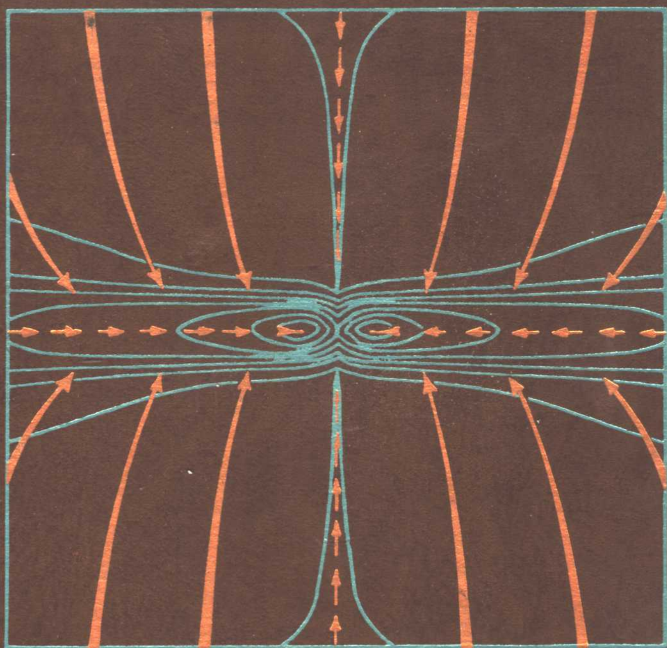
FLUID DYNAMICS

in ASTROPHYSICS and GEOPHYSICS



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LECTURES IN APPLIED MATHEMATICS

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Fluid Dynamics in Astrophysics and Geophysics

Norman R. Lebovitz, Editor

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Preface

The subject of Geophysical Fluid Dynamics (GFD) has a rich history of interplay with mathematics, an interplay that has resurged in recent years. Astrophysical Fluid Dynamics (AFD) can be thought of as the extension of GFD to larger scales, beyond the planetary to the stellar and galactic. This is an oversimplification, but is close enough to the truth that one might expect more interaction among geophysicists, astrophysicists, and mathematicians than has in fact taken place. A step toward facilitating such interactions was taken in the form of the Thirteenth Summer Seminar in Applied Mathematics, sponsored jointly by the American Mathematical Society and the Society for Industrial and Applied Mathematics, which was held on the campus of the University of Chicago from June 29 through July 10, 1981.

Beyond creating an atmosphere promoting interaction among practitioners in the fields of GFD, AFD, and mathematics, the purposes of the seminar were to present, in a form accessible to graduate students and recent Ph.D.'s, background and perspective in these areas and some of the currently exciting research topics. The present volume, which grew out of the seminar, has these same purposes.

The seminar was organized by a committee including (aside from me) Victor Barcilon, Richard di Prima, Peter Goldreich, Joseph Pedlosky, and Alan Toomre. My thanks to all these individuals, with special thanks to Professor Barcilon for devoting time and energy far beyond the call of duty. Further thanks are due to Dr. William LeVeque, Executive Director of the AMS, Ms. Carole Kohanski, and Professor W.H. Meyer for keeping the administrative machinery running smoothly; and to the National Science Foundation, the National Oceanographic and Atmospheric Administration and the National Aeronautics and Space Administration for their financial support. Last but by no means least, my thanks to the speakers and contributors to this volume.

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I. GEOPHYSICAL FLUID DYNAMICS

Lectures in Applied Mathematics
Volume 20, 1983

Lectures in Geophysical Fluid Dynamics

Peter B. Rhines

1. Introduction. I was asked to prepare a presentation of some of the fundamental ideas in geophysical fluid dynamics (GFD) assuming the reader to be competent with fluids but unacquainted with meteorology, oceanography or planetary physics. These five lectures approach from a physical standpoint the motions of planetary fluids, rather than developing the mathematical structures abstracted from them. They lean toward oceanography, which is my home ground.

Historical notes. GFD has long been a part of fluid dynamics. Before the Industrial Revolution, in fact, much of scientific theory was undertaken to explain the natural world. Aristotle, Newton and Laplace all produced theories of the ocean tides. Though less divine than astronomy, "natural" fluid dynamics was prominent in the new continuum theory.

