the aforementioned topics illustrates the justification for providing a means for centralizing and disseminating information from the broader, but still focused, area of harmonic analysis. This will be a key role of *ANHA*. We intend to publish with the scope and interaction that such a host of issues demands.

Along with our commitment to publish mathematically significant works at the frontiers of harmonic analysis, we have a comparably strong commitment to publish major advances in the following applicable topics in which harmonic analysis plays a substantial role:

Antenna theory	*Prediction theory*
Biomedical signal processing	*Radar applications*
Digital signal processing	*Sampling theory*
Fast algorithms	*Spectral estimation*
Gabor theory and applications	*Speech processing*
Image processing	*Time-frequency and time-scale analysis*
Numerical partial differential equations	*Wavelet theory*

The above point of view for the *ANHA* book series is inspired by the history of Fourier analysis itself, whose tentacles reach into so many fields.

In the last two centuries Fourier analysis has had a major impact on the development of mathematics, on the understanding of many engineering and scientific phenomena, and on the solution of some of the most important problems in mathematics and the sciences. Historically, Fourier series were developed in the analysis of some of the classical PDEs of mathematical physics; these series were used to solve such equations. In order to understand Fourier series and the kinds of solutions they could represent, some of the most basic notions of analysis were defined, e.g., the concept of "function." Since the coefficients of Fourier series are integrals, it is no surprise that Riemann integrals were conceived to deal with uniqueness properties of trigonometric series. Cantor's set theory was also developed because of such uniqueness questions.

A basic problem in Fourier analysis is to show how complicated phenomena, such as sound waves, can be described in terms of elementary harmonics. There are two aspects of this problem: first, to find, or even define properly, the harmonics or spectrum of a given phenomenon, e.g., the spectroscopy problem in optics; second, to determine which phenomena can be constructed from given classes of harmonics, as done, for example, by the mechanical synthesizers in tidal analysis.

Fourier analysis is also the natural setting for many other problems in engineering, mathematics, and the sciences. For example, Wiener's Tauberian theorem in Fourier analysis not only characterizes the behavior of the prime numbers, but also provides the proper notion of spectrum for phenomena such as white light; this latter process leads to the Fourier analysis associated with correlation functions in filtering and prediction problems, and these problems, in turn, deal naturally with Hardy spaces in the theory of complex variables.

Nowadays, some of the theory of PDEs has given way to the study of Fourier integral operators. Problems in antenna theory are studied in terms of unimodular trigonometric polynomials. Applications of Fourier analysis abound in signal processing, whether with the Fast Fourier transform (FFT), or filter design, or the adaptive modeling inherent in time-frequency-scale methods such as wavelet theory. The coherent states of mathematical physics are translated and modulated Fourier transforms, and these are used, in conjunction with the uncertainty principle, for dealing with signal reconstruction in communications theory. We are back to the raison d'être of the ANHA series!

John J. Benedetto
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Prologue

John J. Benedetto*

> Like a feather caught in a vortex Williams ran around the square of bases at the center of our beseeching screaming. . . . Gods do not answer letters.
>
> — John Updike, *The New Yorker*, 1960

Claude Elwood Shannon was born on April 30, 1916 and died on February 22, 2001. It is not hyperbole to say that Shannon was a genius and the creator of modern information theory. Hardy relegated Archimedes, Newton, and Gauss to the "Bradman class", named after the Australian cricket batsman, Sir Donald Bradman (1908–2001). Shannon, whose mathematics graduate studies and professorship (after 1958) at MIT placed him diagonally across the Charles River from Fenway Park, has the ineffable distinction of being in my "Ted Williams class".

We celebrated Shannon's accomplishments in May 2001 at the biennial Sampling Theory and Applications (SampTA01) Conference, held in Orlando, Florida and expertly organized by my co-editor of this volume, Ahmed Zayed. The volume itself is the product of wistful impulsiveness as the conference ended with a perceived insufficient paper trail.

Shannon's name is sometimes associated with the first two words in our title: Shannon sampling and Shannon wavelet. I feel this is unfair to Shannon and misleading to the reader. To explain this sentiment, recall that the *Classical Sampling Theorem* is

$$f(t) = T \sum_{n\in\mathbb{Z}} f(nT)\, s(t - nT). \tag{1}$$

This formula is true, uniformly on $\mathbb{R}$ and in $L^2(\mathbb{R})$-norm, for functions f in the Paley-Wiener space of Ω-bandlimited functions, where $2T\Omega \leq 1$, and where the

* The author gratefully acknowledges support from NSF DMS Grant 0139759 and ONR Grant N000140210398.

sampling function s is a $1/(2T)$-bandlimited function satisfying some natural properties. In the special case that $2T\Omega = 1$ and that the Fourier transform of s equals 1 on $[-\Omega, \Omega]$, equation (1) is sometimes referred to as the *Shannon sampling formula* and it gives rise to the so-called *Shannon wavelet* orthonormal basis (ONB) for $L^2(\mathbb{R})$. In this latter case, the wavelet decomposition of functions $f \in L^2(\mathbb{R})$ that are not Ω-bandlimited provides an interpretation of aliasing error.

