REPRESENTATION THEORY AND HARMONIC ANALYSIS ON SEMISIMPLE LIE GROUPS

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Preface

The five papers which appear in this volume were all written more than fifteen years ago. For various reasons, they have never appeared in print. The papers include the theses of J. Arthur, M. S. Osborne, and W. Schmid; the fundamental paper of R. P. Langlands on the classification of irreducible admissible representations of real reductive Lie groups; and an expository paper by P. Trombi on the work of Harish-Chandra on harmonic analysis on real semisimple Lie groups, in particular, the theory of the Eisenstein integral. Three of these works (Arthur, Osborne, Trombi) have received limited circulation through the *Lecture Notes in Representation Theory* published by the Mathematics Department of the University of Chicago. Langlands' paper and Schmid's thesis were distributed as preprints by the authors. However, because of the basic nature of the material contained therein, the Editors have concluded that these papers deserve much broader exposure.

In a real sense, one could say that most of this material has either appeared elsewhere or has been subsumed in later publications. Nonetheless, these later publications cannot replace the vitality and viewpoint of the original manuscripts. We also include a brief introduction to each paper and a synopsis of the major developments which have occurred in the area covered by each paper. The debt owed by the authors to Harish-Chandra and his work is obvious from the contents of their papers. The debt owed by the Editors is equally real. This volume is dedicated to Harish-Chandra, and the royalties will be donated to the Visiting Members Fund at the Institute for Advanced Study.

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Introduction

Arthur and Trombi. The thesis of Arthur and the paper of Trombi are closely related. They both deal with harmonic analysis on real semisimple Lie groups, in particular, analysis on the Schwartz space of Harish-Chandra. Arthur's thesis is concerned with the image under the Fourier transform of the Schwartz space of a semisimple Lie group of real rank one, while Trombi's paper provides an expository account of the harmonic analysis associated to the decomposition of the Schwartz space under the regular representation.

The Schwartz space $\mathscr{C}(G)$ of a real, connected, semisimple Lie group G was introduced by Harish-Chandra [**HC5**] in connection with his profound analysis of the discrete series $\mathscr{E}_2(G)$. For semisimple Lie groups, $\mathscr{C}(G)$ is intended to serve as the analogue of the classical Schwartz space on $\mathbf{R}^n$. While the space $C_c^\infty(G)$ of complex valued, compactly supported, infinitely differentiable functions on G is suitable for many aspects of harmonic analysis, $\mathscr{C}(G)$ is precisely the space of smooth functions needed for L^2 harmonic analysis on G, in particular, for the Plancherel formula. The topology on $\mathscr{C}(G)$ is defined by a family of seminorms, and, with this topology, $\mathscr{C}(G)$ is a Hausdorff, locally convex, complete topological vector space. The inclusion of $C_c^\infty(G)$ (which is given the standard Schwartz topology) into $\mathscr{C}(G)$ is continuous and the image is dense in $\mathscr{C}(G)$. Moreover, $\mathscr{C}(G) \subset L^2(G)$, and the operation of convolution is continuous from $\mathscr{C}(G) \times \mathscr{C}(G)$ to $\mathscr{C}(G)$. In contrast to the Schwartz space of $\mathbf{R}^n$, functions in $\mathscr{C}(G)$, while rapidly decreasing in $L^2(G)$, are not necessarily in $L^p(G)$ for $1 \le p < 2$.

For (π, H) an irreducible, unitary representation of G, $f \in C_c^\infty(G)$, and dx a Haar measure on G, Harish-Chandra proved that $f \mapsto \pi(f) = \int_G f(x)\pi(x)\,dx$ is of trace class, and that there exists a locally integrable function Θ_π on G which is analytic on the regular set in G such that $\hat{f}(\pi) = \operatorname{trace} \pi(f) = \int_G f(x)\Theta_\pi(x)\,dx$. The function Θ_π is called the character of π. A distribution on G is a continuous linear functional on $C_c^\infty(G)$. Among these are the tempered distributions, those which can be extended to continuous linear functionals on $\mathscr{C}(G)$. The map $f \mapsto \hat{f}(\pi)$ is a distribution on G, and π is said to be a tempered representation of G if this extends to a tempered distribution on G. The tempered representations of a real, semisimple Lie group have been classified by Knapp and Zuckerman [**KZ**]. The collection of irreducible, tempered representations of G is called the tempered spectrum of G, and it is this set of representations which supports the Plancherel measure on $\hat{G}$, the unitary dual of G. From another perspective, it is exactly the tempered spectrum of G which is used to decompose the regular representation of G acting on $L^2(G)$ (or $\mathscr{C}(G)$).

