GISBERT WÜSTHOLZ

A Panorama of Number Theory
or The View from Baker's Garden

CAMBRIDGE

A Panorama in Number Theory
or
The View from Baker's Garden

edited by

Gisbert Wüstholz
ETH, Zürich

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014, Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2002

Printed in the United Kingdom at the University Press, Cambridge

Typeface Times 10/13pt. *System* LaTeX 2_ε [DBD]

A catalogue record of this book is available from the British Library

ISBN 0 521 80799 9 hardback

Contributors

Eva Bayer-Fluckiger, Departement de Mathématiques Ecole Polytechnique Federale de Lausanne 1015 Lausanne Switzerland
eva.bayer@epfl.ch

V.V. Beresnevich, Institute of Mathematics of the Belarus Academy of Sciences, 220072, Surganova 11, Minsk, Belarus
beresnevich@im.bas-net.by

V.I. Bernik, Institute of Mathematics of the Belarus Academy of Sciences, 220072, Surganova 11, Minsk, Belarus
bernik@im.bas-net.by

Yuri F. Bilu, A2X Université Bordeaux I, 351, cours de la Liberation, 33405 Talence, France
yuri@math.u-bordeaux.fr

Enrico Bombieri, School of Mathematics, Institute for Advanced Study, Einstein Drive, Princeton NJ 08540, USA
eb@math.ias.edu

Paula B. Cohen, MR AGAT au CNRS, UFR de Mathématiques, Bâtiment M2, Université des Sciences et Technologies de Lille, 59655 Villeneuve d'Ascq cedex, France
Paula.Cohen@univ-lille1.fr

Sinnou David, Université P. et M. Curie (Paris VI), Institut Mathématique de Jussieu, Problèmes Diophantiens, Case 247, 4, Place Jussieu, 75252 Paris CEDEX 05, France
david@math.jussieu.fr

M.M. Dodson, Department of Mathematics, University of York, York YO10 5DD, UK
mmd1@york.ac.uk

Jan–Hendrik Evertse, Universiteit Leiden, Mathematisch Instituut, Postbus 9512, 2300 RA Leiden, The Netherlands
evertse@math.leidenuniv.nl

G. Faltings, Max-Planck-Institut für Mathematik, Vivatgasse 7, 53111 Bonn, Germany
faltings@mpim-bonn.mpg.de

Dorian Goldfeld, Columbia University, Department of Mathematics, New York, NY 10027, USA
goldfeld@columbia.edu

Kálmán Győry, Institute of Mathematics and Informatics, University of Debrecen, H-4010 Debrecen, P.O. Box 12, Hungary
gyory@math.klte.hu

D.R. Heath-Brown Mathematical Institute, Oxford, UK
rhb@maths.ox.ac.uk

Noriko Hirata-Kohno, Department of Mathematics, College of Science and Technology, Nihon University, Suruga-Dai, Kanda, Chiyoda, Tokyo 101-8308, Japan
hirata@math.cst.nihon-u.ac.jp

Ming-Chit Liu, Department of Mathematics, The University of Hong Kong, Pokfulam, Hong Kong; *and* PO Box 625, Alhambra, California CA 91802, USA
matmcliu@yahoo.com

G. Margulis, Yale University, Department of Mathematics, PO Box 208283, New Haven CT 06520-8283, USA
margulis@math.yale.edu

D.W. Masser, Mathematisches Institut, Universität Basel, Rheinsprung 21, 4051 Basel, Switzerland
masser@math.unibas.ch

Yu. Nesterenko, Faculty of Mechanics and Mathematics, Main Building, MSU, Vorobjovy Gory, Moscow, 119899, Russia
nest@trans.math.msu.su

Peter Sarnak, Princeton University, Department of Mathematics, Fine Hall, Princeton NJ 08544-1000, USA
sarnak@math.princeton.edu

Andrzej Schinzel, Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, PO Box 137, 00-950 Warszawa, Poland
schinzel@impan.gov.pl

T. Shioda, Department of Mathematics, Rikkyo University, Nishi-Ikebukuro, Toshima-ku, Tokyo 171, Japan
shioda@rkmath.rikkyo.ac.jp

T.N. Shorey, School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India
shorey@math.tifr.res.in

Tianze Wang, Department of Mathematics, Henan University, Kaifeng 475001, China
wangtz@henu.edu.cn

Jürgen Wolfart, Mathematisches Seminar der Johann Wolfgang Goethe-Universität, Robert-Mayer-Str. 6–10, D–60054 Frankfurt a.M., Germany
wolfart@math.uni-frankfurt.de

