CONTEMPORARY MATHEMATICS

584

Analysis, Geometry and Quantum Field Theory

International Conference
in Honor of Steve Rosenberg's 60th Birthday
September 26–30, 2011
Potsdam University, Potsdam, Germany

Clara L. Aldana
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Analysis, Geometry
and Quantum Field Theory

Preface

Steve Rosenberg has made many important contributions to Differential Geometry, Global Analysis and Mathematical Physics, and found numerous applications of Spectral Theory to these fields. His book *The Laplacian on a Riemannian Manifold* has helped many graduate students to enter the world of Global Analysis.

Steve Rosenberg's 60th birthday was celebrated at the conference "Analysis, Geometry and Quantum Field Theory" organized by Jouko Mickelsson and Sylvie Paycha at Potsdam University in September 2011. The speakers of the conference were internationally renowned experts in Geometry and Analysis; many of them were Steve's collaborators or former students.

The wide range of topics represented in this volume, from Stochastic Analysis to Differential K-theory and from Quantum Field Theory to Mathematical Biology, speaks to the broadness of Steve Rosenberg's mathematical interests.

We would like to thank the authors who contributed to this volume as well as those who served as referees. We are also very grateful to Arthur L. Greenspoon for the very careful editing of most of the papers appearing in this volume.

Contents

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A duality for the double fibration transform

Michael G. Eastwood and Joseph A. Wolf

Abstract. We establish a duality within the spectral sequence that governs the holomorphic double fibration transform. It has immediate application to the questions of injectivity and range characterization for this transform. We discuss some key examples and an improved duality that holds in the Hermitian holomorphic case.

1. Double fibrations

In this article we shall always work in the holomorphic category. By a *double fibration* we shall mean a diagram of the form

(1.1)
$$\begin{array}{ccc} & \mathfrak{X}_D & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ D & & \mathcal{M}_D \end{array}$$

where

(1.2)
- D, $\mathfrak{X}_D$, and $\mathcal{M}_D$ are complex manifolds;
- μ is a holomorphic submersion with contractible fibers;
- ν is a holomorphic submersion with compact fibers;
- $\mathfrak{X}_D \xrightarrow{(\mu,\nu)} D \times \mathcal{M}_D$ is a holomorphic embedding;
- $\mathcal{M}$ is a contractible Stein manifold.

Examples of double fibrations arise naturally as follows. Let G be a complex semisimple (or even reductive) Lie group. There is a beautiful class of complex homogeneous spaces $Z = G/Q$ that can be characterized by any of the following equivalent conditions (see e.g. [**6**] for details).

- Z is a compact complex manifold;
- Z is a compact Kähler manifold;
- Z is a complex projective variety;
- Q is a parabolic subgroup of G.

We shall refer to such compact complex homogeneous spaces Z as *complex flag manifolds.* Now fix a complex flag manifold $Z = G/Q$ and consider a real form G_0 of G. Then it is known [**9**] that the natural action of G_0 on Z has only finitely many orbits and so there is at least one open orbit. If G_0 is compact, then it acts transitively on Z and there are few other exceptional cases when this happens.

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Otherwise, an open G_0-orbit $D \subsetneq Z$ is known as a *flag domain*. As a simple example, let us take $G = \mathrm{SL}(4,\mathbb{C})$ acting on $Z = \mathbb{CP}_3$ in the usual fashion, namely

$$\mathrm{SL}(4,\mathbb{C}) \times \mathbb{CP}_3 \ni (A,[z]) \mapsto [Az] \in \mathbb{CP}_3,$$

where $z \in \mathbb{C}^4$ is regarded as a column vector. If we take

$$\begin{bmatrix} * \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{CP}_3 \text{ as basepoint, then } Q = \left\{ \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix} \in \mathrm{SL}(4,\mathbb{C}) \right\}.$$

If we take $G_0 = \mathrm{SU}(2,2)$, defined as preserving the Hermitian form

$$\langle w, z \rangle \equiv w_1\overline{z_1} + w_2\overline{z_2} - w_3\overline{z_3} - w_4\overline{z_4} \tag{1.3}$$

on $\mathbb{C}^4$, then

$$D = \mathbb{CP}_3^+ \equiv \{[z] \in \mathbb{CP}_3 \mid |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 > 0\}$$

is a flag domain for the action of G_0 on Z.

In general, fixing $K_0 \subset G_0$ a maximal compact subgroup, it is known [**9**] that there is just one K_0-orbit C_0 in D that is a complex submanifold of Z. We regard C_0 as the basepoint of the *cycle space*

$$\mathcal{M}_D \equiv \text{connected component of } C_0 \text{ in } \{gC_0 \mid g \in G \text{ and } gC_0 \subset D\}$$

of D. Evidently, $\mathcal{M}_D$ is an open subset of

$$\mathcal{M}_Z \equiv \{gC_0 \mid g \in G\} = G/J, \quad \text{where } J \equiv \{g \in G \mid gC_0 = C_0\}$$

and hence is a complex manifold. Let us set

$$\mathfrak{X}_Z \equiv G/(Q \cap J) \quad \text{and} \quad \mathfrak{X}_D \equiv \{(z,C) \in D \times \mathcal{M}_D \mid z \in C\}.$$

Then

$$\begin{array}{ccccccc} & \mathfrak{X}_D & & & & \mathfrak{X}_Z & \\ {}^{\mu}\swarrow & & \searrow^{\nu} & \overset{\text{open}}{\subset} & \swarrow & & \searrow \\ D & & \mathcal{M}_D & & Z & & \mathcal{M}_Z \end{array} \tag{1.4}$$

and it is known for any flag domain (see e.g. [**6**] for details) that all the conditions (1.2) of a double fibration are satisfied.

