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Topics in Symbolic Dynamics and Applications

Edited by
F. Blanchard, A. Maass & A. Nogueira

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FOREWORD

This volume contains the courses given at the CIMPA–UNESCO Summer School on Symbolic Dynamics and its Applications held at Universidad de la Frontera, Temuco, Chile, from January 6th to 24th, 1997. This School was devised for graduate students and high level undergraduate students interested in dynamical systems emphasizing symbolic dynamics.

The scientific committee was composed of François Blanchard (IML–CNRS–Marseille), Mike Boyle (University of Maryland at College Park), Mike Keane (CWI–Amsterdam), Alejandro Maass (Universidad de Chile), Servet Martínez (Universidad de Chile) and Arnaldo Nogueira (Universidade Federal do Rio de Janeiro).

The book is divided into eight chapters, each one corresponding to a course given at Temuco and devoted to a particular, active area of symbolic dynamics or some application to ergodic theory or number theory. Each author has used his own notation, so each chapter can be considered as an independent one, even though there exist natural relations between them. Each chapter has its own bibliography; all the references in this foreword can be found in at least one of these bibliographies.

By now the reader has understood that our aim was not to make a textbook on symbolic dynamics – such textbooks exist, one is due to D. Lind and B. Marcus (Cambridge U. Press, 1997) and another to B. Kitchens (Springer-Verlag, 1998) – but rather to present the reader with a sampling of what is going on in the field and around it.

About Symbolic Dynamics

The utilization by Hadamard, at the end of the XIXth century, of infinite sequences of symbols for the study of geodesics on certain surfaces marks the beginning of symbolic dynamics. From that time on symbolic sequences have been used repeatedly for coding various smooth transformations, but the interest for symbolic dynamics among mathematicians was further enhanced by the growing importance of symbolic coding in ergodic information theory; by the discovery of simple but important connections with theoretical computer science; and finally through the upsurge of applications in the fields of engineering and molecular biology, among others.

There are two points of view on symbolic dynamics that almost amount to two distinct definitions of the field. Specialists of smooth dynamics view it as a tool: whenever a smooth system has a "nice" symbolic representation, it is easy to compute its entropy, to obtain informations about the set of invariant measures and other relevant features. Here the basic question is: What symbolic representation can we find for a given smooth system and, if there is one, what are its properties? Of course, it would help to know necessary and sufficient conditions for a system to have symbolic representations at all. On the other hand, "pure" symbolic dynamists study the properties of subshifts in general or those of some class of symbolic systems.

Let us describe briefly the main objects in this theory. One starts with a finite set, say A, often called an alphabet. On this alphabet one construct words, that is, finite concatenations of symbols of A, $w = a_0 ... a_n$; and also infinite sequences of symbols of A, $(a_i)_{i \in \mathbb{N}}$ or $(a_i)_{i \in \mathbb{Z}}$. The set of words on the alphabet A is denoted by A^*; the set of infinite sequences of symbols in A is denoted by $A^{\mathbb{N}}$ for one–sided sequences and $A^{\mathbb{Z}}$ for two–sided sequences. The study of infinite symbolic sequences has been undertaken by looking directly to the sequence and producing different combinatorial objects to describe its complexity, and also by considering the "shift dynamical system" or "subshift" associated to it. Put $K = \mathbb{N}$ or $K = \mathbb{Z}$. We endow the set of infinite sequences of symbols A^K with the natural completely discontinuous product topology and consider the action of the shift transformation $\sigma : A^K \to A^K$, such that $\sigma((a_i)_{i \in K}) = (a_{i+1})_{i \in K}$. The pair (A^K, σ) is called the full shift on A. A subshift is any closed (for the product topology), shift–invariant subset $X \subseteq A^K$ ($\sigma(X) = X$).

A subshift can be described in at least two different ways. First, the closure of the orbit of one given sequence under σ is a subshift; it is called the subshift generated by this sequence. This is the case of the so–called substitutive systems. But a subshift can also be defined as the set of all sequences such that the words appearing in such sequences do not belong to a given set of forbidden words. The most popular subshifts constructed in this way are subshifts of finite type and sofic systems. The techniques developed to analyze these systems nowadays run from pure combinatorial arguments through probability theory to abstract algebraic analysis.