An independent strain influencing "terrestrial" GFD was exploration, commerce, and warfare that involved the sea. Science has often ridden on the worldly ambitions of nations, as with the Royal Society's sponsorship of the first major observational survey of the deep sea, on *H.M.S. Challenger*, 1873, and as with the current scientific outposts in Antarctica. The polar explorations were particulary fruitful for oceanography. Before the days of orbiting satellites, observations of the high-latitude oceans required unusual enterprise. The Norwegian Fridtjof Nansen and colleagues are the best-known example. In the summer of 1893 he and Captain Otto Sverdrup steamed and sailed the specially designed vessel *Fram* into the Arctic Ocean, purposely to become entrapped in the ice as winter descended. Beside their ambition to reach the North Pole lay the far more important notion of *how* they would reach it. They imagined rightly that the winds and ocean circulation would drive the ice westward,

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near the Pole and then carry them south between Greenland and Spitzbergen where they would be freed to sail home. Nansen set off across the ice when they had drifted seemingly to the highest latitude (in fact the *Fram* drifted 100 miles further north thereafter). He never reached the Pole, but walked home via Franz Josef Land. He arrived just as the 3-year drift of the ship ended, more or less as predicted.

This familiar story is worth retelling not merely to add glamor to GFD, but to emphasize that a complex system like the air and sea requires difficult on-site observations built into a long, stable tradition of thought, before it can progress as a deductive science. Beyond Nansen's inspiration of Ekman, which led to a key theory of the wind-driven boundary layer, there were decades of systematic GFD in Scandinavia that produced Bjerknes, Bergeron, Rossby and many others. Even today, you will find oceanographers to be particularly fond of taking their own data at sea. A young theoretician should not be surprised if he soon finds himself unsnarling cable on the fantail, in the stormy wintertime North Atlantic.

The nature of the fluid. The circulation models we abstract are often of homogeneous density, smoothly and steadily flowing around in little gyres driven by mythically steady winds. These are oceans the size of a tea-cup, as judged by the Reynolds number, UL/ν, for which the solutions are valid. U is a characteristic horizontal velocity, L the horizontal scale, and ν the kinematic viscosity. Of course the real terrestrial domain has some ten decades of scale between the largest motions and the smallest, and many different dynamical regimes fit between the extremes. It is a challenge to isolate, understand, and finally to recombine this chain of distinct regimes into a description more successful than the tea-cup oceans and snow-globe atmospheres of classical times.

Here we consider length-scales greater than a few km, and time scales greater than a day, such that the Rossby number $R_0 = U/\Omega L$, is small. R_0 and $R_0 \cdot (\Omega T)$, where T is the time-scale of the dominant motion, are the relative measures of acceleration and Coriolis force. Even within this restricted range there is a great wealth of phenomena. To emphasize how far we have to go, I want to recount a few properties of the earth's ocean and atmosphere.

The 10^9 $(\mathrm{km})^3$ of seawater have a reasonably constant mixing ratio of dissolved salts, for dissolution and river run-off are slow relative to the circulation of the seas themselves. Fortunately the much more variable nutrients (like oxides of phosphorus, nitrogen, silicon) and dissolved oxygen involved in life cycles in the sea rarely affect the dynamics of the fluid (although one could argue in favor of a feedback between plankton blooms, bio-convection, and solar heating of the water). What matters most to the buoyancy of the fluid is the concentration of salts and the temperature, both of which are highly variable. The potential density of seawater varies by a couple of percent across most of the oceans and increases perhaps by 0.2% from the surface to bottom at a given latitude and longitude. Small indeed, but it would require some 10^{29} ergs of work to mix the seas to a uniform state, far more than the total kinetic energy of the currents ($\sim 10^{25}$ ergs). This "heavy" stratification combines with "rapid" planetary rotation to define the essential dynamical problems. Except for the tidal potential, most of the driving of the seas occurs by heating, cooling, evaporation, pressure and turbulent stress exerted at their surface. It is traditional for oceanographers to take as given these boundary conditions, and to proceed with models of the circulation. Meteorologists in the same fashion take sea-surface temperature as a given lower boundary condition, over 7/10 of the earth's surface. Both groups have in several areas reached the limit of this approach, however, for the interaction of the two fluids is great and coupled ocean-atmosphere models will be an important part of the near future. Although usually invoked in the study of long-term climatic variation, this interaction also governs the state of a constant ocean and a constant atmosphere.

The heavy stable stratification makes it difficult to ventilate the deeper regions. The Black Sea, for example is so dense at depth that contact with the surface is denied these layers, which remain anoxic and consequently unable to support life. One of the earliest "GFD" laboratory experiments by Count Luigi Ferdinando Marsigli, in 1681, showed how flow through the Bosporus is driven by the contrast between these waters and the yet denser waters of the Mediterranean.

