CALCULUS · WITH · ANALYTIC · GEOMETRY

FOURTH EDITION

EARL W. SWOKOWSKI

CALCULUS WITH ANALYTIC GEOMETRY

FOURTH EDITION

EARL W. SWOKOWSKI
MARQUETTE UNIVERSITY

PWS·KENT
PUBLISHING COMPANY

BOSTON

PWS-KENT
Publishing Company

20 Park Plaza
Boston, Massachusetts 02116

TO MAUREEN

PWS-KENT Publishing Company is a division of Wadsworth, Inc.

92 91 90 89 88 / 10 9 8 7 6 5 4 3 2

LIBRARY OF CONGRESS CATALOGING-IN-PUBLICATION DATA
Swokowski, Earl W.
Calculus with analytic geometry / Earl W. Swokowski.
—4th ed.
p. cm.
Includes index.
ISBN 0-87150-007-8
1. Calculus. 1. Geometry, Analytic. I. Title
QA303.S94 1988b
515'.15—dc19 87-29161
CIP

COVER IMAGE • Copyright © 1986 Sanjay Kothari
SPONSORING EDITOR • David Geggis
PRODUCTION AND DESIGN • Katherine Townes
TEXT COMPOSITION • Syntax International Pte. Ltd.
TECHNICAL ARTWORK • J & R Art Services; Deborah Schneck
COVER MANUFACTURING • New England Book Components, Inc.
TEXT MANUFACTURING • Arcata Graphics/Hawkins

Printed in the United States of America

PREFACE

This book is a major revision of the Third Edition of *Calculus with Analytic Geometry.* One of my goals was to maintain the mathematical soundness of the previous edition, while making discussions somewhat less formal by rewriting and by placing more emphasis on graphs and figures. Another objective was to stress the usefulness of calculus through a variety of new applied examples and exercises from many different disciplines. Finally, suggestions for improvements from instructors led me to change the order of presentation of certain topics.

A great deal of rewriting, reorganization, and new material has gone into this edition, and to list the changes in detail would make the preface excessively long. The following remarks merely highlight the principal changes from the previous edition.

HIGHLIGHTS OF THIS EDITION

- The review of graphs of functions in Chapter 1 includes vertical and horizontal shifts, stretching, and reflections. Many exercises that involve applications are designed to prepare students for later work with extrema and related rates.
- In Chapter 2 the limit concept is motivated informally, prior to the rigorous approach considered in Section 2.3. To help raise students' level of interest at this early stage of calculus, examples and exercises that involve unusual applications — such as compressed gases, optics, G-forces experienced by astronauts, drug dosage levels, and the theory of relativity — are included throughout the chapter.
- The concept of rate of change (previously in Chapter 4) is introduced in Section 3.3 to provide a greater variety of applications of the derivative early in the text. Related rates are discussed in Section 3.9.
- Chapter 4 consists of concepts pertaining to extrema, graphing, and antiderivatives. Applications to economics (which formerly constituted a separate section) are included, where appropriate, in this and other chapters.
- Properties of the definite integral and the definition of average value of a function are discussed in one section of Chapter 5. Numerical integration involving the use of approximate data is considered at the end of the chapter and in the applications discussed in Chapter 6.
- The concepts of arc length and surfaces of revolution are introduced in Section 6.5 so that mathematical applications of the definite integral are considered in the first half of Chapter 6. The physical applications in the second half of the chapter are independent of one another and may be studied in any order (or omitted), depending on class objectives. Moments and the center of mass of a lamina are discussed in Section 6.8. Nontraditional applications of the definite integral are given in the final section.
- Chapter 7 includes a large number of examples and exercises on applications of the natural logarithmic function and the natural exponential function to diverse fields.
- A review of the trigonometric functions (formerly an appendix) is given at the beginning of Chapter 8. The graphs of all six trigonometric functions are included.

