Chaos in Nonlinear Dynamical Systems

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INTRODUCTION

There is a resurgence of research activity in nonlinear dynamics. Significant progress in qualitative techniques for finite and infinite dimensional systems together with the astronomical advances in computational powers is enabling better understanding of complex phenomena. Such progress until now was either unattainable or too cumbersome to attempt. In recent years, specifically, a great deal of progress has been made in the study of stability and chaotic behavior of nonlinear systems. Still many more challenging problems await resolution. The keen interest in this field is primarily due to the fact that the satisfactory solution of such problems is bound to have significant impact on a wide variety of physical and engineering applications. These include multi-stable chemical and biological systems, electronic and optical switches, structure of turbulence, oscillation in aeroelastic structures, feedback controls and communication and power systems.

Recognizing the importance of this field, the Mathematical Sciences Division, U.S. Army Research Office organized an interdisciplinary workshop on Chaos in Nonlinear Dynamical Systems. The workshop was held on March 13–15, 1984 at the Army Research Office, Research Triangle Park, North Carolina, attended by more than sixty mathematicians, engineers, physicists and chemists. This book is based on the papers presented at this workshop.

In Chapter I, Newhouse discusses various mathematical notions useful in the description of chaotic motions. Pseudorandom phenomena have recently been shown to be relevant to a variety of engineering applications in place of stochastic models commonly used in communication and control theory, and structural analysis. The problem of generating pseudorandom processes using models which are sufficiently simple so as to allow some quantitative analysis is, therefore, drawing much attention. In Chapter II, Brockett and Cebuhar, describe a piecewise linear third order autonomous differential equation to seek new insights into particularly simple mechanisms which appear to generate chaos in such systems.

The next three chapters are devoted to the development and application of the Poincaré-Melnikov-Arnold method. For instance, in Chapter III, Marsden proves existence of chaos in the sense of Poincaré-Birkhoff-Smale horseshoes, followed in the next chapter by Slemrod, who discusses the application of this approach to chaos to the problem of equilibrium distribution of a van der Waals fluid undergoing spatially thermal variations. Using similar techniques, in Chapter V, Salam and Sastry describe the complete dynamics of the Josephson junction circuit and provide a complete bifurcation diagram of the a.c. forced junction. This is done by establishing analytically the existence of chaos for certain parameter ranges in the dynamics of such systems.

In Chapter VI, Levi considers the so-called beating modes in Josephson junction circuits. He provides qualitative analysis of these systems and gives analytical characterization of these modes. The perturbed Sine-Gordon equation models the dynamics of long Josephson tunnel junction. In Chapter VII, Christiansen discusses some of the soliton dynamic states for this equation and describes computer experiments exhibiting hysteresis phenomena and chaotic intermittency between soliton dynamics states occurring as a result of applied external bias. In the following chapter, Kopell discusses the influence of symmetry properties on coherence and chaos in a chain of weakly coupled oscillations. This study is partly motivated by a biological application.

Existence of multiple stable states has importance in a variety of applications. Studies in optical bistability, for instance, have opened up new exciting possibilities of using such logic elements as switches in optical computing. In Chapter IX, McLaughlin, Moloney and

Newell summarize some recent results on coherence and chaos in optical bistable laser cavities. Next, Moss explores two very different types of switches and their behavior when they are subject to large amplitude external interference. In Chapter XI, Arecchi investigates multistability and chaos in quantum optics. Three experimental situations are described that exemplify onset of chaos. Next, Ackerhalt and Milonni describe how chaotic dynamics of molecular vibrations explain the dependence of multi-photon absorption on pulse energy rather than intensity. In Chapter XIII, Casati and Guarneri investigate the chaotic properties of quantum systems under external perturbations.

In the last two chapters, the authors discuss implications of chaotic dynamics to turbulent flows and oscillation in mechanical systems. In Chapter XIV, Manley shows how recent research on the asymptotic properties of the Navier-Stokes equation is valuable in the use of computers as experimental tools in the study of the dynamics of fluids. He shows how the conventional estimate of the number of degrees of freedom of turbulent flows can be obtained from such asymptotic properties. In the last chapter, Dowell and Pierre look at the fundamental mechanisms in nonlinear mechanics that lead to chaotic oscillations. Through specific examples, they identify two categories of systems. In the first category the chaotic oscillations arise as a result of instability of the system to large finite disturbance while in the second category the chaos results from instability with respect to infinitesimal disturbances.