Only with the strongest transfer of buoyancy across the sea surface can the vertical stratification be broken down. To appreciate the range of interesting dynamics that this may lead to, imagine the simple experiment of heating or cooling a beaker of uniformly salty water from above. Heating a simple fluid from above leads to a stable conductive temperature profile, and very little motion (perhaps some shallow cells of flow due to lateral temperature gradients at the top). But the evaporation accompanying the heating of salt water leads to an increased salinity tending of itself to make the surface waters heavy. Who wins? Because the molecular diffusivity of heat is so much greater than that of salt, the hotter, saltier water near the surface can readily give up its thermal, stabilizing buoyancy, leaving its saline, unstable buoyancy to drive deep vertical convection. Separation into distinct species occurs at the molecular level, allowing the salt to "fall out". Just the same phenomenon used to be seen frequently in layers of hot cigarette smoke drifting about a quiet room, with streamers dropping down from the cloud base. The picturesquely named mamma clouds that descend from seemingly stable cloud base may have this same origin; thermal contrasts radiate and diffuse more quickly than water droplets.

Thus, in the eastern Mediterranean Sea, insolation is so great that, with salinity "winning" out over the more mobile heat, dense water is formed. As it passes out toward the Atlantic it is cooled by the Mistral winds whereupon it sinks to the floor of the Mediterranean. These seemingly localized events, very much a quirk of the geometry of geological basins and mountain chains, have a world-wide influence. The water and heat carried to mid-depth by its overburden of salt makes possible further sinking when the water reaches high latitude, where more of its heat is removed by the atmosphere. In this manner the precipitation and river run-off is the sink of salinity, and evaporation and freezing its source. Several of the dominant branches of the general circulation are directly traceable to these sources and sinks. In the atmosphere the complementary circulation of water vapor occurs.

A salinity source just as intricate is the freezing of seawater from above. Once again species are separated, with brine draining through self-determined channels in the relatively fresh ice. The production of water so heavy as to sink right to the sea floor (requiring a surface density of perhaps 1.30 gm/cm^3, in contrast with the normal subtropical surface waters of density ~ 1.26 gm/cm^3) requires extreme atmospheric cooling and the right collaboration of saltiness and coldness. If sea-ice forms it creates some heavy brine just beneath, but also insulates the seawater from any further atmospheric cooling or evaporation.

The net result is that only two sites in the world are known to produce abyssal water: the Greenland-Norwegian Sea in the North Atlantic and the Weddell Sea in the South Atlantic. The entire deep world ocean depends upon this chance confluence of effects for its respiration. You will now appreciate that the evolution of the physical ocean, and life within it must be closely tied to the geological evolution of ocean basins. The closing about 3×10^6 years ago of the Isthmus of Panama, for example, and the separation of Australia from Antarctica some 40×10^6 years ago seem to have caused major, observable changes in the climate and circulation. Such changes are recorded in the faunal distributions preserved in sea-floor sediments; distinct transitions like the onset of ice ages and the sudden cooling of the Southern Ocean have been speculatively linked to the changing topology of the bounding continents.

2. Derivation of equations. The large-scale dynamics of a rotating fluid are best expressed through vorticity relations, because a modified "potential" vorticity is a scalar which is exactly conserved following fluid elements, in the limit of no molecular diffusion. A number of different versions of the equation exist, for different applications. The spherical geometry alone presents major complications, and forces us to make some slightly worrisome approximations. An interesting way to write the momentum equations is

$$\mathbf{u}_t + (\boldsymbol{\zeta} + 2\boldsymbol{\Omega}) \times \mathbf{u} + \nabla\left(\frac{p}{\rho} + 1/2|\mathbf{u}|^2\right) = \mathbf{g} \qquad \text{(mom)}$$

where p is pressure, ρ density, $\mathbf{g}$ is the geopotential gradient (gravity + centrifugal), $\boldsymbol{\zeta} \equiv \nabla \times \mathbf{u}$, $\boldsymbol{\Omega}$ is the polar rotation vector, $\mathbf{u}$ the velocity (u, v, w). Also $\mathbf{x} \equiv (x, y, z)$ are (east, north, up) while r, λ, ϕ are spherical polar coordinates. This shows the local acceleration to be balanced by Bernoulli gradient, the "vortex force" due to the *absolute* (inertial-space) vorticity $\boldsymbol{\zeta} + 2\boldsymbol{\Omega}$, and $\mathbf{g}$.

The vorticity is just the antisymmetric part of the rate-of-strain tensor, or the average angular velocity of "Pooh-sticks" cast on the water, times two. This spin-like quantity must be expected whenever streamlines curve, or when there is shear across streamlines.