- The discussion in Chapter 9 is limited to techniques of integration. Applications from earlier chapters are reconsidered in exercises that require advanced methods of integration.
- Chapter 10 contains numerous new examples and exercises that involve applications of indeterminate forms and improper integrals.
- The approach to infinite sequences in Chapter 11 provides a geometric motivation for the concepts of convergence and divergence. The ratio test for positive-term series is introduced early, and alternating series and absolute convergence are discussed in one section. A new table summarizes all the tests discussed in the chapter.
- In Chapter 12 the notion of eccentricity of conic sections is prominent. Applications include LORAN navigation and orbits of planets and comets.
- The topics of tangent lines, arc length, and surfaces of revolution that are associated with parametrized curves are consolidated in one section in Chapter 13. Polar equations of conics are considered in the last section.
- Chapter 14 has been streamlined so that vectors in three dimensions immediately follow vectors in two dimensions. Lines and planes are discussed in one section, as are cylinders and quadric surfaces.
- In Chapter 15 the geometric significance of vector-valued functions is stressed by means of many figures, examples, and exercises.
- Chapter 16 contains fifty new figures (including computer graphics) in the exposition and exercises. Tree diagrams are introduced to help visualize chain rules. The concept of gradient is given special attention in later sections of the chapter.
- Double integrals in polar coordinates and the concept of surface area appear early in Chapter 17. Moments and the center of mass of a nonhomogeneous solid are discussed near the end of the chapter. The general theorem on change of variables in multiple integrals (formerly in Section 18.9) is stated in Section 17.9.
- In Chapter 18 one definition and one evaluation theorem unify the three types of line integrals in two dimensions. Conservative vector fields are emphasized. Evaluation formulas for surface integrals are given in one theorem. Divergence and curl of a vector field are motivated through examples.
- Chapter 19 contains a discussion of first- and second-order linear differential equations with applications.

FEATURES OF THE TEXT

APPLICATIONS Every calculus book has applied problems from engineering, physics, chemistry, biology, and economics. This revision also includes exercises from specialized fields such as physiology, sociology, psychology, ecology, oceanography, meteorology, radiotherapy, astronautics, and transportation.

EXAMPLES Each section contains carefully chosen examples to help students understand and assimilate new concepts. Whenever feasible, applications are included to demonstrate the usefulness of a topic.

EXERCISES Exercise sets begin with routine drill problems and progress gradually to more difficult types. Applied problems generally appear near the end of a set to allow students to gain confidence in manipulations and new ideas before attempting questions that require analyses of practical situations.

Over 300 new exercises involving applications are included to stress the diversity and power of calculus. Many applications are novel, and differ greatly from standard applications that have been used traditionally in calculus books.

A review section at the end of each chapter consists of a list of important topics and pertinent exercises.

Answers to the odd-numbered exercises are given at the end of the text.

CALCULATORS Calculators are referred to where appropriate. It is possible to work most of the exercises without a calculator; however, instructors may wish to encourage its use for computations involving approximate data.

TEXT DESIGN A new use of color makes discussions easier to follow and highlights major concepts. All figures have been redrawn for this edition and, wherever possible, are placed in the margin next to the discussion. Graphs are usually labeled and color-coded to clarify complex figures. Drawings have been added to many exercise sets to help students visualize applied problems.

FLEXIBILITY Syllabi from schools that used the previous edition attest to the flexibility of the text. Sections and chapters can be rearranged in different ways, depending on the objectives and length of the course.

SUPPLEMENTS FOR THE INSTRUCTOR

The following teaching aids may be obtained from the publisher:

Complete Solutions Manual, Volumes I and II
by Jeff Cole, Anoka-Ramsey Community College, and Gary Rockswold, Mankato State University

Even-numbered answer booklet

Computerized test generator (for IBM-PCs and compatibles)

Printed test bank

PWS-KENT GradeDisk

SUPPLEMENTS FOR THE STUDENT

The following supplements are available:

Student Supplement, Volumes I and II
by Thomas A. Bronikowski, Marquette University, which contains solutions for every third problem in the text

Programmed Study Guide
by Roy A. Dobyns, Carson-Newman College, which is keyed to the first nine chapters of the text

True BASIC™ Calculus Software by True BASIC, Inc. for IBM-PCs and compatibles and the Apple MacIntosh

Calculus and the Computer
by Sheldon Gordon, Suffolk Community College

ACKNOWLEDGMENTS

I wish to thank Michael R. Cullen of Loyola Marymount University for supplying most of the new exercises dealing with applications. This large assortment of problems provides strong motivation for the mathematical concepts introduced in the text. Professor Cullen also furnished the computer graphics that accompany some of these exercises. Christian C. Braunschweiger of Marquette University contributed many suggestions that improved the exposition.