In our example, we may take

$$K_0 = \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2)) = \left\{ \begin{bmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \in \mathrm{SU}(2,2) \right\} \tag{1.5}$$

whence

$$C_0 = \left\{ \begin{bmatrix} * \\ * \\ 0 \\ 0 \end{bmatrix} \in \mathbb{CP}_3 \right\}, \quad J = \left\{ \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \in \mathrm{SL}(4,\mathbb{C}) \right\},$$

and $\mathcal{M}_Z = \mathrm{Gr}_2(\mathbb{C}^4)$, the Grassmannian of 2-planes in $\mathbb{C}^4$. The base cycle C_0 and, therefore, every other cycle is intrinsically a Riemann sphere $\mathbb{CP}_1$. Geometrically,

$$\mathcal{M}_D = \{\Pi \in \mathrm{Gr}_2(\mathbb{C}^4) \mid \langle\cdot\,,\cdot\rangle|_\Pi \text{ is positive definite}\} \equiv \mathbb{M}^{++}$$

and analytically we may realize $\mathcal{M}_D$ as a convex tube domain in $\mathbb{C}^4$

$$\mathcal{M}_D \cong \{\zeta = x + iy \in \mathbb{C}^4 \mid {x_1}^2 > {x_2}^2 + {x_3}^2 + {x_4}^2 \text{ and } x_1 > 0\}$$

by means of

$$\mathbb{C}^4 \ni (\zeta_1, \zeta_2, \zeta_3, \zeta_4) \longmapsto \Pi \equiv \operatorname{span}\left\{ \begin{bmatrix} 1+\zeta_1+\zeta_2 \\ \zeta_3 + i\zeta_4 \\ 1-\zeta_1-\zeta_2 \\ -\zeta_3 - i\zeta_4 \end{bmatrix}, \begin{bmatrix} \zeta_3 - i\zeta_4 \\ 1+\zeta_1-\zeta_2 \\ -\zeta_3 + i\zeta_4 \\ 1-\zeta_1+\zeta_2 \end{bmatrix} \right\}.$$

Notice that, in this particular case, the cycle space $\mathcal{M}_D$ is itself a flag domain (for the action of $\mathrm{SU}(2,2)$ on $\mathrm{Gr}_2(\mathbb{C}^4)$). This is unusual.

For our second example, let us start with another of the open orbits of $\mathrm{SU}(2,2)$ on $\mathrm{Gr}_2(\mathbb{C}^4)$, namely

$$D = \{\Pi \in \mathrm{Gr}_2(\mathbb{C}^4) \mid \langle\cdot\,,\,\cdot\rangle|_\Pi \text{ is strictly indefinite}\} \equiv \mathbb{M}^{+-}.$$

With the same choice (1.5) of maximal compact subgroup K_0, the base cycle C_0 is

$$\left\{\Pi \in \mathrm{Gr}_2(\mathbb{C}^4) \mid \Pi = \operatorname{span}\{\alpha, \beta\} \text{ for some } \left\{ \begin{array}{l} \alpha \text{ of the form } [*,*,0,0]^t \\ \beta \text{ of the form } [0,0,*,*]^t \end{array} \right\}\right\}.$$

Hence the base cycle and, therefore, every other cycle is intrinsically $\mathbb{CP}_1 \times \mathbb{CP}_1$. By definition, we always have $J \supseteq K$, the complexification of K_0, but often they are equal, and this is the case here. The cycle space $\mathcal{M}_D$ is $\mathbb{M}^{++} \times \mathbb{M}^{--}$, where $\mathbb{M}^{--}$ denotes the set of planes in $\mathbb{C}^4$ on which $\langle\cdot\,,\,\cdot\rangle$ is negative definite. As a product of two Stein manifolds it is Stein. For $(\Pi_1, \Pi_2) \in \mathcal{M}_D$, the corresponding cycle is

$$\{\Pi \in \mathrm{Gr}_2(\mathbb{C}^4) \mid \Pi = \operatorname{span}\{\alpha, \beta\} \text{ for some } \alpha \in \Pi_1 \text{ and } \beta \in \Pi_2\}.$$

2. The transform

Consider a general double fibration (1.1), satisfying the conditions (1.2), and suppose we are given a holomorphic vector bundle E on D and a cohomology class $\omega \in H^r(D; \mathcal{O}(E))$. We shall continue to refer to the compact complex submanifolds $\mu(\nu^{-1}(x))$ for $x \in \mathcal{M}_D$ as *cycles* in D and now consider the restriction of ω to these cycles:

$$\omega|_{\mu(\nu^{-1}(x))} \in H^r\big(\mu(\nu^{-1}(x)); \mathcal{O}(E|_{\mu(\nu^{-1}(x))})\big), \text{ as } x \in \mathcal{M}_D \text{ varies.}$$

As ν has compact fibers, these cohomology spaces are finite-dimensional and we shall suppose that their dimension is constant as $x \in \mathcal{M}_D$ varies (generically this is the case and in the homogeneous setting, as discussed above, this is manifest if one starts with E a G-homogeneous vector bundle). Then, as $x \in \mathcal{M}_D$ varies we obtain a vector bundle E' on $\mathcal{M}_D$ and a holomorphic section $\mathcal{P}\omega \in \Gamma(\mathcal{M}_D, \mathcal{O}(E'))$ thereof. This is the *double fibration transform* of ω. It is often most interesting starting with cohomology in the same degree as the dimension of the fibers of ν. Two natural questions associated with this transform are

- is it injective?
- what is its range?

There are clear parallels with the Radon transform and other transforms from real integral geometry, especially when integrating over compact cycles.

The complex version, however, benefits from the following general result.