Now, what is a symbolic representation? Suppose C is a compact metric space (or a manifold), and $T : C \to C$ a homeomorphism (a diffeomorphism). A subshift (X, σ) of $A^{\mathbb{Z}}$ is said to be a representation of (C, T) if there exists a continuous, surjective map $\pi : C \to X$ such that $\pi \circ T = \sigma \circ \pi$. Of course one wishes π to have some suitable properties; the best situation is when π is one–to–one except on a set of universal measure 0.

Two families of subshifts have been subject to particularly deep investigations. Both families provide symbolic representations for some natural smooth maps; for both families there exist underlying, most of the time algebraic, sometimes arithmetic, structures that provide tools for their study.

The first consists of subshifts of finite type, sofic systems and coded systems. In these subshifts the set of allowed (or, for that matter, forbidden) words is defined with the help of a matrix. A subshift of finite type (SFT) is one that can be described by local constraints, that is, by forbidding a finite set of words; a sofic system is any homomorphic image of a SFT (this definition also provides a description in terms of a finite matrix); and a coded system is defined with the help of a matrix with countable entries. Such subshifts provide natural symbolic representations of smooth transformations of tori or the unit interval, for instance Anosov diffeomorphisms. They display many chaotic properties : a dense set of periodic points, and also, except for a trivial subclass, positive entropy ("high complexity") and a wide variety of shift–invariant measures; SFTs even have equilibrium states for continuous potentials. Owing to their positive entropy and the simple structure of their word combinatorics they have been used for the construction of some error–avoiding codes.

A long–standing conjecture on subshifts of finite type was due to R. Williams. It stated that conjugacy between two subshifts of finite type was equivalent to the shift equivalence of their defining matrices. It was recently proved to be false by Kim and Roush (Annals of Math.).

On the other hand the motivations for the study of systems of "very low complexity" like Sturmian subshifts or substitution dynamical systems arise from their connections with geometry (billiards, interval exchanges, tilings), number theory (irrational rotations of the 1-torus, numeration systems), formal language theory (DOL– and TAG–systems) and C^*-algebras. Subshifts in this area have entropy zero and often rigid features like minimality and/or unique ergodicity. They have given rise to sophisticated numeration systems – representations of real numbers by infinite sequences of symbols – and they provide exciting interpretations of almost periodic tilings and quasi–crystal models.

Between these extreme and rather well–known classes of systems, there can be found an infinite variety of dynamical behaviours; a typical family is that of minimal subshifts, some of which have positive and some of which zero entropy, and which range from very rigid to very chaotic. One typical class is that of Toeplitz systems: they are all minimal and some have positive, some others zero entropy; some are uniquely ergodic,others have uncountably many ergodic measures; they provide an almost inexhaustible source of examples and counter–examples for symbolic dynamics and ergodic theory.

The Contents

The chapters in this volume show different aspects of "high complexity" and "low complexity" symbolic sequences.

In Chapter 1 the author investigates symbolic sequences of "low complexity", in particular Sturmian sequences introduced by Hedlund and Morse in the thirties, and automatic sequences. Both classes of sequences have been used to model systems in crystallography and biology. The chapter begins with the definition of the so-called complexity function of a sequence, $p(n)$: for each positive integer n this function counts the number of words of length n appearing in the sequence. In particular, Sturmian sequences are exactly those for which $p(n) = n+1$. Then the author describes the main techniques and results that serve to study these systems: the graph of words, the notion of special factors, and the formal power series approach.

Chapter 2 continues the study of "low complexity" symbolic systems. Here substitution dynamical systems are considered. A substitution is any map $\tau : A \to A^*$, extended to sequences of $A^{\mathbb{N}}$ in a natural way. A substitution dynamical system is any subshift generated by a fixed point of a substitution. The existence of such a fixed point is ensured by a standard condition called primitivity. In the literature, the main reference in this area is the book of Queffélec (LNM-1294-Springer-Verlag) about the spectral properties of these systems, where some general introduction is also provided. The main discussion in this chapter is focussed on the topological dynamical properties of these systems and their representation by means of Bratteli–Vershik diagrams. This recent approach derives from the remarkable work of Herman, Putnam and Skau (Int. J. of Math., 1992) about the representation of minimal Cantor systems. After reviewing the main dynamical properties of primitive substitution systems, the author describes all the steps that are necessary to construct this representation.

The two following chapters are devoted to "high complexity" systems. In particular Chapter 3, after presenting a more detailed view of these systems than the one given above, describes the main algebraic tools that have been developed in order to investigate the problem of shift equivalence between subshifts of finite type, in other words, the Williams conjecture and related questions. The main algebraic invariants that are considered are matrix invariants and dimension group type invariants. The chapter ends with a description of the most recent results about the Williams conjecture.