I also wish to thank the following individuals, who reviewed the manuscript:

H.S. Butts, *Louisiana State University*
Dawson Carr, *Sandhills Community College*
Mark P. Hale, Jr., *University of Florida*
Dale T. Hoffman, *Bellevue Community College*
Joseph E. Hyman, *El Camino College*
G. Philip Johnson, *Oakland University*
Elgin Johnston, *Iowa State University*
Herbert M. Kamowitz, *University of Massachusetts—Boston*
James T. Loats, *Metropolitan State College*
Robert H. Lohman, *Kent State University*
Stanley M. Lukawecki, *Clemson University*
Francis E. Masat, *Glassboro State College*
Judith R. McKinney, *California State Polytechnic University—Pomona*
J. Osterburg, *University of Cincinnati*
Neal C. Raber, *University of Akron*
Dennis Ryan, *Wright State University*
Nancy M. Thompson, *Metropolitan State College*
John R. Unbehaun, *University of Wisconsin—La Crosse*
R. Voronka, *New Jersey Institute of Technology*
Alan Wiederhold, *San Jacinto College*
Dennis H. Wortman, *University of Massachusetts—Boston*

and mathematics educators, who met with me and representatives from PWS-KENT for several days in the summer of 1986, and who later reviewed portions of the manuscript:

Cliff Clarridge, *Santa Monica College*
Jeff Cole, *Anoka-Ramsey Community College*
Michael Cullen, *Loyola Marymount University*
Bruce Edwards, *University of Florida*
Michael Schneider, *Belleville Area College*

Their comments on pedagogy and their specific recommendations about the content of calculus courses helped me to improve the book.

I am grateful for the excellent cooperation of the staff at PWS-KENT. Two people in the company deserve special mention. My production editor Kathi Townes did a truly exceptional job in designing the book and taking care of an enormous number of details associated with the production. I cannot thank her enough for her assistance. Managing Editor Dave Geggis supervised the project, contacted many reviewers and users of my texts, and was a constant source of information and advice.

In addition to all the persons named here, I express my sincere appreciation to the many unnamed students and teachers who have helped shape my views on how calculus should be presented in the classroom.

EARL W. SWOKOWSKI

CONTENTS

FOR THE STUDENT

Calculus was invented in the seventeenth century as a tool for investigating problems that involve motion. Algebra and trigonometry may be used to study objects that move at constant speeds along linear or circular paths; but calculus is needed if the speed varies or if the path is irregular. An accurate description of motion requires precise definitions of *velocity* (the rate at which distance changes per unit time) and *acceleration* (the rate at which velocity changes). These definitions may be obtained by using one of the fundamental concepts of calculus—the *derivative*.

Although calculus was developed to solve problems in physics, its power and versatility have led to uses in many diverse fields of study. Modern-day applications of the derivative include investigating the rate of growth of bacteria in a culture, predicting the outcome of a chemical reaction, measuring instantaneous changes in electrical current, describing the behavior of atomic particles, estimating tumor shrinkage in radiation therapy, forecasting economic profits and losses, and analyzing vibrations in mechanical systems.

The derivative is also useful in solving problems that involve maximum or minimum values, such as manufacturing the least expensive rectangular box that has a given volume, calculating the greatest distance a rocket will travel, obtaining the maximum safe flow of traffic across a long bridge, determining the number of wells to drill in an oil field for the most efficient production, finding the point between two light sources at which illumination will be greatest, and maximizing corporate revenue for a particular product. Mathematicians often employ derivatives to find tangent lines to curves and to help analyze graphs of complicated functions.

Another fundamental concept of calculus—the *definite integral*—is motivated by the problem of finding areas of regions that have curved boundaries. Definite integrals are employed as extensively as derivatives and in as many different fields. Some applications are finding the center of mass or moment of inertia of a solid, determining the work required to send a space probe to another planet, calculating the blood flow through an arteriole, estimating depreciation of equipment in a manufacturing plant, and interpreting the amount of dye dilution in physiological tests that involve tracer methods. We can also use definite integrals to investigate mathematical concepts such as area of a curved surface, volume of a geometric solid, or length of a curve.