I thank all of those whose efforts have helped both to make the workshop successful and to bring this book into its final form. I particularly appreciate the contributors to this volume. Special thanks are due to Mrs. Frances Atkins, Mrs. Mary Mitchell and Mrs. Brenda Hunt of the Army Research Office for the diligent cooperation through all phases of this workshop and the preparation of this book.

JAGDISH CHANDRA
September 1984

CONTENTS

UNDERSTANDING CHAOTIC DYNAMICS

S. E. NEWHOUSE*

Abstract: Various mathematical notions useful in the description of chaotic motion are discussed. Emphasis is given to hyperbolic attractors and Bowen-Ruelle-Sinai measures.

We describe here certain mathematical structures, results, and methods which are useful for understanding chaotic dynamical systems. Roughly speaking, a chaotic dynamical system is one which has presumably many) solutions which display highly aperiodic or erratic time dependence. A glance at recent physical and engineering literature (e.g. Holmes[3)], Swinney[8)]) reveals an abundance of physical systems with such motions. For reasons of space we will mainly deal with discrete dynamical systems (i.e. mappings from a subset M of Euclidean space to itself). Most of the results we discuss here have counterparts for systems with continuous time. For more information on this subject as well as related ideas, we refer to Newhouse[4,5,6,7)], Farmer et al[1)], and Guckenheimer and Holmes[2)].

We consider a smooth manifold M (possibly with boundary) and a twice differentiable mapping f from M to M. We assume f is one-to-one and the inverse map $f^{-1}: f(M) \to M$ is also twice

*Department of Mathematics, University of North Carolina, Chapel Hill, NC 27514

differentiable. Given $x \in M$, consider the positive orbit $0_+(x) = \{x, f(x), f^2(x), \ldots\}$ of x and the ω-limit set $\omega(x) = \omega(x,f) = \{y \in M$: there is a sequence $n_1 < n_2 < \ldots$ such that $f^{n_i}x \to y$ as $i \to \infty\}$. Note that if $0_+(x)$ is bounded, then $\omega(x)$ is a closed, bounded set, and $f(\omega(x)) = \omega(x)$. In general, we are interested in describing $\omega(x,f)$ for as many x and f as possible. A simple situation arises when each $\omega(x,f)$ is a periodic orbit for $x \in M$. Then each initial state tends toward a periodic orbit. Such an f is not chaotic.

A first instance in which f might be called chaotic is when there are uncountably many points x for which $\omega(x,f)$ is also uncountable. This can occur for relatively mild f: let S^1 be the unit circle in $\mathbb{R}^2$, and let $f = S^1 \to S^1$ be defined by $f(u,v) = (u \cos 2\pi\alpha - v \sin 2\pi\alpha, u \sin 2\pi\alpha + v \cos 2\pi\alpha)$ for $(u,v) \in S^1$ and α irrational. Thus, f is just a rotation through angle $2\pi\alpha$. It is well-known that $\omega((u,v),f) = S^1$ for each $(u,v) \in S^1$. An additional condition frequently encountered in chaotic systems is "sensitive dependence on initial conditions." This could be defined as follows. Let $d(x,y)$ denote the distance between x and y. A point x exhibits sensitive dependence on intitial conditions if there are an $\alpha > 0$ and a constant $C > 0$ such that for any $\varepsilon > 0$ and any positive integer $n > 0$, there is a point y such that

(1) $d(x,y) < \varepsilon$

(2) $d(f^jx, f^jy) \geq Ce^{\alpha j}d(x,y)$ for $0 \leq j \leq n$.

The infinitesimal version of this is more easily defined: there is a vector v tangent to M at x such that $\limsup_{n \to \infty} \frac{1}{n} \log |T_x f^n v| > 0$. Here $T_x f^n$ is the derivative (linear approximation) to f^n at x and $|w|$ the norm of a vector w. The latter condition on x and v is frequently referred to as "(x,v) has a positive Lyapunov exponent." Sometimes one says that x has a positive Lyapunov exponent if there is a v such that (x,v) has one. Again there are simple mappings which satisfy this condition. Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(u,v) = (\lambda^{-1}u, \lambda v)$ with $1 < \lambda$. Any point in $\mathbb{R}^2$ has a positive exponent. All the points in $\mathbb{R} \times \{0\}$ have in addition, bounded positive orbits.