The analysis of the Schwartz space of a real, semisimple Lie group, in particular, the theory of the constant term, wave packets, the Maass-Selberg relations, and the Plancherel formula is contained in three long papers by Harish-Chandra [**HC10, HC11, HC12**]. These papers represent the culmination of Harish-Chandra's work on harmonic analysis on semisimple Lie groups. An illuminating acount of this work is conained in three (shorter) papers by Harish-Chandra [**HC6, HC7, HC8**] which, upon careful reading, yield a remarkable overview of Harish-Chandra's philosophy of harmonic analysis.

The paper of Trombi contained in this volume is an exposition (with many proofs) of the material described above. It is based on lectures given by Harish-Chandra at the Institute for Advanced Study and provides a very readable introduction to Harish-Chandra's work on harmonic analysis. Another version written by Trombi may be found in [**Tr**].

Without reproducing Harish-Chandra's (or Trombi's) papers in their entirety, we will attempt to describe in more detail some of the components in the analysis of the Schwartz space. A parabolic subgroup P of G can be decomposed in the form $P = MAN$, where M is a reductive group, A is the split component of P, and N is the unipotent radical of P. The group P is said to be a cuspidal parabolic subgroup of G if M has a compact Cartan subgroup or, equivalently, $\mathscr{E}_2(M) \neq \emptyset$. The tempered spectrum of G consists of the irreducible components of representations induced (unitarily) from the cuspidal parabolic subgroups of G. Specifically, suppose that $P = MAN$ is a cuspidal parabolic subgroup of G. Let (σ, V) be in $\mathscr{E}_2(M)$, and let ν be an element of $\mathfrak{A}^*$, the real dual of the Lie algebra of A. Then, the map

$$(\sigma, \nu)\colon man \mapsto \exp(i\nu(\log a))\sigma(m)$$

defines a representation of P on V. The (unitary) representation $\pi_P(\sigma, \nu) = \mathrm{Ind}_P^G(\sigma, \nu)$ is a (generalized) principal series representation which depends only on the class of A (i.e., the set of all parabolic subgroups whose split component is A). Subject to certain equivalences, the tempered spectrum of G is made up of the irreducible components of the representations $\pi_P(\sigma, \nu)$. Of course, if $\mathscr{E}_2(G) \neq \emptyset$, then G itself is a cuspidal parabolic subgroup to which the discrete series of G are attached as part of the tempered spectrum.

For $P = MAN$ a parabolic subgroup of G, $f \in \mathscr{C}(G)$, and dn a Haar measure on N, define $f^P(x) = \int_N f(xn)\,dn$, $x \in G$. This integral converges, and f is called a cusp form on G if $f^P = 0$ for all proper parabolic subgroups of G. The subspace of cusp forms on G is denoted by ${}^0\mathscr{C}(G)$. It is a closed subspace of $\mathscr{C}(G)$, and ${}^0\mathscr{C}(G) \neq \{0\}$ if and only if $\mathscr{E}_2(G) \neq \emptyset$ in which case ${}^0\mathscr{C}(G)$ is spanned by the K-finite matrix coefficients of the discrete series of G. If λ denotes the left regular representation of G on $H = L^2(G)$, and ${}^0H$ denotes the closure of ${}^0\mathscr{C}(G)$ in $L^2(G)$, then ${}^0H$ is invariant under λ and $(\lambda, {}^0H)$ is the orthogonal direct sum of the discrete series of G each occurring with infinite multiplicity.

The next task in the analysis of $\mathscr{C}(G)$ is the decomposition of the orthogonal complement of ${}^0\mathscr{C}(G)$ in $\mathscr{C}(G)$. To this end, let $\{A_i\}$ be a set of representatives for the split components of classes of cuspidal parabolic subgroups of G. Each A_i indexes a closed subspace $\mathscr{C}_i(G)$ of $\mathscr{C}(G)$ which is, roughly speaking, generated by the matrix coefficients of the principal series representations $\pi_{P_i}(\sigma, \nu)$ induced from a cuspidal parabolic subgroup $P_i = MA_iN$. This last statement must be properly interpreted since these matrix coefficients are not square integrable (unless $P = G$) and hence not in $\mathscr{C}(G)$ (or $L^2(G)$). It is here that Harish-Chandra introduced the Eisenstein integrals and, with these, wave packets to yield the contribution of the representations $\pi_{P_i}(\sigma, \nu)$ to $\mathscr{C}_i(G)$. The Eisenstein integrals may be regarded as a form of the matrix coefficients of the principal series representations. Wave packets are obtained by integrating the Eisenstein integral against a smooth, compactly supported function in the ν parameter. It is the wave packets which lie in the Schwartz space and, ultimately, yield the decomposition of $\mathscr{C}(G)$.