G. Wüstholz, ETH Zürich, ETH Zentrum, D-MATH, HG G 66.3 Raemistrasse 101, CH-8092 Zürich, Switzerland.
wustholz@math.ethz.ch

Kunrun Yui, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong
makryu@ust.hk

Introduction

The millennium, together with Alan Baker's 60th birthday offered a singular occasion to organize a meeting in number theory and to bring together a leading group of international researchers in the field; it was generously supported by ETH Zurich together with the Forschungsinstitut für Mathematik. This encouraged us to work out a programme that aimed to cover a large spectrum of number theory and related geometry with particular emphasis on Diophantine aspects. Almost all selected speakers we able to accept the invitation; they came to Zurich from many parts of the world, gave lectures and contributed to the success of the meeting. The London Mathematical Society was represented by its President, Professor Martin Taylor, and it sent greetings to Alan Baker on the occasion of his 60th birthday.

This volume is dedicated to Alan Baker and it offers a panorama in number theory. It is as exciting as the scene we enjoyed, during the conference, from the cafeteria on top of ETH overlooking the town of Zurich, the lake and the Swiss mountains as well as the spectacular view that delighted us on our conference excursion to Lake Lucerne in central Switzerland. The mathematical spectrum laid before us in the lectures ranged from sophisticated problems in elementary number theory through to diophantine approximations, modular forms and varieties, metrical diophantine analysis, algebraic independence, arithmetic algebraic geometry and, ultimately, to the theory of logarithmic forms, one of the great achievements in mathematics in the last century. The articles here document the present state of the art and suggest possible new directions for research; they can be expected to inspire much further activity. Almost all who were invited to contribute to the volume were able to prepare an article; with very few exceptions, the promised papers were eventually submitted and the result turns out itself to be like a very colourful panoramic picture of mathematics taken on a beautiful clear day in the autumn of the year 1999.

It is not easy to group together the different contributions in a systematic way. Indeed we appreciate that any attempt at categorisation can certainly be disputed and may invoke criticism. Nonetheless, for the reader who is not an expert in this area, we think that it would be helpful to have some guidelines. Accordingly we shall now discuss briefly the various subjects covered in the book.

Since one of the main motivations for the conference – as already said at the beginning – was the 60th birthday of Alan Baker, the theory of **logarithmic forms** was a very important and significant part of the proceedings. The article *One Century of Logarithmic Forms* by Gisbert Wüstholz is an overview of the evolution of the subject beginning with the famous seventh problem of Hilbert. It describes the history of its solution, the subsequent development of the theory of logarithmic forms and then goes on to explain how the latter is now regarded as an integral part of a general framework relating to group varieties. This overview should be seen as a homage to Alan Baker and his work. Several contributions are directly connected with it: we mention Yu Kunrui's paper *Report on p-adic Logarithmic Forms* which surveys the now very extensive p-adic aspects of the theory, and the article *Recent Progress on Linear Forms in Elliptic Logarithms* by Sinnou David and Noriko Hirata-Kohno which gives a detailed exposition of elliptic aspects, especially with regard to important quantitative results. The theory of logarithmic forms has found numerous applications in very different areas. One of the earliest and most direct of these has been to diophantine equations of classical type coming in part from problems in algebraic number theory. This side of the subject is explained in Kalman Győry's paper *Solving Diophantine Equations by Baker's Theory*.

One domain, seemingly very far from the area of logarithmic forms but in fact surprisingly strongly related to it, is **modular forms and varieties**. A good illustration is provided by the article *Baker's Method and Modular Curves* by Yuri F. Bilu, which shows how Baker's theory can be applied in the context of Siegel's theorem to give effective estimates for the heights of integral points on a large class of modular curves. Another example is the paper *Application of the André–Oort Conjecture to Some Questions in Transcendence* by Paula Cohen and Gisbert Wüstholz. Here it is not so much the classical logarithmic theory that is involved but the considerably wider framework mentioned earlier on group varieties. And Jürgen Wolfart's contribution *Regular Dessins, Endomorphisms of Jacobians, and Transcendence* connects modular geometry with modern logarithmic theory, in particular with abelian varieties. Quite differently, Peter Sarnak's article *Maass Cusp Forms with Integer Coefficients* spans the bow from cuspidal eigenforms of Laplacians for congruence subgroups of $SL(2, \mathbb{Z})$ and automorphic cuspidal representations to classical logarithmic theory and transcendence.