Using Lagrange interpolation, Cauchy proved the Classical Sampling Theorem in 1827 and 1841. It was used by Borel, Hadamard, and de la Vallée-Poussin in the late 19th century. Of course, Hadamard and de la Vallée-Poussin proved the prime number theorem (PNT) in 1896; so it should also be mentioned that in 1894 von Koch constructed functions by means of a discrete version of (1) in his attempt to prove the PNT. Later, equation (1) played a prominent role in Steffensen's major work in Acta Mathematica in 1914. So the Classical Sampling Theorem was a venerable tool in interpolation theory and analytic number theory before E. T. Whittaker's (1915), Kotel'nikov's (1933), and Shannon's (1949) versions of (1).

There are several proofs of the Classical Sampling Theorem, and they are all elementary. In 1949, in his paper dealing with sampling, Shannon references the proof in J. M. Whittaker's book (1935). What is certainly true is that Shannon was instrumental in popularizing (1) in the engineering community. Shannon himself wrote in the late 1940s that the sampling theorem "has been given previously in other forms by mathematicians but in spite of its evident importance seems not to have appeared in the literature of communication theory". My only quibble would be that the formula he invokes to quantify his sampling theorem is in fact precisely the "19th century equation (1)" for the special case $2T\Omega = 1$.

When I wrote above that it was unfair to Shannon to refer to (1) as the Shannon sampling formula, I meant, tongue-in-cheek, that someone of Shannon's virtuosity should not be known for a result whose original luster from an earlier mathematical era had long since been assimilated into various mathematical toolboxes. (This is analogous to the Dirac δ-function, which is neither a function nor discovered by the dazzingly deep Dirac.) Similarly, Shannon had nothing to do with wavelets; and the Shannon wavelet ONB is just as easily, and more accurately, called the *sinc* or Littlewood-Paley wavelet ONB.

Notwithstanding my grousing about origins, it is still natural to ask where Shannon used the special case of equation (1) in which $2T\Omega = 1$.

In his monumental, *A Mathematical Theory of Communication* (1948) (*"A"* changed to *"The"* in the 1949 book version), Shannon expressed his indebtedness to Norbert Wiener (1942 classified, 1949) for establishing the "basic philosophy and theory" of communication theory, and for formulating communication theory as a statistical problem. In this latter regard, Shannon was also greatly influenced by R. V. L. Hartley (1928), who formulated communication problems in terms of "rate of communication" and "capacity of a system to transmit information".

As such, Shannon's model of the communication process involved the following well-known concepts: messages as samples of a stochastic process governed by an *a priori* distribution (Wiener); received signals subject to errors governed by a stochastic process – noise (the common man); and the use of a logarithmic measure, e.g., Boltzmann's entropy, to quantify the size of a "universe of messages" (Hartley). On the other hand, the effectiveness of modern information theory is a consequence of Shannon's significant developments on this last point, where he introduced average conditional entropy as a measure of information transfer, and of his concepts of channel and channel capacity.

It may be reassuring to mere mortals that "Shannon's original paper was reviewed, somewhat dismissively, in Mathematical Reviews" (Brockway McMillan at the Wiener Centenary Conference organized by Pesi Masani and his Sancho Panza). In any case, in order to find an expression for channel capacity C in the case of continuous channels, i.e., continuous time ideal strictly bandlimited white Gaussian channels, Shannon required the Classical Sampling Theorem for $2T\Omega = 1$. There are still open and important problems in this area, but Shannon's computation of C began with (1) in order to show the equivalence of such continuous channels with discrete time channels sampled at twice the bandwidth.

Shannon was aware of Gabor's *Theory of communication* (1946), see Shannon's *Communication in the presence of noise* in Proceedings IRE 37 (1949), 10-21 (the first version was submitted in 1940). Gabor's research is a basis of the theory of Gabor frames, which, in turn, plays a significant role in modern sampling theory. Gabor's analysis (1965) of the roles of Shannon and Wiener in communication theory was first published in the Proceedings of the Symposia in Applied Mathematics 52 (1997), which is the proceedings of the aforementioned Wiener Centenary Conference. Gabor's ideas, coupled with the advent of wavelet theory, have added unimaginable scope to classical sampling theory and applications developed through the first three fourths of the 20th century, see the Introduction of *Fundamental Papers in Wavelet Theory* (Princeton University Press, 2004). The SampTA tradition emerged from this setting.

The first SampTA conference did not occur until 1997 with SampTA97 in Jurmala, Latvia; and then there was SampTA99 in Aveiro, Portugal, SampTA01 in Orlando, and SampTA03 in Strobl, Austria. The topics of these conferences, along with many of the transcendent ideas of harmonic analysis, are the subject of Birkhäuser's Applied and Numerical Harmonic Analysis (ANHA) book series, of which this volume is now a member.

In the present era of mathematical engineering we are all beneficiaries of Birkhäuser's enlightened support of harmonic analysis, and, in particular, of its publication of the ANHA series. Ann Kostant of Birkhäuser has the élan and incomparable editorial breadth and depth to keep us sanguine for publishing opportunities at the highest level. For this volume, Tom Grasso of Birkhäuser has suffered our delays, eccentricities, and impositions with patience, and extraordinary skill and professionalism; and we truly appreciate his expert guidance.

The first chapter of this volume was written by my co-editor, Ahmed Zayed. It is a beautiful and broad exposition of the main themes of the book, as well as an integrated introduction to the remaining chapters. It also presents a clear and concise background for appreciating the scientific triptych that follows. Enjoy!

College Park, MD
October, 2003

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