In Arthur's thesis, he determines the image of $\mathscr{C}(G)$ under the map $f \mapsto \pi(f)$, $f \in \mathscr{C}(G)$, π a tempered representation of G. There is a natural candidate for this image which is denoted $\mathscr{C}(\hat{G})$. The main problem is the proof of surjectivity which requires the full range of harmonic analysis on $\mathscr{C}(G)$ as outlined above. As noted by Harish-Chandra [**HC8**], Arthur's thesis develops an interesting connection between Eisenstein integrals and the reducibility of principal series representations. While Arthur's thesis applies only to groups of real rank one, a number of his results are valid for groups of general rank. In later work [**A1**], Arthur proved similar results for groups of general rank. These results were announced in [**A2**].

The Schwartz space of Harish-Chandra has been generalized by Herb and Wolf to a general semisimple Lie group with infinite center. In a series of papers [**He1, He2, HW1, HW2, HW3, HW4, HW5**], they have carried out a program of harmonic analysis on these groups which is analogous to that of Harish-Chandra for groups with finite center. While Herb and Wolf use much of the full power of Harish-Chandra's work, their investigation requires considerable care because of the difficulties which arise from the existence of an infinite center.

At the present time, there is nothing approaching a classification of the discrete series for p-adic groups. Recent work by Kazhdan and Lusztig [**KL**] and Morris [**Mo**] is a promising start. Despite the lack of knowledge of the discrete series, Harish-Chandra was able to carry out a major portion of his program on the harmonic analysis of the Schwartz space and the Plancherel formula for p-adic groups [**HC9, HC13**]. (The text of [**HC13**] may be found in the *Collected Papers* of Harish-Chandra [**HC14**].) The book of Silberger [**Sl2**] contains many of the details of this work.

Langlands. The idea of studying a class of irreducible representations of a semisimple Lie group which is larger than the collection of unitary representations originated in Harish-Chandra's papers [**HC1**] and [**HC2**] in 1953 and 1954. (Harish-Chandra also attributes this idea to Chevalley.) Harish-Chandra proved a series of powerful general results about admissible representations—for example, two irreducible admissible representations are infinitesimally equivalent if and only if they have the same global character. He also proved the (startlingly concrete) subquotient theorem which states that any irreducible admissible representation must be (infinitesimally equivalent to) a composition factor—that is, a quotient of two subrepresentations—of a principal series attached to a minimal parabolic subgroup. Thus, to study arbitrary abstract representations, one had to look no further than some finite dimensional vector bundles on a compact homogeneous space.

Unfortunately, principal series representations turned out to be extraordinarily complicated objects. Harish-Chandra never realized his hope of studying discrete series representations as subquotients of principal series. By the late 1960s, complete lists of irreducible admissible representations were available for only a few noncompact semisimple groups. In general, it was not even known which principal series representations were themselves irreducible. In 1974, Zhelobenko [**Zh**] showed how to sharpen Harish-Chandra's subquotient theorem into a classification of admissible representations of complex semisimple Lie groups. Specifically, he showed that certain principal series representations have distinguished irreducible quotient representations, and that every irreducible admissible representation appears in an essentially unique way as such a distinguished quotient. This established a bijection between some principal series (which are given by simple explicit parameters) and irreducible admissible representations (all of this for complex groups, of course).

Motivated also by Hirai's work [**Hi**] on $SU(n,1)$, Langlands saw how to extend Zhelobenko's work to all real semisimple Lie groups. The key idea was not to sharpen the subquotient theorem, but rather (in a sense) to weaken it. Langlands considered not only principal series representations induced from the minimal parabolic subgroup, but also series induced from tempered representations on other parabolic subgroups. He showed that any irreducible, admissible representation could be realized in a distinguished way as a quotient of such a "generalized principal series" representation. The tempered representations had already been parameterized in a finite-to-one way by Harish-Chandra. This was refined to a one-to-one parameterization by Knapp and Zuckerman in 1982 [**KZ**]. Thus, the work of Knapp and Zuckerman combined with that of Langlands gives an explicit parameterization of the irreducible admissible representations of a real semisimple Lie group.