THEOREM 2.1. *For any double fibration* (1.1), *and holomorphic vector bundle* E *on* D, *there is a spectral sequence*

$$E_1^{p,q} = \Gamma(\mathcal{M}_D; \nu_*^q \Omega_\mu^p(E)) \Longrightarrow H^{p+q}(D; \mathcal{O}(E)), \tag{2.1}$$

where

- $\Omega_\mu^1 \equiv \Omega_{\mathfrak{X}_D}^1/\mu^*\Omega_D^1$, *the holomorphic* 1*-forms along the fibers of* μ;
- $\Omega_\mu^p \equiv \Lambda^p\Omega_\mu^1$, *the holomorphic* p*-forms along the fibers of* μ;
- $\Omega_\mu^p(E) \equiv \Omega_\mu^p \otimes \mu^*E$.

PROOF. There are two stages to the proof, the details of which may be found in [**1**]. The first uses that the fibers of μ are contractible to conclude that

$$H^r(D; \mathcal{O}(E)) \cong H^r(\mathfrak{X}_D; \mu^{-1}\mathcal{O}(E))$$

where $\mu^{-1}\mathcal{O}(E)$ denotes the sheaf of germs of holomorphic sections of μ^*E on $\mathfrak{X}_D$ that are locally constant along the fibers of μ. The second stage uses the resolution $0 \to \mu^{-1}\mathcal{O}(E) \to \Omega_\mu^\bullet(E)$ to construct a spectral sequence

$$E_1^{p,q} = H^q(\mathfrak{X}_D; \Omega_\mu^p(E)) \Longrightarrow H^{p+q}(\mathfrak{X}_D; \mu^{-1}\mathcal{O}(E)),$$

which combines with the natural isomorphisms $H^q(\mathfrak{X}_D, \mathcal{O}(F)) \cong \Gamma(\mathcal{M}_D, \nu_*^q\mathcal{O}(F))$, valid for any holomorphic vector bundle F on $\mathfrak{X}_D$ because $\mathcal{M}_D$ is Stein. □

For the rest of this article we shall suppose that the direct images $\nu_*^q\Omega_\mu^p(E)$ are locally free and therefore may be regarded as holomorphic vector bundles on $\mathcal{M}_D$. From this viewpoint, the E_1-differentials become first order differential operators on $\mathcal{M}_D$ and, more generally, the spectral sequence ideally interprets the cohomology $H^r(D; \mathcal{O}(E))$ in terms of systems of holomorphic differential equations on $\mathcal{M}_D$. This is especially interesting when D is a flag domain, $\mathcal{M}_D$ is its cycle space, and E is G-homogeneous because then this double fibration transform can provide useful alternative realizations of the G_0-representations afforded by $H^r(D; \mathcal{O}(E))$.

3. Examples

Let us now return to the flag domains introduced in §1 and see how the spectral sequence (2.1) works out for the first of these domains, namely $D = \mathbb{CP}_3^+$. The main issue in executing (2.1) is in computing the direct images $\nu_*^q\Omega_\mu^p(E)$. We need a notation for the irreducible homogeneous vector bundles on the flag manifold Z. For this we shall follow [**1**], recording both the parabolic subgroup Q and the representation of Q by annotating the appropriate Dynkin diagram (it turns out to be most convenient to record the lowest weight of the representation). For our first domain, in which $Z = \mathbb{CP}_3$, the irreducible homogeneous vector bundles are

$$\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet} \quad \text{for integers } a, b, c \text{ with } b, c \geq 0.$$

The details are in [**1**] but some particular cases are

$$\overset{0}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{0}{\bullet} = \text{the trivial bundle} \equiv \mathcal{O}$$
$$\overset{1}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet} = \text{the holomorphic tangent bundle} \equiv \Theta$$
$$\overset{-2}{\times}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\!-\!\!\overset{0}{\bullet} = \text{the holomorphic cotangent bundle} \equiv \Omega^1$$
$$\overset{-3}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet} = \text{the bundle of holomorphic 2-forms} \equiv \Omega^2$$
$$\overset{1}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{0}{\bullet} = \text{the tautological bundle} \equiv \mathcal{O}(1)$$
$$\overset{k}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{0}{\bullet} = \text{the } k^{\text{th}} \text{ power of the tautological bundle} \equiv \mathcal{O}(k).$$

Similarly, the irreducible homogeneous vector bundles on $\mathfrak{X}_Z$, the flag manifold

$$\mathrm{F}_{1,2}(\mathbb{C}^4) = \{(L, \Pi) \in \mathbb{CP}_3 \times \mathrm{Gr}_2(\mathbb{C}^4) \mid L \subset \Pi\}$$

are given by

$$\overset{a}{\times}\!-\!\overset{b}{\times}\!-\!\overset{c}{\bullet} \quad \text{for integers } a, b, c \text{ with } c \geq 0.$$

For computational purposes, it is always better to consider the diagram

$$\begin{array}{ccc} & \mathfrak{X}_Z = G/(Q \cap J) & \\ & {}^{\mu}\swarrow \quad \searrow^{\nu} & \\ G/Q = Z & & \mathcal{M}_Z = G/J \end{array} \tag{3.1}$$

from (1.4), where we have extended the definition of μ and ν to $\mathfrak{X}_Z$ as shown. The point is that, with this enhanced definition of ν, we have $\mathfrak{X}_D = \nu^{-1}(\mathcal{M}_D)$ and so the fibers over $\mathcal{M}_D$ are unchanged. In particular, the direct images $\nu_*^q \Omega_\mu^p(E)$, as required in the spectral sequence (2.1), can be computed from (3.1) and then simply restricted to the open Stein subset $\mathcal{M}_D \subset \mathcal{M}_Z$. The advantage of (3.1) is that all three spaces are G-homogeneous and the two mappings are G-equivariant. Hence, we may use representation theory to compute $\nu_*^q \Omega_\mu^p(E)$ etc.