If we put the above definitions together we get a better definition of a chaotic f: say $f: M \to M$ is chaotic if there is a closed bounded subset V of M with non-empty interior such that

(1) f maps V into its interior .

(2) there are uncountably many points x in V for which $\omega(x)$ is uncountable and x has a positive Lyapunov exponent.

In the known examples where f is chaotic in this last sense and the chaos is persistent in the sense that any g C^1 near f also satisifies (1) and (2), one has some interesting behavior for f. For instance, f must have infinitely many periodic orbits in V, and f must have transverse homoclinic points (see Newhouse[4]), and Guckenheimer-Holmes[2]) in V.

We wish to consider further notions of chaos. There is the frequently used term "strange attractor." This term was created by

Ruelle and Takens[9], and has been modified by various authors subsequently. One could define such an object as follows. A closed set $\Lambda \subset M$ is <u>invariant</u> if $f(\Lambda) = \Lambda$. The set Λ is an <u>attractor</u> if there is an open set $U \supset \Lambda$ such that $f(U) \subset U$ and $\bigcap_{n\geq 0} f^n(U) = \Lambda$. The attractor is <u>strange</u> if it has a complicated topology; e.g. if it is not a manifold. Frequently, one adds further indecomposability conditions as well. One such condition is that there is a point x in Λ whose orbit is dense in Λ. The basin of the attractor Λ is the set $B(\Lambda) = \bigcap_{n\leq 0} f^n(U) = \{x \in M: \omega(x) \subset \Lambda\}$. The idea of Ruelle and Takens was that typical points in the basin of a strange attractor would have a complicated asymptotic behavior, and that such objects could provide models for chaotic motion. In particular, the continuous time versions of such objects in certain infinite dimensional spaces could provide models for turbulence in fluids. Experiments in a number of situations have lent credence to this idea. Our above notions of chaos are independent of the notion of a strange attractor but they are certainly consistent with it: one merely has to consider V as a subset of $B(\Lambda)$.

In addition to (1) and (2) above, there are further conditions one could put on the set V. For instance, one might require the points x in (2) to be dense in V or to have positive or full volume in V. If the latter condition holds, one could ask for a description of statistical properties of $O_+(x)$ for typical x in V. For example, how are the points $\{x, fx, \ldots, f^n x\}$ distributed for typical x in V and large n? What is the structure of time averages $\frac{1}{n}\sum_{k=0}^{n-1}\psi(f^k x)$ for continuous ψ, large n, and typical x? These questions and many more have been answered in the case when V is in the basin of attraction of a hyperbolic attractor. Such objects provide mathematically understood models of chaotic behavior. A large amount of current research in dynamics attempts to extend the mathematics of hyperbolic attractors to more general situations. See Newhouse[7)] for related ideas.

It is our contention that much can be gained toward understanding chaotic dynamics by the careful study of hyperbolic attractors. That is, many of the phenomena arising in such attractors also occur in many other chaotic situations. There are several rigorous results supporting this contention although much work remains to be done.

Let us now recall the notion of a hyperbolic attractor, and give some typical examples.

A closed bounded invariant set Λ is <u>hyperbolic</u> if there are constants $C > 0$, $0 < \lambda < 1$, and for each $x \in \Lambda$, there is a splitting $T_x M = E_x^s \oplus E_x^u$ such that

(a) $v \in E_x^s \Longrightarrow T_x f(v) \in E_{f(x)}^s$ and

$v \in E_x^u \longrightarrow T_x f(v) \in E_{f(x)}^u$.

(b) $n \geq 0$ and $v \in E_x^s \Longrightarrow |T_x f^n(v)| \leq C\,\lambda^n |v|$ and

$n \geq 0$ and $v \in E_x^u \Longrightarrow |T_x f^{-n}(v)| \leq C\,\lambda^n |v|$.