Zhelobenko characterized the distinguished quotient representations in several ways, notably, as cokernels of certain intertwining operators and in terms of their restrictions to a maximal compact subgroup K. Langlands generalized only the first of these characterizations. In 1975, Schmid [**Sc1**] showed how to characterize the discrete series in terms of their restrictions to K. Using that work, Vogan [V1], in 1979, showed how to characterize Langlands quotients in terms of restrictions to K. (The proof is not difficult, but it is not illuminating either. There is still no really satisfactory understanding of the equivalence of the analytic and algebraic characterizations.)

Vogan's work suggested the existence of an entirely different way to construct admissible representations, using certain parabolic subalgebras of the complexified Lie algebra in place of real parabolic subgroups. Zuckerman found such a construction; it is described in [**V2**]. Current work on the classification of irreducible unitary representations relies heavily on the interplay of Zuckerman's construction with Langlands. There is a survey in [**V4**].

The constructions of Langlands and Zuckerman have a common generalization in that of Beilinson and Bernstein [**BB**]. Roughly speaking, that paper provides a sort of geometric description of any irreducible representation of the complexified Lie algebra $\mathfrak{G}$, in terms of the algebraic variety of all Borel subalgebras of $\mathfrak{G}$. For admissible representations, this description can be made into an explicit parameterization, which can, in turn, be shown to coincide with that of Langlands and Knapp and Zuckerman. The paper [**HMSW**] of Heckt, Milicic, Schmid, and Wolf treats these relationships in depth.

Langlands' formulation as sketched above can be carried over to p-adic semisimple groups without difficulty. The corresponding results were proved by Borel and Wallach in [**BW**], and Silberger in [**Si1**]. Unfortunately, there is not yet a classification of tempered representations in the p-adic case like that of Harish-Chandra and Knapp and Zuckerman in the real case. Thus, the p-adic result is more in the nature of an abstract structure theorem than a classification. Nevertheless, it has been used by Tadic in [**Ta**] to describe all the irreducible unitary representations of $\mathrm{GL}(n)$ over a p-adic field in terms of the discrete series of $\mathrm{GL}(m)$, $m \leq n$.

Using Harish-Chandra's finite-to-one parameterization of tempered representations, Langlands obtains (from his realization of admissible representations in terms of generalized principal series) a finite-to-one parameterization of irreducible admissible representations. He formulates this parameterization in terms of maps of the Weil group $W_{\mathbf{R}}$ of $\mathbf{R}$ into a group called the *L-group of G*. This formulation, with some modification, also makes sense in the p-adic case, and is currently an area of intensive investigation.

Osborne. Because it is much less well known than some of the other papers in this volume, we will describe Osborne's thesis at some length before turning to other developments.

Osborne was motivated by the fixed point theorem of Atiyah and Bott, and by their application of it to deduce the Weyl character formula from the Bott-Borel-Weil theorem. The Atiyah-Bott theorem concerns an elliptic complex $\mathscr{E}$ on a compact manifold M, and an automorphism f of M and $\mathscr{E}$. (The prototypical example is the deRham complex on any compact M; any automorphism of M

automatically extends to an automorphism of this complex.) Then f defines automorphisms f^i of the cohomology $H^i(M, \mathcal{E})$. Since these spaces are finite dimensional, it makes sense to form the alternating sum $\sum(-1)^i \operatorname{tr}(f^i)$. Under an appropriate hypothesis on f (that it should have only simple fixed points), the theorem computes this alternating sum of traces as a sum of local terms at the fixed points of f.

Suppose now that G is a semisimple Lie group and Γ is a discrete subgroup with $\Gamma\backslash G$ compact. The representation of G on $L^2(\Gamma\backslash G)$ has a distribution character. One can ask for a formula which computes this character in terms of the local data at fixed points. The fixed points of G acting on $\Gamma\backslash G$ are never simple, so nothing like the Atiyah-Bott result can apply directly. What is desired is an analogous result. An element of G has a fixed point on $\Gamma\backslash G$ if and only if it is conjugate to an element of Γ, so one is asking to compute the character in terms of data on the conjugacy classes of elements of Γ. This is exactly what is accomplished by the Selberg trace formula.