With this enhanced viewpoint in place, the bundles of holomorphic forms along the fibers of μ are

$$\Omega_\mu^0 = \overset{0}{\times}\!-\!\overset{0}{\times}\!-\!\overset{0}{\bullet}, \qquad \Omega_\mu^1 = \overset{1}{\times}\!-\!\overset{-2}{\times}\!-\!\overset{1}{\bullet}, \qquad \Omega_\mu^2 = \overset{2}{\times}\!-\!\overset{-3}{\times}\!-\!\overset{0}{\bullet}. \tag{3.2}$$

Now let us consider the double fibration transform for $H^r(D; \mathcal{O}(k))$. Line bundles are straightforward because

$$\mu^*\mathcal{O}(k) = \mu^* \overset{k}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet} = \overset{k}{\times}\!-\!\overset{0}{\times}\!-\!\overset{0}{\bullet},$$

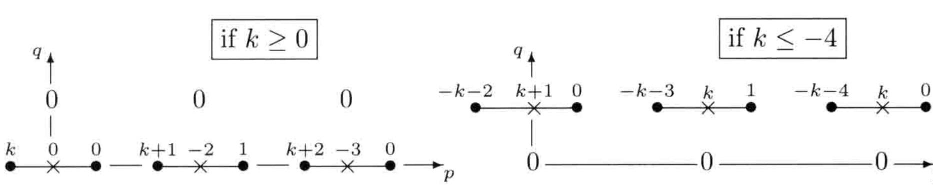
which is irreducible. Writing $\Omega_\mu^p(k)$ for $\Omega_\mu^p \otimes \mu^*\mathcal{O}(k)$, we have

$$\Omega_\mu^0(k) = \overset{k}{\times}\!-\!\overset{0}{\times}\!-\!\overset{0}{\bullet} \qquad \Omega_\mu^1(k) = \overset{k+1}{\times}\!-\!\overset{-2}{\times}\!-\!\overset{1}{\bullet} \qquad \Omega_\mu^2(k) = \overset{k+2}{\times}\!-\!\overset{-3}{\times}\!-\!\overset{0}{\bullet}. \tag{3.3}$$

The direct images are computed in accordance with the Bott-Borel-Weil Theorem along the fibers of ν, which reads

$$\begin{array}{lll} \nu_* \overset{a}{\times}\!-\!\overset{b}{\times}\!-\!\overset{c}{\bullet} & = \overset{a}{\bullet}\!-\!\overset{b}{\times}\!-\!\overset{c}{\bullet} & \text{for } a \geq 0 \\ \nu_*^1 \overset{a}{\times}\!-\!\overset{b}{\times}\!-\!\overset{c}{\bullet} & = \overset{-a-2}{\bullet}\!-\!\overset{a+b+1}{\times}\!-\!\overset{c}{\bullet} & \text{for } a \leq -2, \end{array} \tag{3.4}$$

with all other direct images vanishing (see e.g. [**1**] for details). In particular, there are the spectral sequences (2.1) of the following form.



Let us say that the spectral sequence $E_1^{p,q}$ is *concentrated in degree zero* if and only if $E_1^{p,q} = 0$ for $q > 0$ and *strictly concentrated in degree zero* if and only if, in addition, $E_1^{p,0} \neq 0$, $\forall p$. Similarly, let us say we have *concentration in top degree* if and only if $E_1^{p,q} = 0$ for $q < s$, where $s = \dim_{\mathbb{C}}(\text{fibers of } \nu)$ and *strictly* so if and only if $E_1^{p,s} \neq 0$, $\forall p$. Thus, strict concentration occurs in this example for $k \geq 0$ or $k \leq -4$. In fact, it is easily verified that

$$
\begin{array}{rcl}
k \geq 0 & \Longrightarrow & \text{strict concentration in degree zero,} \\
k = -1 & \Longrightarrow & \text{concentration in degree zero,} \\
k = -2 & \Longrightarrow & \text{no concentration,} \\
k = -3 & \Longrightarrow & \text{concentration in top degree } (s = 1), \\
k \leq -4 & \Longrightarrow & \text{strict concentration in top degree.}
\end{array} \tag{3.5}
$$

The double fibration transform in this case is known as the *Penrose transform* [**4**]. Always, the spectral sequence is most easily interpreted when it concentrates in top degree for then it collapses to yield, in particular, an isomorphism

$$\mathcal{P} : H^s(D; \mathcal{O}(E)) \xrightarrow{\cong} \ker : \Gamma(\mathcal{M}_D; \nu_*^s \Omega_\mu^0(E)) \to \Gamma(\mathcal{M}_D; \nu_*^s \Omega_\mu^1(E)).$$

In our example

$$\mathcal{P} : H^s(D; \mathcal{O}(k)) \xrightarrow{\cong} \ker : \Gamma(\mathcal{M}_D; \overset{-k-2}{\bullet}\!\!-\!\!\overset{k+1}{\times}\!\!-\!\!\overset{0}{\bullet}) \to \Gamma(\mathcal{M}_D; \overset{-k-3}{\bullet}\!\!-\!\!\overset{k}{\times}\!\!-\!\!\overset{1}{\bullet}),$$

for $k \leq -3$ and the right hand side has an interpretation in physics as so-called *massless fields of helicity* $-1 - k/2$ (see e.g. [**4**] for details).

The main aim of this article is to show that concentration in zero versus top degree are related by a duality. This will turn out to be useful because the spectral sequence has simple consequences when concentrated in top degree whereas criteria for concentration in degree zero are more easily determined.