Condition (a) describes invariance of the infinitesimal subspaces E_x^s and E_x^u, and condition (b) describes exponential forward contraction of vectors in E_x^s and exponential backward contraction of vectors in E_x^u. An attractor which is also a hyperbolic set is a hyperbolic attractor. Let us give some examples.

1. Let $\mathbb{R}^2 = \{(u,v)\}$ be the Euclidean plane, and let

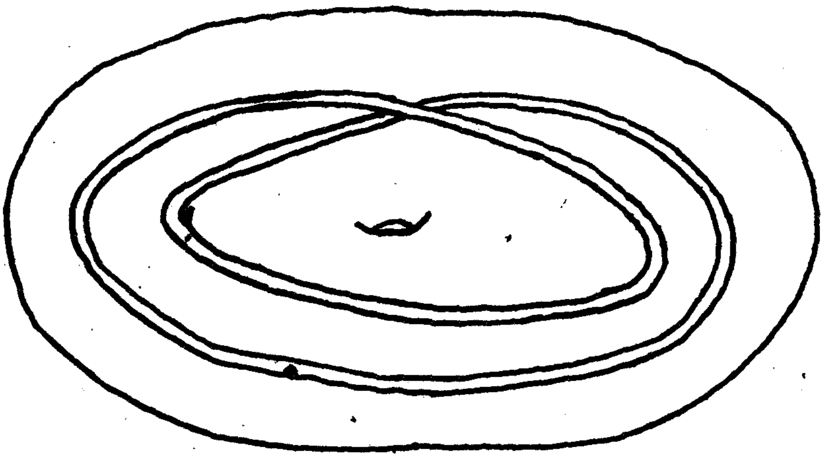
   $A(u,v) = (2u + v, u + v)$.

   Thus A is a linear map with matrix $\binom{21}{11}$, determinant 1, and two real eigenvalues $\lambda = \frac{3+\sqrt{5}}{2}$, $\lambda^{-1} = \frac{3-\sqrt{5}}{2}$, with eigenspaces E_λ, and $E_{\lambda^{-1}}$, respectively. Let $T^2 = \mathbb{R}^2/Z^2$ be the two-dimensional torus where Z^2 is the integer lattice in $\mathbb{R}^2$. The map A induces a diffeomorphism $\bar{A}$ from T^2 to itself. Let $\pi: = \mathbb{R}^2 \to T^2$ be the natural projection. Let $x \in T^2$ and let $\bar{x} \in \mathbb{R}^2$ be such that $\pi(\bar{x}) = x$. Let $E_\lambda(\bar{x})(E_{\lambda^{-1}}(\bar{x}))$ be the line in $\mathbb{R}^2$ through $\bar{x}$ parallel to $E_\lambda(E_{\lambda^{-1}})$. Let $E_x^s = \pi(E_{\lambda^{-1}}(\bar{x}))$ and $E_x^u = \pi(E_\lambda(\bar{x}))$. It is easily checked that E_x^s and E_x^u are independent of the choice of $\bar{x} \in \pi^{-1}(x)$ and satisfy conditions (a) and (b) with $\bar{A} = f$. Then, all of T^2 is a hyperbolic attractor, and $B(T^2) = T^2$.

2. Let $\mathbb{C}$ be the set of complex numbers. Let $D = \{w \in \mathbb{C} = |w| \le 1\}$, and let $S^1 = \{z \in \mathbb{C}: |z| = 1\}$. Thus, S^1 is the unit circle and D is the closed unit disk. Let $S^1 \times D$ be the product space which we think of as a solid torus of revolution in $\mathbb{R}^3$. Letting (x_1, y_1, z_1) be coordinates in $\mathbb{R}^3$, we consider $S^1 \times D =$

   $$\{(x_1,y_1,z_1): (x_1^2 + y_1^2)^{1/2} - a)^2 + z_1^2 \le b^2\}$$

where $0 < b < a$, and $(x_1, y_1, 0)$ corresponds to $(x_1 + \sqrt{-1}\, y_1, 0)$ in $S^1 \times D$. The mapping $f(z,w) = (z^2, \frac{z}{2} + \frac{w}{4})$ maps $S^1 \times D^2$ into its interior as in the next figure.