In the articles of Wüstholz and of Yu Kunrui mentioned above, the very intimate relation between the theory of linear forms in logarithms and the so-called *abc*-conjecture is explained. The latter is now recognised as one of the most central problems in mathematics. Dorian Goldfeld succeeds in giving an extraordinarily broad picture of the topic in his fine paper *Modular Forms, Elliptic Curves and the abc-Conjecture*. And in Yu.V. Nesterenko's survey article *On Algebraic Independence of Numbers* we have an excellent reference for research and achievements during the last millennium on algebraic independence questions; the emphasis is on recent significant progress concerning modular forms and their connection with hypergeometric functions and it can be confidently predicted that the article will be very influential for further investigations.

Another important subject is the theory of **lattices** and the contribution of Eva Bayer-Fluckiger on *Ideal Lattices* can be seen as a splendid introduction. Applications are described, for instance, to Knot theory and Arakelov theory. Another series of applications can be found in the paper of Tetsuji Shioda entitled *Integral Points and Mordell–Weil Lattices*. Here the author demonstrates a close connection with Lie theory by explaining, amongst other things, how integral points on elliptic curves can be regarded as roots of root lattices associated with Lie groups of exceptional type like E_8.

Diophantine approximations and equations are the general subject of a further series of papers. We mention the very interesting overview of Enrico Bombieri *Forty Years of Effective Results in Diophantine Theory*. Bombieri has for many years sought to square the circle in the sense of making the Thue–Siegel–Dyson–Schneider–Roth theory effective, and he has met with much success. A substantial part of his article is devoted to describing the state of the art here and, in particular, how it relates to Baker's theory. Complementing Bombieri's point of view, the paper *Points on Subvarieties of Tori* by Jan-Hendrik Evertse explains how the non-effective theory has made progress in the last four decades especially with regard to diophantine geometry. Gerd Faltings' article *A New Application of Diophantine Approximation* indicates the path along which some very modern diophantine theory might develop in the near future; it contains very exciting new geometrical ideas and tools and it can be expected to serve as a valuable source for further research. Finally there is the contribution of David Masser entitled *Search Bounds for Diophantine Equations*; here the author takes a very fundamental point of view which leads to an important new topic, namely the existence of *a priori* (or what Masser calls *search*) bounds for the solutions of equations. It seems likely to attract the interest not only of number theorists but also of theoretical and possibly even practical computer scientists.

A totally different point of view on diophantine approximation is taken by the so-called **metrical theory**. This deals with approximations on very general classes of manifolds and spaces typically concerning points on a geometric space outside some fixed sets of measure zero. In an early paper Alan Baker considered such questions and there has been considerable progress in the field since then. Apart from its significance to number theory, applications have been given in the context of Hausdorff dimensions and so-called small denominator questions related to stability problems for dynamical systems. In *Regular Systems, Ubiquity and Diophantine Approximation*, V.V. Beresnevich, V.I. Bernik and M.M. Dodson present a careful and valuable report on the development of the theory. The article by Gregory Margulis entitled *Diophantine Approximation, Lattices and Flows on Homogeneous Spaces* connects with this and these two contributions taken together can be expected to be very influential for future research. In Margulis' paper the direction is the study of homogeneous spaces rather than arbitrary manifolds and it furnishes the framework for investigating orbits in the space of lattices. This point of view has made it possible to successfully apply techniques from differential geometry and Lie theory and it shows again the remarkable range of tools and techniques currently used in number theory.

Broadly speaking, one can place under the heading **analytic number theory** the article of Ming-Chit Liu and Tianze Wang *On Linear Ternary Equations with Prime Variables – Baker's Constant and Vinogradov's Bound*, the paper by T.N. Shorey *Powers in Arithmetic Progression*, the contribution 'On the Greatest Common Divisor of Two Univariate Polynomials' by Andrzej Schinzel and the short note of D.R. Heath-Brown entitled *Heilbronn's Exponential Sum and Transcendence Theory*. Apart from the classical sphere of ideas which one traditionally associates with analytic number theory, these papers have the extra quality of bringing in methods from outside the field. The Heath-Brown contribution gives a particularly good example in which transcendence techniques similar to those introduced by Stepanov in the context of the Weil conjectures become central for the study of exponential sums.

Returning to the beautiful Swiss landscape, in the same way that it invites one to climb this or that grand mountain or to explore some of the host of picturesque features, so we hope that the panorama exhibited in this volume invites visits to and explorations of a more abstract but nonetheless beautiful and colourful region of mathematics. Many of the exciting sites owe their existence to Alan Baker.