Take $\nabla \times$ (mom), equivalent to cross-differentiating the scalar equations to remove pressure:

$$\boldsymbol{\zeta}_t + (\mathbf{u}\cdot\nabla)(\boldsymbol{\zeta} + 2\boldsymbol{\Omega}) = ((\boldsymbol{\zeta} + 2\boldsymbol{\Omega})\cdot\nabla)\,\mathbf{u} + \nabla p \times \nabla\frac{1}{\rho},$$

or

$$\underset{①}{\frac{D(\boldsymbol{\zeta} + 2\boldsymbol{\Omega})}{Dt}} = \underset{②}{[(\boldsymbol{\zeta} + 2\boldsymbol{\Omega})\cdot\nabla]\,\mathbf{u}} + \underset{③}{\nabla p \times \nabla\frac{1}{\rho}}. \qquad (2.1)$$

D/Dt is the rate of change following the fluid. We take the fluid to be incompressible, with basic vertical stratification, $\rho = \bar{\rho}(z) + \rho'(x, y, z, t)$.

$$D\rho/Dt = 0, \qquad D\rho'/Dt = -w\bar{\rho}_z, \qquad \nabla\cdot\mathbf{u} = 0. \qquad \text{(cont)}$$

The absolute vorticity changes by stretching and bending of vortex lines (②) or due to intersections of isobars and isopycnals (③). The effect of ② can be seen by realizing that (if ③ = 0) this same equation is satisfied by vectors representing dyed lines of fluid; call such a "dye-arrow" $\delta\mathbf{S}$. $D\delta\mathbf{S}/Dt$ is the rate of change of $\delta\mathbf{S}$ seen by an observer following the fluid and $(\delta\mathbf{S}\cdot\nabla)\mathbf{u}$ is a vector expressing the rotation and stretching of $\delta\mathbf{S}$ by gradients of $\mathbf{u}$. The classical result is thus that (if ③ = 0) vortex tubes behave just like dye-tubes until dissipation occurs (a vortex tube lies everywhere parallel to $\boldsymbol{\zeta}$, and has cross-section area proportional to $|\boldsymbol{\zeta}|$).

The $p \leftrightarrow \rho$ effect, term ③, is the production of spin by pressure forces. Clearly, if $\rho \equiv$ constant, the center of mass and center of pressure of a small sphere of fluid coincide, and so no spin results. If $\nabla\rho \neq 0$, this is no longer true. Imagine, for example, a uniform horizontal ∇p acting on a vertically stratified density field. The same force acts at each level, z, yet the inertia varies in z, so the acceleration $\mathbf{u}_t$ is greater above than below: this yields horizontal vorticity. The buoyancy twisting may be written

$$-\frac{1}{\rho^2}\nabla p \times \nabla\rho = -\frac{1}{\rho^2}J(p, \rho).$$

Thus, when isolines of p and ρ intersect, the twisting effect is proportional to the number of intersections ("solenoids").

In the geostrophic case, ∇p and $\nabla\rho$ are nearly vertical, implying that the vorticity they can produce is horizontal, with $\nabla p \times \nabla(1/\rho)$ normal to and perpendicular to the z-axis, almost. This helps to explain why the large-scale vertical vorticity ($\lesssim 0\,(10^{-6})\ \mathrm{sec}^{-1}$) remains small, yet dominates the dynamics of the large-scale flow. The horizontal vorticity in these currents may readily exceed f ($\sim 10^{-4}\ \mathrm{sec}^{-1}$), so that one must be wary of departures from geostrophy that could twist horizontal vorticity into vertical.

The most general potential vorticity principle is due to Ertel. Form $\nabla\rho\cdot$(2.1) and manipulate;

$$\frac{D}{Dt}\frac{(\zeta + 2\Omega)\cdot\nabla\rho}{\rho} = 0. \tag{2.2}$$

This resolves the vorticity equation in a direction normal to isopycnal (ρ = const) surfaces. In that direction, $\nabla p \times \nabla\rho \equiv 0$, so that no buoyancy twist is felt. This suggests a local Kelvin's theorem taking a circuit lying in a ρ = const surface. The result is $(\partial/\partial t)\iint(\boldsymbol{\zeta} + 2\Omega)\cdot\mathbf{n}\,dS = 0$ or, roughly, $|\boldsymbol{\zeta}_n + 2\boldsymbol{\Omega}_n|\cdot S$ = const where S is the area enclosed, $(\ )_n$ are normal components. Now S is directly proportional to $\nabla\rho$ (by (cont)), hence we have (2.2).