The concepts of derivative and definite integral are defined by limiting processes. The notion of *limit* is the initial idea that separates calculus from elementary mathematics. Sir Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716) independently discovered the connection between derivatives and integrals and are both credited with the invention of calculus. Many other mathematicians have added greatly to its development in the last 300 years.

The applications of calculus mentioned here represent just a few of the many considered in this book. We can't possibly discuss all the uses of calculus, and more are being developed with every advance in technology. Whatever your field of interest, calculus is probably used in some pure or applied investigations. Perhaps *you* will discover a new application for this branch of science.

CHAPTER 1

FUNCTIONS AND GRAPHS

This chapter contains a review of topics required for the study of calculus. After a brief discussion of real numbers, coordinate systems, and graphs in two dimensions, we will turn our attention to one of the most important concepts in mathematics—the notion of *function*.

1.1 REAL NUMBERS

Calculus is based on properties of **real numbers.** If we add the real number 1 successively to itself we obtain the **positive integers** 1, 2, 3, 4, ... The **integers** consists of all positive and negative integers together with the real number 0. We sometimes list the integers as follows:

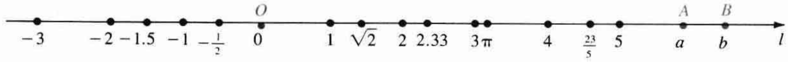
$$\ldots, \quad -4, \quad -3, \quad -2, \quad -1, \quad 0, \quad 1, \quad 2, \quad 3, \quad 4, \quad \ldots$$

A **rational number** is a real number that can be expressed as a quotient a/b, for integers a and b with $b \neq 0$. Real numbers that are not rational are **irrational.** For example, the ratio of the circumference of a circle to its diameter is irrational. This real number is denoted by π and the notation $\pi \approx 3.1416$ is used to indicate that π is *approximately equal* to 3.1416. Another example of an irrational number is $\sqrt{2}$.

Real numbers may be represented by *nonterminating decimals*. For example, the decimal representation for the rational number 177/55 is found by division to be 3.2181818 . . . , where the digits 1 and 8 repeat indefinitely. Rational numbers may always be represented by *repeating* decimals. Irrational numbers may be represented by nonterminating and *nonrepeating* decimals.

A *one-to-one correspondence* exists between the real numbers and the points on a line l, in the sense that to each real number a there corresponds one and only one point P on l and, conversely, to each point P there corresponds one real number. Such a correspondence is illustrated in Figure 1.1, where we have indicated several points corresponding to real numbers. The

FIGURE 1.1



point O that corresponds to the real number 0 is the **origin.**

The number a that is associated with a point A on l is the **coordinate** of A. An assignment of coordinates to points on l is a **coordinate system** for l, and l is called a **coordinate line,** or a **real line.** A direction can be assigned to l by taking the **positive direction** to the right and the **negative direction** to the left. The positive direction is noted by placing an arrowhead on l as shown in Figure 1.1.

Real numbers that correspond to points to the right of O in Figure 1.1 are **positive real numbers,** whereas those that correspond to points to the left of O are **negative real numbers.** The real number 0 is neither positive nor negative.

If a and b are real numbers, and $a - b$ is positive, we say that ***a* is greater than *b*** and write $a > b$. An equivalent statement is ***b* is less than *a*** ($b < a$). The symbols $>$ and $<$ are **inequality signs,** and expressions such as $a > b$ and $b < a$ are **inequalities.** Referring to Figure 1.1, if A and B are points with coordinates a and b, respectively, then $b > a$ (or $a < b$) *if and only if A lies to the left of B*. Since $a - 0 = a$, it follows that $a > 0$ if and only if a is positive. Similarly, $a < 0$ means that a is negative. We can prove the following properties.

PROPERTIES OF INEQUALITIES (1.1)

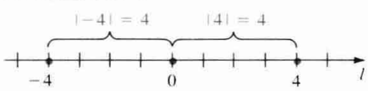
(i) If $a > b$ and $b > c$, then $a > c$.
(ii) If $a > b$, then $a + c > b + c$.
(iii) If $a > b$, then $a - c > b - c$.
(iv) If $a > b$ and c is positive, then $ac > bc$.
(v) If $a > b$ and c is negative, then $ac < bc$.

Analogous results are true if the inequality signs are reversed. Thus, if $a < b$ and $b < c$, then $a < c$; if $a < b$, then $a + c < b + c$, and so on.