I express my gratitude to the Schulleitung of ETH and especially to Alain Sznitman, the director of the Forschungsinstitut when the conference took place, for making possible the whole project which has led to this volume. The assistance of Renate Leukert has been invaluable in the editing of the work. Finally, I am sincerely grateful to my secretary Hedi Oehler and the secretaries of the Forschungsinstitut at the time, in particular Ruth Ebel, without whose help the organization of the conference would certainly have been impossible.

Contents

1

One Century of Logarithmic Forms

G. Wüstholz

1 Introduction

At the turn of any century it is very natural on the one hand for us to look back and see what were great achievements in mathematics and on the other to look forward and speculate about which directions mathematics might take. One hundred years ago Hilbert was in a similar situation and he raised on that occasion a famous list of 23 problems that he believed would be very significant for the future development of the subject. Hilbert's article on future problems in mathematics published in the *Comptes Rendus du Deuxième Congrès International des Mathématiciens* stimulated tremendous results and an enormous blossoming of the mathematical sciences overall. A significant part of Hilbert's discussion was devoted to number theory and Diophantine geometry and we have seen some wonderful achievements in these fields since then. In this survey, we shall recall how transcendence and arithmetical geometry have grown into beautiful and far-reaching theories which now enhance many different aspects of mathematics. Very surprisingly three of Hilbert's problems, which at first seemed very distant from each other, have now come together and have provided the catalyst for a vast interplay between the subjects in question. We shall concentrate on one of them, namely the seventh, and describe the principal developments in transcendence theory which it has initiated. This will lead us to the theory of linear forms in logarithms and to the generalization of the latter in the context of commutative group varieties. The theory has evolved to be the most crucial instrument towards a solution of the tenth problem of Hilbert on the effective solution of diophantine equations as well as many other well-known questions. The intimate relationships in this field become especially evident through a simple conjecture, the *abc*-conjecture, which seems to hold the key to much of the future direction of number theory. We shall discuss this at the end of this article.

2 Hilbert's seventh problem

Hilbert remarked in connection with the seventh problem that he believed that the proof of the transcendence of α^β for algebraic $\alpha \neq 0, 1$ and algebraic irrational β would be extremely difficult and that certainly the solution of this and analogous problems would lead to valuable new methods. Surprisingly, the problem was eventually solved independently, by different methods, by Gelfond and Schneider in 1934. Gelfond and Kuzmin had solved some particular cases of the conjecture a few years earlier and the solutions of Gelfond and Schneider used similar methods together with techniques introduced by Siegel in his well-known investigations on Bessel functions. The Gelfond–Schneider theorem shows that for any non-zero algebraic numbers α_1 and α_2 with $\log \alpha_1$ and $\log \alpha_2$ linearly independent over the rationals we have

$$\beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 \neq 0.$$

In 1935 Gelfond considered the problem of establishing a lower bound for the absolute value of the linear form $L = \beta_1 T_1 + \beta_2 T_2$ evaluated at $(\log \alpha_1, \log \alpha_2)$ and succeeded in proving that its value Λ is bounded below by

$$\log |\Lambda| \gg -h(L)^\kappa$$

where $h(L)$ denotes the logarithmic height of the linear form and $\kappa > 5$. It was realized by Gelfond around 1940 that an extension of the theorem to linear forms in more than two variables would enable one to solve some of the most challenging problems in number theory and in the theory of diophantine equations. We mention here the *Liouville problem* of establishing effective lower bounds for the approximation of an algebraic number by rationals sharper than the bound obtained by Liouville himself. Other examples were the *Thue equation* and effective bounds for the size of solutions in *Siegel's great theorem on integral points* on algebraic curves. To great surprise one of the oldest and most exciting problems in number theory, Euler's famous *numeri idonii* problem, has also turned out to be intimately related to the theory of logarithmic forms.