Geophysical scaling. (2.1) and (2.2) apply to all scales of motion from internal waves, 3-D turbulence on up. They become simple for time-scales $T \gg \Omega^{-1}$, and length scales $L \gg H$, the fluid depth. Then the vertical acceleration is so slight ($\sim W/T \sim (R_0 H/L)U/T$ where $R_0 \equiv U/\Omega L$) that vertical (mom) is hydrostatic, and "traditional"

$$\frac{Dw}{Dt} + p_z/\rho = -g - 2\boldsymbol{\Omega}\times\mathbf{u}\,|_z.$$

Horizontal (mom) is geostrophic,

$$\underset{O(\Omega T)^{-1}}{\mathbf{u}_t} + \underset{O(U/\Omega L)}{\boldsymbol{\zeta}\times\mathbf{u}} + \underset{O(1)}{2\Omega\times\mathbf{u}} = \underset{O(1)}{-\nabla p/\rho} - \underset{O(U/\Omega L)}{\nabla\tfrac{1}{2}|\mathbf{u}|^2|_h}$$

(A subscript h denotes the horizontal compontent.) The geostrophic balance occurs when each of the two Rossby numbers, $(\Omega T)^{-1}$ and $U/\Omega L$ is small.

Now, p acts as a stream-function for $\mathbf{u}$,

$$\mathbf{u}|_h = \frac{\hat{\mathbf{z}} \times \nabla p}{2\Omega_z \rho}(1 + O(R_0)); \qquad (\hat{\mathbf{z}} = \text{vertical unit vector}).$$

$$R_0 \equiv U/fL + (\Omega T)^{-1}.$$

The horizontal divergence $\partial u/\partial x + \partial v/\partial y \sim U/L \cdot R_0 \equiv -\partial w/\partial z$ is very small. $\nabla\rho$ is nearly vertical as is ∇p.

Thence (2.1) becomes the *quasi-geostrophic vorticity equation*, with Cartesian components

$$\text{vertical:} \quad \frac{D}{Dt}\zeta_z - 2\Omega_z \frac{\partial w}{\partial z} = 0 \qquad 2\Omega_z \equiv 2\mathbf{\Omega}|_{\text{vertical}}, \tag{2.3a}$$

$$\text{horizontal:} \quad -2\Omega_z \frac{\partial v}{\partial z} = -\rho^{-1} g \frac{\partial \rho}{\partial x}, \qquad -2\Omega_z \frac{\partial u}{\partial z} = \rho^{-1} g \frac{\partial \rho}{\partial y}. \tag{2.3b}$$

Here

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}_h \cdot \nabla_h = \frac{\partial}{\partial t}(\cdot) + \frac{\nabla p x \nabla(\cdot)}{2\Omega_z \rho} \cdot \hat{\mathbf{z}} = \frac{\partial}{\partial t}(\cdot) + \frac{J(p, (\cdot))}{2\Omega_z \rho},$$

since w is so small. The neglect of $D\zeta_x/Dt$ relative to $2\Omega_z \partial v/\partial z$ is an error $O(\Omega T)^{-1}$.

The dominant effect of spherical geometry is included in this tangent "β-plane" approximation by allowing the local vertical component of $\mathbf{\Omega}$ to vary linearly in y. The Coriolis frequency becomes $2\Omega_z = f = f_0 + \beta y$. See Pedlosky (1979).

The horizontal parts give the thermal-wind balance to $O(R_0)$ (which is thus a vorticity-statement), while the vertical gives a simple balance between stretching of planetary (z-comp) vorticity and production of relative vorticity. No production of vertical vorticity by density twisting (to $O(\Omega^2 L^2/gH)$) occurs.

We now manipulate (2.3) into an equation in a single variable, the *quasi-geostrophic potential vorticity equation*.

$$\frac{D}{Dt}\left[\nabla_h^2 \psi + \rho \frac{\partial}{\partial z}\left(\frac{f_0^2}{\rho N^2}\frac{\partial \psi}{\partial z}\right) + \beta y\right] = 0; \qquad \nabla_h^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{2.4}$$

Here ψ is $p/\rho f_0$, and N the buoyancy frequency, $N^2 \equiv -(g/\rho)(\partial\bar{\rho}/\partial z)$. This equation can be derived from Ertel also, by careful application of geophysical scaling. It is closely related to the *barotropic potential vorticity equation* which applies under similar scaling to an unstratified fluid.