The symbol $a \geq b$, which is read ***a* is greater than or equal to *b*,** means that either $a > b$ or $a = b$. The symbol $a < b < c$ means that $a < b$ and $b < c$, in which case we say that ***b* is *between* *a* and *c*.** The notations $a \leq b$, $a < b \leq c$, $a \leq b < c$, $a \leq b \leq c$, and so on, can be interpreted from the preceding definitions.

If a real number a is the coordinate of a point A on a coordinate line l, the symbol $|a|$ is used to denote the number of units (or distance) between A and the origin, without regard to direction. The nonnegative number $|a|$ is the *absolute value* of a. Referring to Figure 1.2, we see that for the point with coordinate -4, we have $|-4| = 4$. Similarly, $|4| = 4$. In general, *if a is negative we change its sign to find $|a|$. If a is nonnegative, then $|a| = a$.* The next definition summarizes this discussion.

FIGURE 1.2



DEFINITION (1.2)

Let a be a real number. The **absolute value** of a, denoted by $|a|$, is

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

EXAMPLE 1 Find $|3|$, $|-3|$, $|0|$, $|\sqrt{2}-2|$, and $|2-\sqrt{2}|$.

SOLUTION Since 3, $2-\sqrt{2}$, and 0 are nonnegative,

$$|3| = 3, \quad |2-\sqrt{2}| = 2-\sqrt{2}, \quad \text{and} \quad |0| = 0.$$

Since -3 and $\sqrt{2}-2$ are negative, we use the formula $|a| = -a$ to obtain

$$|-3| = -(-3) = 3 \quad \text{and} \quad |\sqrt{2}-2| = -(\sqrt{2}-2) = 2-\sqrt{2}. \quad \bullet$$

We can show that for all real numbers a and b,

$$|a| = |-a|, \qquad |ab| = |a||b|, \qquad -|a| \le a \le |a|.$$

The following properties can also be proved.

PROPERTIES OF ABSOLUTE VALUES ($b > 0$) **(1.3)**

(i) $|a| < b$ if and only if $-b < a < b$.
(ii) $|a| > b$ if and only if either $a > b$ or $a < -b$.
(iii) $|a| = b$ if and only if $a = b$ or $a = -b$.

Properties (ii) and (iii) are also true if $b = 0$. Thus, if $b \ge 0$, then

$$|a| \le b \quad \text{if and only if} \quad -b \le a \le b$$

and

$$|a| \ge b \quad \text{if and only if} \quad a \ge b \quad \text{or} \quad a \le -b.$$

THE TRIANGLE INEQUALITY **(1.4)**

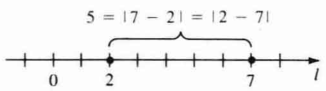
$$|a+b| \le |a| + |b|$$

PROOF Consider $-|a| \le a \le |a|$ and $-|b| \le b \le |b|$. Adding corresponding sides, we obtain

$$-(|a| + |b|) \le a + b \le |a| + |b|.$$

Using the remark preceding this theorem gives us the Triangle Inequality. • •

FIGURE 1.3



We shall use absolute values to define the distance between any two points on a coordinate line. Note that the distance between the points with coordinates 2 and 7 shown in Figure 1.3 equals 5 units on l. This distance is the difference, $7-2$, obtained by subtracting the smaller coordinate from the larger. If we employ absolute values, then, since $|7-2| = |2-7|$, the order of subtraction is irrelevant.

DEFINITION (1.5)

Let a and b be the coordinates of two points A and B, respectively, on a coordinate line l. The **distance between A and B,** denoted by $d(A, B)$, is

$$d(A, B) = |b - a|$$

The number $d(A, B)$ denotes the **length of the line segment AB.** Observe that, since $d(B, A) = |a-b|$ and $|b-a| = |a-b|$,

$$d(A, B) = d(B, A).$$

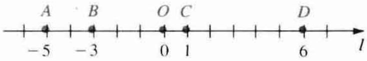
Also note that the distance between the origin O and the point A is

$$d(O, A) = |a - 0| = |a|.$$

which agrees with the geometric interpretation of absolute value illustrated in Figure 1.2. The formula $d(A, B) = |b - a|$ is true regardless of the signs of a and b, as illustrated in the next example.