4. The duality

THEOREM 4.1. *Let κ_D and $\kappa_{\mathcal{M}_D}$ denote the canonical bundles on D and $\mathcal{M}_D$, respectively. Let $d = \dim_{\mathbb{C}}$(fibers of μ) and recall that $s = \dim_{\mathbb{C}}$(fibers of ν). Then there are canonical isomorphisms*

$$\nu_*^q \Omega_\mu^p(\kappa_D \otimes E^*) = \kappa_{\mathcal{M}_D} \otimes (\nu_*^{s-q} \Omega_\mu^{d-p}(E))^*, \ \ \forall\, 0 \leq p \leq d,\ 0 \leq q \leq s. \tag{4.1}$$

The spectral sequence (2.1) *for the vector bundle E is (strictly) concentrated in top degree if and only if the corresponding spectral sequence for $\kappa_D \otimes E^*$ is (strictly) concentrated in degree zero.*

PROOF. Certainly, the last statement follows immediately from (4.1): as $\mathcal{M}_D$ is contractible and Stein, if $\nu_*^q \Omega_\mu^p(E)$ is non-zero then neither is $\Gamma(\mathcal{M}_D; \nu_*^q \Omega_\mu^p(E))$.

Notice that (4.1) generalizes Serre duality [**8**]. Specifically, if D is an arbitrary compact manifold, then we may take $\mathfrak{X}_D = D$ and $\mathcal{M}_D$ to be a point. Then $d = 0$, direct images revert to cohomology, and (4.1) becomes

$$H^q(D; \mathcal{O}(\kappa_D \otimes E^*)) = H^{s-q}(D; \mathcal{O}(E))^*.$$

Conversely, Serre duality along the fibers of ν is the essential ingredient in proving (4.1) as follows. Let $\kappa_{\mathfrak{X}_D}$ denote the canonical bundle on $\mathfrak{X}_D$. Since μ and ν are submersions, we can write $\kappa_{\mathfrak{X}_D}$ in two different ways:

$$\kappa_{\mathfrak{X}_D} = \mu^*(\kappa_D) \otimes \kappa_\mu \quad \text{and} \quad \kappa_{\mathfrak{X}_D} = \nu^*(\kappa_{\mathcal{M}_D}) \otimes \kappa_\nu, \tag{4.2}$$

where κ_μ and κ_ν are the canonical bundles along the fibers of μ and ν, respectively. Thus, bearing in mind the Hodge isomorphism $\Omega_\mu^p = \kappa_\mu \otimes (\Omega_\mu^{d-p})^*$ along the fibers of μ, we find that

$$\begin{aligned}
\nu_*^q\Omega_\mu^p(\kappa_D\otimes E^*) &= \nu_*^q\big(\mu^*(\kappa_D)\otimes\Omega_\mu^p\otimes\mu^*(E^*)\big)\\
&= \nu_*^q\big(\kappa_{\mathfrak{X}_D}\otimes\kappa_\mu^*\otimes\Omega_\mu^p\otimes\mu^*(E^*)\big)\\
&= \nu_*^q\big(\nu^*(\kappa_{\mathcal{M}_D})\otimes\kappa_\nu\otimes(\kappa_\mu^*\otimes\Omega_\mu^p)\otimes\mu^*(E^*)\big)\\
&= \kappa_{\mathcal{M}_D}\otimes\nu_*^q\big(\kappa_\nu\otimes(\Omega_\mu^{d-p})^*\otimes\mu^*(E^*)\big)\\
&= \kappa_{\mathcal{M}_D}\otimes\nu_*^q\big(\kappa_\nu\otimes(\Omega_\mu^{d-p}\otimes\mu^*(E))^*\big),
\end{aligned}$$

which may be identified by Serre duality along the fibers of ν to give

$$\nu_*^q\Omega_\mu^p(\kappa_D\otimes E^*) = \kappa_{\mathcal{M}_D}\otimes\big(\nu_*^{s-q}(\Omega_\mu^{d-p}\otimes\mu^*(E))\big)^* = \kappa_{\mathcal{M}_D}\otimes(\nu_*^{s-q}\Omega_\mu^{d-p}(E))^*,$$

as required. □

5. Applications

Let us firstly show how Theorem 4.1 yields (3.5) with minimal calculation. It is clear from (3.3) that strict concentration in degree zero occurs if $k\geq 0$. Indeed, since $\overset{-1}{\times}\!-\!\overset{0}{\times}\!-\!\overset{0}{\bullet}$ is singular along the fibers of ν it is also clear that concentration in degree zero also occurs when $k=-1$. But now

$$\kappa_D\otimes\big(\overset{k}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\big)^* = \overset{-4}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\otimes\overset{-k}{\times}\!-\!\overset{0}{\times}\!-\!\overset{0}{\bullet} = \overset{-k-4}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}$$

and Theorem 4.1 tells us that we have strict concentration in top degree if and only if $-k-4\geq 0$, which gives $k\leq -4$ as expected. Similarly, $-k-4=-1$ if and only if $k=-3$.

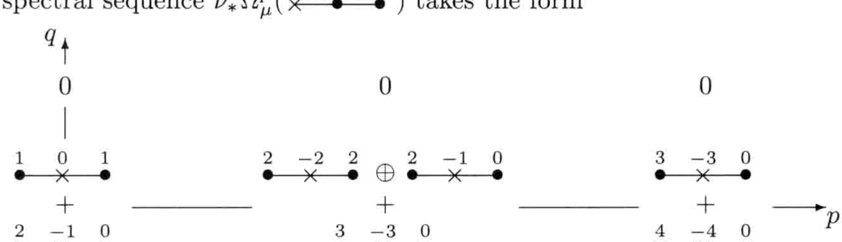
To extend this analysis to vector bundles there are two issues to be overcome. The first is that the pullback $\mu^*(\overset{a}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet})$ is reducible in general. Specifically,