The Liouville problem was mentioned by Davenport to Baker as a research topic in the early 60s. Baker's early papers made the first breakthrough in this area. The approach was through hypergeometric functions and Padé approximation theory and was related to some work of Thue and Siegel. After several significant results in diverse branches of transcendence theory, Baker was led to the famous class number problem of Gauss. A careful study of the work of Heilbronn, Gelfond, Linnik and others in this field convinced him that the most promising approach to this and many other fundamental questions in number theory was through linear forms in logarithms. Despite the fact that no significant progress had been made in this subject for many years, Baker succeeded in 1966 in establishing a definitive result; namely if $\alpha_1, \ldots, \alpha_n$ are non-zero algebraic numbers such that $\log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over the rationals, then $1, \log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over the field of all algebraic numbers. The result and the method of proof was one of the most significant advances in number theory made in 20th century. Baker's theorem includes both the Hermite–Lindemann and the Gelfond–Schneider theorems as special cases. Baker's original paper also contained a quantitative result on lines similar to Gelfond's two-variable estimate mentioned earlier; this was sufficient to deal with the class number problem, the Thue problem, the Liouville problem, the elliptic curve problem and clearly had great potential for future research. It soon became clear that further progress on many critical problems would depend on sharp estimates for logarithmic forms, and between 1966 and 1975 Baker wrote a series of important papers on this subject. It was here that many of the instruments now familiar to specialists in the field were introduced, among them Kummer theory, the so-called Kummer descent and delta functions. In this context one should mention that both Stark and Feldman made substantial contributions.

3 Elliptic theory

Having published his solution to Hilbert's seventh problem, Schneider started to study elliptic and later also abelian functions. He had been motivated by a paper of Siegel on periods of elliptic functions. There Siegel had proved that for an elliptic curve with algebraic invariants g_2, g_3 not all periods can be algebraic. In particular he obtained the transcendence of non-zero periods when the elliptic curve has complex multiplication. In a fundamental paper Schneider established the transcendence of elliptic integrals of the first and second kind taken between algebraic points. As a special instance one obtains the transcendence of the value of the linear form $L = \alpha T_1 + \beta T_2$ at $(\omega, \eta(\omega))$ where ω is a non-zero period and $\eta(\omega)$ the corresponding quasi-period. Plainly the result gives Siegel's theorem without the additional hypothesis on complex multiplication. Schneider also applied the result to the modular j-function and found that it takes algebraic values at algebraic arguments if and only if the argument is imaginary quadratic. As was realized only recently, this opened a close connection with Hilbert's twelfth problem which emerged out of the famous *Jugendtraum* of Kronecker. For some forty years there was no obvious progress; then in 1970 Baker realized that the method which he had developed for dealing with linear forms in logarithms could be adapted to give the transcendence of non-zero values of linear forms in two elliptic periods and their associated quasi-periods. With this fundamental contribution he opened a new field of research. Masser and Coates succeeded a few years later in including the period $2\pi i$ and determining the dimension of the vector space generated by the numbers $1, 2\pi i, \omega_1, \omega_2, \eta(\omega_1), \eta(\omega_2)$. The main difficulty in going further and extending the results to an arbitrary number of periods lay in the use of determinants in Baker's method. It became clear that to achieve a breakthrough such aspects had to be modified significantly.

The situation was very similar in the case of abelian varieties which was first studied by Schneider in 1939 and which was subsequently investigated by Masser, Coates and Lang between 1975 and 1980. Again difficulties relating to the use of determinants presented a severe obstacle for progress. It was realized about that time that transcendence theory has much to do with algebraic groups and with the exponential map of a Lie group in particular. Lang was the first to consider a reformulation of the Gelfond–Schneider theorem and other classical results in the language of group varieties. With advice from Serre this was taken up by Waldschmidt and, amongst other things, he interpreted Schneider's result on elliptic integrals in the new language. This prepared the ground for the first successful attack, by Laurent, on Schneider's third problem concerning elliptic integrals of the third kind. However the difficulties relating to determinants referred to earlier blocked the passage to a complete solution. Likewise, for abelian integrals, Schneider had raised the question of extending his elliptic theorems to abelian varieties. As Arnold pointed out in his monograph on Newton, Hooke and Huyghens, the question is closely related to an unsolved problem of Leibniz emerging from celestical mechanics.

4 Group varieties

The situation changed completely when it was realized almost simultaneously by Brownawell, Chudnovsky, Masser, Nesterenko and Wüstholz that one way to deal with the difficulties referred to in the previous section was to use commutative algebra. Masser and Wüstholz started to apply the theory to commutative group varieties and the breakthrough was obtained by Wüstholz in 1981 when he succeeded in establishing the correct multiplicity estimates for group varieties. As a consequence he was able to formulate and prove the Analytic Subgroup Theorem. It says that if G is a commutative and connected algebraic group defined over $\overline{\mathbb{Q}}$ then an analytic subgroup defined over $\overline{\mathbb{Q}}$ contains a non-trivial algebraic point if and only if it contains a non-trivial algebraic subgroup over $\overline{\mathbb{Q}}$. The theorem generalizes that of Baker in a natural way and hence includes, as special cases, the classical theorems of Hermite, Lin-