The largest invariant set $\Lambda = \bigcap_{n\geq 0} f^n(S^1 \times D)$ is a hyperbolic attractor of topological dimension 1. In fact, it looks locally like the product of a Cantor set and an interval.

3. Geodesic (inertial motion on a surface of negative curvature is such that the motion on each positive energy surface is the continuous time analog of a hyperbolic attractor.

Theorem 1 below presents some statistical results which have been obtained for hyperbolic attractors.

Recall that a Borel probability measure μ on M is a non-negative countably additive real-valued set function defined on σ-field of Borel sets in M such that $\mu(M) = 1$. The measure μ is invariant if $\mu(f^{-1}A) = \mu(A)$ for every Borel set A. One calls μ ergodic if whenever E is a Borel set such that $f(E) = E$ it follows that $\mu(E) = 0$ or 1. If μ is ergodic, then there is a set E with $\mu(E) = 1$ such that each orbit in E is dense in E. Thus, ergodicity is a kind of strong indecomposability condition. We recall that Λ is called topologically mixing if for any open sets U, V in M such that $U \cap \Lambda \neq \phi$ and $V \cap \Lambda \neq \phi$, there is an integer $N > 0$ such that $n \geq N$ implies $f^n(U \cap \Lambda) \cap (V \cap \Lambda) \neq \phi$. Thus, topological mixing means that all large iterates of open sets in Λ (in the relative topology) get spread around well in Λ. A hyperbolic attractor containing a fixed point is, in fact, topologically mixing. We will say that the hyperbolic attractor is non-trivial if it contains more than one orbit. It follows that it must be uncountable and even have Hausdorff dimension greater than or equal to 1.

Theorem 1 Assume $f: M \to M$ has a topologically mixing non-trivial hyperbolic attractor. Then,

(a) there is a set $A \subset B(\Lambda)$ such that the Lebesque measure of $B(\Lambda) - A$ is zero, and for every $x \in A$ and every continuous function $\phi: M \to \mathbb{R}$, $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k x) = \hat{\phi}(x)$ exists and is independent of x in A

(b) the operator $\phi \longrightarrow \mu(\phi) = \hat{\phi}(x)$ for $x \in A$ determines an invariant measure called the Bowen-Ruelle-Sinai measure (or natural measure[1]) for Λ.

(c) μ is exponentially mixing: given C^1 real-valued functions ϕ and ψ, $\left|\int \phi(f^n x)\, \psi(x)\, d\mu - \int \phi d\mu \cdot \int \psi d\mu\right| \le Ce^{-n\alpha}$

for $n \ge 0$ and some constant $C > 0$.

(d) μ satisfies a central limit theorem[11]: for typical C^1 ψ there is a constant $\sigma = \sigma(\psi) > 0$ so that

$$\mu\{x \in V: \frac{1}{\sqrt{n}} (\sum_{k=0}^{n-1} \psi(f^k x) - n \int \psi du) < r\}$$

$$\longrightarrow \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{r} \exp(-x^2/2\sigma^2)\, dx \quad \text{as} \quad n \to \infty.$$

(e) μ is "stable" under random perturbations[12].

Parts (a), (b), (c) of theorem 1 are due to Ruelle[10].

Remarks:

1. There are instances of chaotic f with no non-trivial hyperbolic attractors. However, the properties in (a) through (e) of theorem 1 may still hold in much generality. Even in the absence of rigorous proofs, theorem 1 suggests numerical tests which can be applied. For instance, one could compute time averages of various fuctions at various points to see if limiting time averages exist. Given a

function $\psi: M \to \mathbb{R}$, and an integer $n > 0$, one could compute

$$A(n,x,\psi) = \frac{1}{i} \sum_{k=0}^{i-1} \psi(f^{n+k}x)\, \psi\,(f^k x)) - (\frac{1}{i} \sum_{k=0}^{i-1} \psi(f^k x))^2$$

for typical x and i much larger than n. Suppose this were nearly independent of x and large i and decreased exponentially with n. Then, one would have evidence of the existence of a measure satisfying (a) and (c). Similarly, one could test for (d).

2. There are necessary and sufficient conditions for the existence of measures (with non-zero Lyapunov exponents) satisfying (a) for points x in sets of positive Lebesque measure. Precise statements are found in Newhouse[7] and the proofs will appear in the future.

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