At first glance, the Selberg trace formula does not look like a very good analogue of the Atiyah-Bott fixed point formula, because the character of $L^2(\Gamma\backslash G)$ does not appear to be an alternating sum of traces on cohomology groups. Osborne's most difficult technical goal was therefore to realize such distribution characters as alternating sums. He did this "one representation at a time," then patched the results together using some more or less formal methods. More precisely, let $P = MAN$ be a minimal parabolic subgroup of G, (π, H) an irreducible unitary representation of G, and Θ_π the character of π. Write H^∞ for the space of smooth vectors of π. This is a representation π^∞ of the Lie algebra $\mathfrak{G}$ of G. The complex computing the Lie algebra cohomology of $\mathfrak{N}$ with coefficients in π^∞ has as its jth term $\operatorname{Hom}(\bigwedge^j \mathfrak{N}, H^\infty)$. The group MA acts on this complex and on its cohomology. Roughly speaking, Osborne conjectured that, for g a regular element of G belonging to MA, $\Theta_\pi(g)$ is equal to the alternating sum of the traces of g on the cohomology groups of this complex, divided by the alternating sum of the traces of g on $\bigwedge^i \mathfrak{N}$. (For H finite-dimensional, this fact is a consequence of elementary linear algebra.) For technical reasons, one has to require g to lie in a certain Weyl chamber for this to have a chance to be true in general. In this form, Osborne proved the result for $\mathrm{SL}(2, \mathbf{R})$ and $\mathrm{SL}(2, \mathbf{C})$, and for "most" principal series for $\mathrm{SO}(n, 1)$.

Finally, Osborne constructed a complex of vector bundles on $\Gamma\backslash G$, with fiber $(\bigwedge^j \mathfrak{N})^*$. Very roughly speaking, the alternating sum of the traces of a regular element g of MA on the cohomology groups of this complex is (assuming the conjecture in the previous paragraph) the value at g of the character of $L^2(\Gamma\backslash G)$. Combining this with the Selberg trace formula, one gets a reasonable analogue of the Atiyah-Bott theorem.

To the best of our knowledge, Osborne's results have not been used in later work on $L^2(\Gamma\backslash G)$. (Exactly the same ideas are at the foundation of almost all work on the occurrence of elliptic representations (such as discrete series). The point is that then one can identify the complexes that arise with Dolbeault complexes on compact, complex manifolds, so a lot is known about them. This line of research predates Osborne's thesis, however.) What has survived is the conjectural relationship between Lie algebra cohomology and character formulas. A version of this relationship was established in the late 1970s by Hecht and Schmid, and appears in **[HS2]**. This paper reflects enormous advances in the understanding of $\mathfrak{N}$-homology (or cohomology) over the previous decade. At the beginning of these developments was a result of Casselman and Osborne **[CO]** relating the $\mathfrak{N}$-cohomology with coefficients in a representation to the infinitesimal character of that representation. What finally emerged was this. It had been understood since the 1950s that representations of a semisimple group should be attached in some way to characters of Cartan subgroups, and, in discussing the papers of Langlands and Schmid elsewhere in this Introduction, we see a bit of how that was accomplished. Osborne's conjecture helped to establish that $\mathfrak{N}$-cohomology is a kind of inverse of that process; it starts with a representation and produces characters of Cartan subgroups. The culmination of all these ideas is the Beilinson-Bernstein theory, which brings all constructions and their inverses into one geometric setting.

Osborne's thesis also contains nice accounts of a number of more or less well-known (but rarely proved) technical results. For example, there is a complete proof that the space of smooth vectors in an induced representation is (sometimes) the space of smooth sections of the corresponding vector bundle. All of this material makes splendid reading for a graduate student.

Schmid. In **[HC4]** and **[HC5]**, Harish-Chandra gave a complete parameterization of the discrete series of a semisimple Lie group G. In particular, for each parameter λ, he gave an explicit formula for part of the character Θ_λ of the corresponding representation π_λ, and an algorithm for computing the remaining values of Θ_λ. He also computed the formal degrees of the discrete series and proved enough about their matrix coefficients to complete his proof of the Plancherel formula for G.

In Harish-Chandra's work, he did not give explicit realizations of the representations π_λ. In this respect, his work paralleled that of H. Weyl on the characters of compact Lie groups. In 1955, A. Borel and A. Weil constructed a realization of the representations of a compact Lie group K on holomorphic sections of appropriate line bundles over K/T, T a maximal torus in K. In **[HC3]**, for noncompact semisimple G, Harish-Chandra found all the discrete series which admit analogous realizations in sections of holomorphic vector bundles. These are the holomorphic discrete series for G. However, except in a few special cases, such discrete series are quite rare.