EXAMPLE 2 If A, B, C, and D have coordinates -5, -3, 1, and 6, respectively, find $d(A, B)$, $d(C, B)$, $d(O, A)$, and $d(C, D)$.

FIGURE 1.4



SOLUTION The points are sketched in Figure 1.4. By Definition (1.5):

$$d(A, B) = |-3 - (-5)| = |-3 + 5| = |2| = 2$$
$$d(C, B) = |-3 - 1| = |-4| = 4$$
$$d(O, A) = |-5 - 0| = |-5| = 5$$
$$d(C, D) = |6 - 1| = |5| = 5$$

For some topics, such as inequalities, it is convenient to use the notation and terminology of *sets*. We may think of a **set** as a collection of objects of some type. The objects are **elements** of the set. In our work, $\mathbb{R}$ will denote the set of real numbers. If S is a set, then $a \in S$ means that a is an element of S, whereas $a \notin S$ signifies that a is not an element of S. If every element of a set S is also an element of a set T, then S is a **subset** of T. Two sets S and T are **equal,** and we write $S = T$, if S and T contain precisely the same elements. The notation $S \neq T$ means that S and T are not equal. If S and T are sets, their **union** $S \cup T$ consists of the elements that are either in S, in T, or in *both* S and T. The **intersection** $S \cap T$ consists of the elements that the sets have in common.

We frequently use letters to represent arbitrary elements of a set. For example, we may use x to denote a real number when we do not wish to specify a *particular* real number. A letter that is used to represent *any* element of a given set is sometimes called a **variable.** A symbol that represents a *specific* element is a **constant.** In most of our work, letters near the end of the alphabet, such as x, y, and z, will be used for variables. Letters such as a, b, and c will denote constants. Throughout this text, unless otherwise specified, variables represent real numbers.

The **domain of a variable** is the set of real numbers represented by the variable. To illustrate, $\sqrt{x}$ is a real number if and only if $x \geq 0$, and hence the domain of x is the set of nonnegative real numbers. Similarly, given the expression $1/(x - 2)$, we must exclude $x = 2$ in order to avoid division by zero; consequently, in this case the domain is the set of all real numbers different from 2.

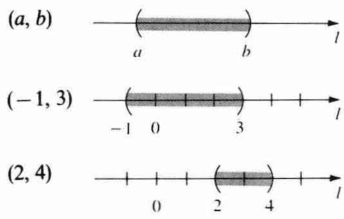
If the elements of a set S have a certain property, we sometimes write $S = \{x: \quad \}$ and state the property describing the variable x in the space after the colon. For example, $\{x: x > 3\}$ denotes the set of all real numbers greater than 3. Finite sets are sometimes identified by listing all the elements within braces. Thus, if the set T consists of the first five positive integers, we may write $T = \{1, 2, 3, 4, 5\}$.

Of major importance in calculus are certain subsets of $\mathbb{R}$ called **intervals.** If $a < b$, the set of all real numbers between a and b is an **open interval** and is denoted by (a, b), as in the following definition.

OPEN INTERVAL (1.6)

$$(a, b) = \{x: a < x < b\}$$

FIGURE 1.5



The numbers a and b are the **endpoints** of the interval.

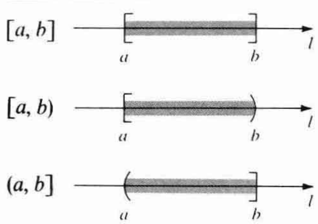
The **graph** of a set of real numbers is defined as the points on a coordinate line that correspond to the numbers in the set. In particular, the graph of the open interval (a, b) consists of all points between the points corresponding to a and b. In Figure 1.5 we have sketched the graphs of a general open interval (a, b) and two specific open intervals $(-1, 3)$ and $(2, 4)$. The parentheses on each graph indicate that the endpoints of the interval are not included. For convenience, we shall use the terms *interval* and *graph of an interval* interchangeably.

To denote the inclusion of an endpoint in an interval, we use a bracket instead of a parenthesis. If $a < b$, then a **closed interval,** denoted by $[a, b]$, and a **half-open interval,** denoted by $[a, b)$ or $(a, b]$, are defined as follows.