(5.1)
$$\mu^*(\overset{a}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet}) = \overset{a}{\times}\!-\!\overset{b}{\times}\!-\!\overset{c}{\bullet} + \begin{matrix}\overset{a+1}{\times}\!-\!\overset{b-2}{\times}\!-\!\overset{c+1}{\bullet}\\ \oplus\\ \overset{a+1}{\times}\!-\!\overset{b-1}{\times}\!-\!\overset{c-1}{\bullet}\end{matrix} + \begin{matrix}\overset{a+2}{\times}\!-\!\overset{b-4}{\times}\!-\!\overset{c+2}{\bullet}\\ \oplus\\ \overset{a+2}{\times}\!-\!\overset{b-3}{\times}\!-\!\overset{c}{\bullet}\\ \oplus\\ \overset{a+2}{\times}\!-\!\overset{b-2}{\times}\!-\!\overset{c-2}{\bullet}\end{matrix} + \cdots + \overset{a+b+c}{\times}\!-\!\overset{-b-c}{\times}\!-\!\overset{b}{\bullet}.$$

The second is that, even for an irreducible bundle $V=\overset{a}{\times}\!-\!\overset{b}{\times}\!-\!\overset{c}{\bullet}$ on $\mathbb{F}_{1,2}(\mathbb{C}^4)$, the bundle $\Omega_\mu^1\otimes V$ may be reducible. For example, these two issues in combination imply that

$$\Omega_\mu^1(\overset{1}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}) = \overset{1}{\times}\!-\!\overset{-2}{\times}\!-\!\overset{1}{\bullet}\otimes\big(\overset{1}{\times}\!-\!\overset{0}{\times}\!-\!\overset{1}{\bullet}+\overset{2}{\times}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\bullet}\big) = \begin{matrix}\overset{2}{\times}\!-\!\overset{-2}{\times}\!-\!\overset{2}{\bullet}\\ \oplus\\ \overset{2}{\times}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\bullet}\end{matrix} + \overset{3}{\times}\!-\!\overset{-3}{\times}\!-\!\overset{1}{\bullet}$$

and the spectral sequence $\nu_*^q\Omega_\mu^p(\overset{1}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet})$ takes the form

(5.2)


In particular, it is concentrated in degree zero. This is a general feature as follows.

THEOREM 5.1. *The spectral sequence for* $H^r(D;\mathcal{O}(\overset{a}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet}))$ *associated to the double fibration*

$$
\begin{array}{ccccccc}
 & \mathfrak{X}_D & & & & \mathfrak{X}_Z = \mathrm{F}_{1,2}(\mathbb{C}^4) & \\
 & {}^{\mu}\swarrow \quad \searrow^{\nu} & & \overset{\text{open}}{\subset} & & {}^{\mu}\swarrow \quad \searrow^{\nu} & \\
\mathbb{CP}_3^+ = D & & \mathcal{M}_D = \mathbb{M}^{++} & & \mathbb{CP}_3 = Z & & \mathcal{M}_Z = \mathrm{Gr}_2(\mathbb{C}^4)
\end{array}
\tag{5.3}
$$

is strictly concentrated in degree zero if $a \geq 0$.

PROOF. Firstly, notice that all the composition factors occurring in (5.1) are dominant with respect to the first node if $a \geq 0$. Although clear by inspection, the underlying reason for this is that the composition factors are obtained from the leading term $\overset{a}{\times}\!-\!\overset{b}{\times}\!-\!\overset{c}{\bullet}$ by adding simple negative roots for $\times\!-\!\bullet\!-\!\bullet$, namely

$$
\overset{1}{\times}\!-\!\overset{-2}{\bullet}\!-\!\overset{1}{\bullet} \quad \text{and} \quad \overset{0}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{-2}{\bullet}
\tag{5.4}
$$

(minus the second two rows of the Cartan matrix for $\mathfrak{sl}(4,\mathbb{C})$) both of which have a non-negative coefficient over the first node. Secondly, there is the question of tensoring these composition factors with Ω^p_μ from (3.2). Each of Ω^p_μ is dominant with respect to the first node and, more generally,

$$
\begin{array}{rcl}
\Omega^0_\mu \otimes \overset{a}{\times}\!-\!\overset{b}{\times}\!-\!\overset{c}{\bullet} & = & \overset{a}{\times}\!-\!\overset{b}{\times}\!-\!\overset{c}{\bullet} \\
\Omega^1_\mu \otimes \overset{a}{\times}\!-\!\overset{b}{\times}\!-\!\overset{c}{\bullet} & = & \overset{a+1}{\times}\!-\!\overset{b-2}{\times}\!-\!\overset{c+1}{\bullet} \oplus \overset{a+1}{\times}\!-\!\overset{b-1}{\times}\!-\!\overset{c-1}{\bullet} \quad (\text{if } c \geq 1) \\
\Omega^2_\mu \otimes \overset{a}{\times}\!-\!\overset{b}{\times}\!-\!\overset{c}{\bullet} & = & \overset{a+2}{\times}\!-\!\overset{b-3}{\times}\!-\!\overset{c}{\bullet}
\end{array}
\tag{5.5}
$$

(noting that it is $\overset{0}{\times}\!-\!\overset{1}{\times}\!-\!\overset{-2}{\bullet}$ (cf. (5.4)) that is responsible for the second direct summand of $\Omega^1_\mu(\overset{a}{\times}\!-\!\overset{b}{\times}\!-\!\overset{c}{\bullet})$). Clearly, all the various composition factors occurring in $\Omega^p_\mu(\overset{a}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet})$ are dominant with respect to the first node if $a \geq 0$ and, therefore, all direct images are concentrated in degree zero, as required. □

COROLLARY 5.2. *The spectral sequence for* $H^r(\mathbb{CP}_3^+;\mathcal{O}(\overset{a}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet}))$ *associated to the double fibration* (5.3) *is strictly concentrated in top degree (namely, first degree) if* $a \leq -4-b-c$.