There is a deep connection between Chapters 2 and 3. Some Bratteli diagrams can be endowed with two dynamics, one of them describing a subshift of finite type, the other a substitution dynamical system or an odometer; this establishes a kind of duality between these two classes. It is not surprising that dimension groups play an essential rôle in the two settings, with a different topological interpretation in each (see Durand-Host-Skau, Erg. Th. Dyn. Syst. 1999).

Chapter 4 is an introduction to the study of dynamical properties of $\mathbb{Z}^d$–actions of automorphisms over a compact topological group. The topic of $\mathbb{Z}^d$–actions is a difficult one, on which fewer results are known than for $\mathbb{Z}$–symbolic actions. This corresponds to the study of multidimensional Markov shifts (or subshifts of finite type) which have a group structure, also called Markov subgroups. The chapter concentrates on the "zero–dimensional" case. In particular, a complete characterization of expansive automorphisms of zero–dimensional topological groups is provided. In the transitive and positive entropy case, they correspond to full shifts. After that, the author presents one of the main techniques used in the study of the multidimensional case: the so–called descending chain condition. The symbolic dynamics of

$(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$ provides the main example.

In Chapter 5 the author considers symbolic systems from a probabilistic point of view. Using the concept of an asymptotic rare event to mean a sequence of measurable sets of asymptotically zero probability, the author presents some results concerning asymptotic laws for the random times of occurrence of various asymptotically rare events in the case of shifts of finite type endowed with a Hölder equilibrium state. There is also a brief review of known results in this direction for more general ergodic dynamical systems.

The last three chapters of the book deal with applications of symbolic dynamics to ergodic theory, number theory and one-dimensional dynamics.

Chapter 6 deals with the connection between dynamics, the area of number theory called diophantine problems and combinatorial Ramsey theory. Van der Waerden's theorem (for any finite partitioning of $\mathbb{N}$, one of the elements contains arbitrarily long arithmetic progression) can actually be interpreted as a statement in topological dynamics and proved in this setting; likewise Szemerédi's theorem (a subset of $\mathbb{N}$ of positive density contains arbitrarily long arithmetic progressions), first proved in a purely combinatorial way, is also an ergodic statement, with an ergodic proof due to Furstenberg. The aim of this section is to present the fundamental ideas that led to these proofs and to many further developments; one of the most important is considering symbolic sequences.

Chapter 7 is devoted to the dynamics of symbolic systems that arise when one develops numbers in some particular numeration systems. First the author studies β–expansions of numbers and the relations between the defining real number $\beta > 1$ and the associated symbolic system: the set of lexicographically maximal β–expansions happens to possess interesting symbolic properties. Here a symbolic representation of an interval transformation plays an important part. Afterwards the author considers a closely related type of integer expansions: expansions with respect to an increasing sequence of integers $\{U_0, U_1, U_2, ...\}$. The basic example is the Fibonacci sequence.

Finally, Chapter 8 gives a complete description of the symbolic dynamics of Lorenz maps. This is another instance of a symbolic representation. Lorenz maps are piecewise increasing maps of the interval. A symbolic representation helps solve the problem of their topological conjugacy: in particular, for expanding Lorenz maps the conditions of Hubbard and Sparrow solve the topological classification problem.

Acknowledgments

We would like to express our gratitude to all the participants of the school and to the people who contributed to its organization. In particular, to Servet Martínez who promote the idea of this school and contributed to its achievement in a significant way; and to Cesar Burgueño whose help at Temuco was priceless.

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The Editors

Contents

Chapter 1

SEQUENCES OF LOW COMPLEXITY: AUTOMATIC AND STURMIAN SEQUENCES

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The complexity function is a classical measure of disorder for sequences with values in a finite alphabet: this function counts the number of factors of given length. We introduce here two characteristic families of sequences of low complexity function: automatic sequences and Sturmian sequences. We discuss their topological and measure-theoretic properties, by introducing some classical tools in combinatorics on words and in the study of symbolic dynamical systems.

1.1 Introduction

The aim of this course is to introduce two characteristic families of sequences of low "complexity": automatic sequences and Sturmian sequences (complexity is defined here as the combinatorial function which counts the number of factors of given length of a sequence over a finite alphabet). These sequences