In **[Bt]**, Bott generalized the Borel-Weil realization of representations of compact Lie groups by considering higher Dolbeault cohomology with coefficients in a holomorphic vector bundle. (The zero cohomology is just the space of holomorphic sections.) He obtained no new representations, but rather new constructions of old ones. Soon thereafter, B. Kostant and R. P. Langlands realized that Bott's construction made sense for some noncompact semisimple Lie groups as well. Specifically, suppose that T is a compact Cartan subgroup of G. Any character μ of T defines a homogeneous complex line bundle $\mathscr{L}_\mu$ on G/T. A choice of positive roots R^+ of T in the complexified Lie algebra of G gives rise to a complex structure on G/T, and $\mathscr{L}_\mu$ acquires the structure of a holomorhic line bundle. We may therefore speak of the Dolbeault cohomology $H^i(G/T; R^+; \mathscr{L}_\mu)$ of G/T with coefficients in $\mathscr{L}_\mu$. This is a vector space on which G acts. It may be infinite-dimensional (since G/T is noncompact). That it is a Hausdorff, topological vector space requires a difficult proof; one must show that a certain $\overline{\partial}$ operator has closed range.

The simplest example is $G = \mathrm{SL}(2, \mathbf{R})$. Here, the homogeneous space G/T may be identified with the unit disk, and $H^0(G/T; \mathscr{L}_\mu)$ is essentially the space of holomorphic functions on the unit disk. Such a space is too big to be given a reasonable Hilbert space structure; one must impose growth conditions at the boundary. In general, the representations $H^i(G/T; R^+; \mathscr{L}_\mu)$ are too big to be isomorphic to discrete series representations, but they may be infinitesimally equivalent to such representations. Kostant and Langlands offered two conjectures. The first is a generalization of the Borel-Weil theorem: *each discrete series representation π_λ is infinitesimally equivalent to a certain Dolbeault cohomology representation $H^s(G/T; R^+; \mathscr{L}_\mu)$.* Here, s depends only on G, and R^+ and μ depend in a simple way on Harish-Chandra's parameter λ. The second conjecture generalizes Bott's theorem. One replaces Dolbeault cohomology by certain spaces of L^2-harmonic forms $\mathbf{H}^i(G/T; R^+; \mathscr{L}_\mu)$. (When G/T is compact, $\mathbf{H}^i$ coincides with H^i by Hodge theory.) The conjecture is that each nonzero $\mathbf{H}^i(G/T; R^+; \mathscr{L}_\mu)$ is isomorphic to a certain discrete series π_λ. We note that only the second of these two conjectures was precisely formulated.

Schmid's thesis is concerned with (formulating and) proving the first of these conjectures. He introduced powerful algebro-geometric techniques for studying the Dolbeault cohomology representations and their restrictions to a maximal compact subgroup. In particular, he proved that Blattner's conjectured multiplicity formula for the restriction of π_λ to a maximal compact subgroup K was usually valid, at least for that $H^s(G/T; R^+; \mathscr{L}_\mu)$ which was a conjectured model for π_λ.

Schmid's work provided concrete models for most discrete series representations—the first new ones of any generality since Harish-Chandra's work on the holomorphic discrete series ten years earlier. Over the following few years, Schmid went on to prove completely both of the conjectures of Kostant and Langlands, and (with H. Hecht) the conjecture of Blattner (see [**Sc2**] and [**HS1**]). This wealth of information has proved very useful in a variety of problems in harmonic analysis in which discrete series arise, especially in the theory of automorphic forms.

Once Schmid had established techniques for studying Dolbeault cohomology on G/T, it was natural to consider cohomology on larger classes of complex homogeneous spaces. That idea has several descendents today, each with its own claim to the throne. One such is the theory of cohomological induction (see [**V3**] and the references contained therein). This now allows one to construct many wonderful families of unitary representations that, like the discrete series, are more or less inaccessible by parabolic induction. A more direct descendent is the work of Rawnsley, Schmid, and Wolf [**RSW**] on the direct geometric construction of unitary structures on Dolbeault cohomology representations. This has proven to be an almost intractable problem. The best general results say only that the Dolbeault cohomology construction is well-behaved on admissible representations. Finally, the work of Beilinson and Bernstein on D-modules and representation theory ([**BB**, **HMSW**]) is very much in the spirit of Schmid's thesis. Recent work of Hecht and Taylor [**HT**] makes an explicit connection.

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