PROOF. By standard weight considerations

$$
(\overset{a}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet})^* = \overset{-a-b-c}{\times}\!-\!\overset{c}{\bullet}\!-\!\overset{b}{\bullet}
$$

and so

$$
\kappa_{\mathbb{CP}_3} \otimes (\overset{a}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet})^* = \overset{-4}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet} \otimes \overset{-a-b-c}{\times}\!-\!\overset{c}{\bullet}\!-\!\overset{b}{\bullet} = \overset{-4-a-b-c}{\times}\!-\!\overset{c}{\bullet}\!-\!\overset{b}{\bullet},
$$

and we require $-4-a-b-c \geq 0$ in accordance with Theorem 4.1. □

Further discussion of this example is postponed until §6.

Now let us consider the other example from §1, namely the flag domain $\mathbb{M}^{+-}$. As usual, for computational purposes, we should extend the double fibration to the diagram (3.1). In this case we obtain

$$
\begin{array}{ccccccc}
 & \mathfrak{X}_D & & & & \mathfrak{X}_Z = G/(Q \cap K) & \\
 & {}^{\mu}\swarrow \quad \searrow^{\nu} & & \overset{\text{open}}{\subset} & & \swarrow \quad \searrow & \\
\mathbb{M}^{+-} = D & & \mathcal{M}_D = \mathbb{M}^{++} \times \mathbb{M}^{--} & & \mathrm{Gr}_2(\mathbb{C}^4) = Z & & \mathcal{M}_Z = G/K
\end{array}
$$

where

$$G/K = \mathrm{SL}(4,\mathbb{C})/\left\{\begin{bmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}\right\} = \big\{(\Pi_1,\Pi_2) \in \mathrm{Gr}_2(\mathbb{C}^4)\times\mathrm{Gr}_2(\mathbb{C}^4) \mid \Pi_1 \pitchfork \Pi_2\big\}.$$

An additional difficulty in effecting the transform in this case is that, having taken $\mathrm{SU}(2,2)$ to preserve the standard Hermitian form (1.3), the usual basepoint for $\mathrm{Gr}_2(\mathbb{C}^4)$ is not in the domain $\mathbb{M}^{+-}$. Instead, as basepoints we may take

$$\left(\begin{bmatrix} * \\ * \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ * \\ * \end{bmatrix}\right) \in \mathcal{M}_Z \qquad \begin{bmatrix} * \\ 0 \\ * \\ 0 \end{bmatrix} \in Z \qquad \Longrightarrow Q = \left\{\begin{bmatrix} * & * & * & * \\ 0 & * & 0 & * \\ * & * & * & * \\ 0 & * & 0 & * \end{bmatrix}\right\}.$$

On the other hand, we would like to denote the homogeneous vector bundles on $\mathrm{Gr}_2(\mathbb{C}^4)$ by $\overset{a}{\bullet}\!\!-\!\!\overset{b}{\times}\!\!-\!\!\overset{c}{\bullet}$ as usual. In order to reconcile these two viewpoints, notice that we may conjugate $Q \subset \mathrm{SL}(4,\mathbb{C})$ into standard form: explicitly,

$$(5.6)\qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} * & * & * & * \\ 0 & * & 0 & * \\ * & * & * & * \\ 0 & * & 0 & * \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \equiv \widetilde{Q}.$$

We are therefore confronted with the diagram

$$(5.7)\qquad \begin{array}{ccccc} & & & \mathfrak{X}_Z = G/(Q\cap K) & \\ & \swarrow^{\widetilde{\mu}} & & {}^{\mu}\swarrow \quad \searrow^{\nu} & \\ \mathrm{Gr}_2(\mathbb{C}^4) = G/\widetilde{Q} \cong G/Q = Z & & & & \mathcal{M}_Z = G/K \end{array}$$

as the computational key to the double fibration transform. The first consequence of this additional feature appears in pulling back an irreducible vector bundle from Z to $\mathfrak{X}_Z$. As already mentioned, we shall identify Z as $G/\widetilde{Q}$ and write the irreducible bundles thereon as $\overset{a}{\bullet}\!\!-\!\!\overset{b}{\times}\!\!-\!\!\overset{c}{\bullet}$. Irreducible bundles on

$$\mathfrak{X}_Z = G/(Q\cap K) = \mathrm{SL}(4,\mathbb{C})/\left\{\begin{bmatrix} * & * & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}\right\},$$

however, are carried by representations of the diagonal subgroup of $Q \cap K$. Hence, pullback by $\widetilde{\mu}$ may be achieved by restriction to the subgroup

$$\widetilde{Q} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \supset \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \equiv B$$

followed by conjugation as in (5.6). Explicitly,

$$\widetilde{\mu}^*(\overset{a}{\bullet}\!\!-\!\!\overset{b}{\times}\!\!-\!\!\overset{0}{\bullet}) = \sigma_2\big(\overset{a}{\times}\!\!-\!\!\overset{b}{\times}\!\!-\!\!\overset{0}{\times} + \overset{a-2}{\times}\!\!-\!\!\overset{b+1}{\times}\!\!-\!\!\overset{0}{\times} + \overset{a-4}{\times}\!\!-\!\!\overset{b+2}{\times}\!\!-\!\!\overset{0}{\times} + \cdots + \overset{-a}{\times}\!\!-\!\!\overset{a+b}{\times}\!\!-\!\!\overset{0}{\times}\big)$$

where σ_2 denotes the effect of the conjugation (5.6) on weights. This is a simple Weyl group reflection, the effect of which is computed in [**1**], for example, to obtain

$$\widetilde{\mu}^*(\overset{a}{\bullet}\!\!-\!\!\overset{b}{\times}\!\!-\!\!\overset{0}{\bullet}) = \overset{a+b}{\times}\!\!-\!\!\overset{-b}{\times}\!\!-\!\!\overset{b}{\times} + \overset{a+b-1}{\times}\!\!-\!\!\overset{-b-1}{\times}\!\!-\!\!\overset{b+1}{\times} + \overset{a+b-2}{\times}\!\!-\!\!\overset{-b-2}{\times}\!\!-\!\!\overset{b+2}{\times} + \cdots + \overset{b}{\times}\!\!-\!\!\overset{-a-b}{\times}\!\!-\!\!\overset{a+b}{\times}.$$