CLOSED AND HALF-OPEN INTERVALS (1.7)

$$[a, b] = \{x: a \le x \le b\}$$
$$[a, b) = \{x: a \le x < b\}$$
$$(a, b] = \{x: a < x \le b\}$$

FIGURE 1.6



Typical graphs are sketched in Figure 1.6, where a bracket indicates that the corresponding endpoint is part of the graph.

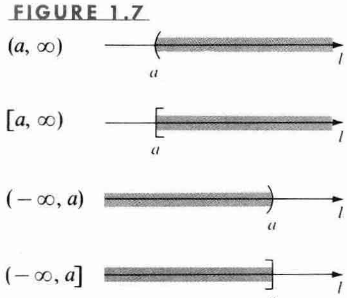
In future discussions of intervals, whenever the numbers a and b are not stated explicitly we will always assume that $a < b$. If an interval is a subset of another interval I, it is a **subinterval** of I. For example, the closed interval $[2, 3]$ is a subinterval of $[0, 5]$.

We use the following notation for **infinite intervals.**

INFINITE INTERVALS (1.8)

$$(a, \infty) = \{x: x > a\}$$
$$[a, \infty) = \{x: x \ge a\}$$
$$(-\infty, a) = \{x: x < a\}$$
$$(-\infty, a] = \{x: x \le a\}$$

FIGURE 1.7



For example, $(1, \infty)$ represents all real numbers greater than 1. The symbol ∞ is read *infinity* and is merely a notational device. It does not represent a real number. Typical graphs of infinite intervals for an arbitrary real number a are sketched in Figure 1.7. The absence of a parenthesis or bracket on the right for (a, ∞) or $[a, \infty)$ and on the left for $(-\infty, a)$ or $(-\infty, a]$ indicates that the graph, shown as the colored portion, extends indefinitely. The set $\mathbb{R}$ of real numbers is sometimes denoted by $(-\infty, \infty)$.

We will often consider inequalities that involve variables, such as

$$x^2 - 3 < 2x + 4.$$

If numbers, such as 4 or 5, are substituted for x in $x^2 - 3 < 2x + 4$, we obtain false statements, such as $13 < 12$ or $22 < 14$, respectively. Other numbers, such as 1 or 2, produce true statements: $-2 < 6$ or $1 < 8$, respectively. If a true statement is obtained when x is replaced by a real number a, then a is a **solution** of the inequality. Thus 1 and 2 are solutions of the inequality $x^2 - 3 < 2x + 4$, but 4 and 5 are not solutions. To **solve** an inequality means to find all solutions. Two inequalities are **equivalent** if they have exactly the same solutions.

To solve an inequality, we usually replace it with a list of equivalent inequalities, terminating in one for which the solutions are obvious. The main tools used in applying this method are properties of inequalities and absolute value. For example, if x represents a real number, then adding the same expression in x to both sides of an inequality leads to an equivalent inequality. We may multiply both sides by an expression containing x if we are certain that the expression is positive for all values of x under consideration. If we multiply both sides of an inequality by an expression that is always negative, such as $-7 - x^2$, then we reverse the inequality sign in the equivalent inequality.

EXAMPLE 3 Solve the inequality $4x + 3 > 2x - 5$ and represent the solutions graphically.

SOLUTION The following inequalities are equivalent (supply reasons):

$$4x + 3 > 2x - 5$$
$$4x > 2x - 8$$
$$2x > -8$$
$$x > -4$$

FIGURE 1.8

Hence the solutions consist of all real numbers greater than -4, that is, the numbers in the infinite interval $(-4, \infty)$. The graph is sketched in Figure 1.8. •

EXAMPLE 4 Solve the inequality $-5 \leq \dfrac{4 - 3x}{2} < 1$ and sketch the graph corresponding to the solutions.

SOLUTION We may proceed as follows:

$$-5 \leq \frac{4 - 3x}{2} < 1$$
$$-10 \leq 4 - 3x < 2$$
$$-14 \leq -3x < -2$$
$$\frac{14}{3} \geq x > \frac{2}{3}$$
$$\frac{2}{3} < x \leq \frac{14}{3}$$

FIGURE 1.9

Hence the solutions are the numbers in the half-open interval $(\frac{2}{3}, \frac{14}{3}]$. The graph is sketched in Figure 1.9. •