More generally, we may write $\overset{a}{\bullet}\!-\!\overset{b}{\times}\!-\!\overset{c}{\bullet} = \overset{a}{\bullet}\!-\!\overset{b}{\times}\!-\!\overset{0}{\bullet} \otimes \overset{0}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{c}{\bullet}$ to compute

$$\widetilde{\mu}^*(\overset{a}{\bullet}\!-\!\overset{b}{\times}\!-\!\overset{c}{\bullet}) = \overset{a+b}{\times}\!-\!\overset{-b}{\times}\!-\!\overset{b+c}{\times} + \begin{array}{c}\overset{a+b-1}{\times}\!-\!\overset{-b-1}{\times}\!-\!\overset{b+c+1}{\times}\\ \oplus \\ \overset{a+b+1}{\times}\!-\!\overset{-b-1}{\times}\!-\!\overset{b+c-1}{\times}\end{array} + \cdots + \overset{b+c}{\times}\!-\!\overset{-a-b-c}{\times}\!-\!\overset{a+b}{\times}. \tag{5.8}$$

The following proposition is almost immediate by inspection.

PROPOSITION 5.3. *The direct images* $\nu^q_*\widetilde{\mu}^*(\overset{a}{\bullet}\!-\!\overset{b}{\times}\!-\!\overset{c}{\bullet})$ *are strictly concentrated in degree zero if* $b \geq 0$.

PROOF. It remains to observe that, recording the irreducible homogeneous vector bundles on $\mathcal{M}_Z$ by irreducible representations of K in the usual manner,

$$\begin{array}{rcll} \nu_*(\overset{r}{\times}\!-\!\overset{s}{\times}\!-\!\overset{t}{\times}) & = & \overset{r}{\bullet}\!-\!\overset{s}{\times}\!-\!\overset{t}{\bullet} & \text{if } r \geq 0 \quad \text{and } t \geq 0 \\ \nu^1_*(\overset{r}{\times}\!-\!\overset{s}{\times}\!-\!\overset{t}{\times}) & = & \overset{r}{\bullet}\!-\!\overset{s+t+1}{\times}\!-\!\overset{-t-2}{\bullet} & \text{if } r \geq 0 \quad \text{and } t \leq -2 \\ \nu^1_*(\overset{r}{\times}\!-\!\overset{s}{\times}\!-\!\overset{t}{\times}) & = & \overset{-r-2}{\bullet}\!-\!\overset{r+s+1}{\times}\!-\!\overset{t}{\bullet} & \text{if } r \leq -2 \quad \text{and } t \geq 0 \\ \nu^2_*(\overset{r}{\times}\!-\!\overset{s}{\times}\!-\!\overset{t}{\times}) & = & \overset{-r-2}{\bullet}\!-\!\overset{r+s+t+2}{\times}\!-\!\overset{-t-2}{\bullet} & \text{if } r \leq -2 \quad \text{and } t \leq -2 \end{array} \tag{5.9}$$

and all other direct images vanish; i.e., the usual formulæ [**1**] pertain. □

LEMMA 5.4. *The holomorphic* 1*-forms along the fibers of* μ *from the diagram* (5.7) *are given by*

$$\Omega^1_\mu = \left(\overset{1}{\times}\!-\!\overset{-2}{\times}\!-\!\overset{1}{\times} + \begin{array}{c}\overset{-1}{\times}\!-\!\overset{-1}{\times}\!-\!\overset{1}{\times}\\ \oplus \\ \overset{1}{\times}\!-\!\overset{-1}{\times}\!-\!\overset{-1}{\times}\end{array}\right) \oplus \left(\overset{1}{\times}\!-\!\overset{0}{\times}\!-\!\overset{1}{\times} + \begin{array}{c}\overset{-1}{\times}\!-\!\overset{1}{\times}\!-\!\overset{1}{\times}\\ \oplus \\ \overset{1}{\times}\!-\!\overset{1}{\times}\!-\!\overset{-1}{\times}\end{array}\right) \tag{5.10}$$

PROOF. This is simply a matter of identifying the weights of $\mathfrak{q}/(\mathfrak{q} \cap \mathfrak{k})$. □

THEOREM 5.5. *The spectral sequence for* $H^r(D; \mathcal{O}(\overset{a}{\bullet}\!-\!\overset{b}{\times}\!-\!\overset{c}{\bullet}))$ *associated to the double fibration*

$$\begin{array}{ccccc} & & \mathfrak{X}_D & & \\ & {}^{\mu}\swarrow & & \searrow^{\nu} & \\ \mathrm{Gr}_2(\mathbb{C}^4) \supset^{\mathrm{open}} \mathbb{M}^{+-} = D & & & & \mathcal{M}_D = \mathbb{M}^{++} \times \mathbb{M}^{--} \end{array} \tag{5.11}$$

is strictly concentrated in degree zero if $b \geq 0$.

PROOF. As for Proposition 5.3, we should inspect the composition factors in $\Omega^p_\mu(\overset{a}{\bullet}\!-\!\overset{b}{\times}\!-\!\overset{c}{\bullet}) = \Omega^p_\mu \otimes \widetilde{\mu}^*(\overset{a}{\bullet}\!-\!\overset{b}{\times}\!-\!\overset{c}{\bullet})$ and determine, with respect to the first and last nodes, whether they are dominant or singular (i.e. whether the integer over that node is non-negative or -1, respectively) since, according to (5.9), such an inspection will determine whether we have (strict) concentration in degree zero. As the composition factors are all line bundles, this is straightforward arithmetic. If $b \geq 1$, then it is easy to check that the leading terms are dominant and the rest are mostly dominant but occasionally singular. In this case, the conclusion of the theorem is clear. When $b = 0$ there are just two exceptions, both of which occur