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Lie Groups, Convex Cones, and Semigroups

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AND

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Preface

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he wrote and kindly permitted us to use. We have used plain TeX on VAXes at the Technische Hochschule Darmstadt, Louisiana State University, Tulane University, and the University of New Orleans, and the version ST-TeX written by KLAUS GUNTERMANN of THD for the Atari 1040ST. The help we received from the System Manager for Computing at the Fachbereich Mathematik der Technischen Hochschule Darmstadt, KLAUS-THOMAS SCHLEICHER, through these years has been invaluable. We also thank GUDRUN SCHUMM of THD for her assistance in managing the laser printer and WOLFGANG WEIKEL for sharing with us the programs he wrote for editing, file management, transmission, and PC-operation. MICHAEL MISLOVE of Tulane University indefatigably assisted Hofmann during his sabbatical with all computer related problems. He also introduced him to the Chemical Engineering Department of Tulane University. The first TeX program at Tulane was mounted and operated with the assistance of ANIL MENAWAT and MICHAEL HERMANN on the computer of this department. First printouts were done at the University of New Orleans through the generosity of its Computer Science Department and the patient help of WILLIAM A. GREENE. Also, NEAL STOLTZFUS at Louisiana State University was very helpful in the management of file transfer between Tulane University and LSU.

Dr. MARTIN GILCHRIST of Oxford University Press has organized the publication of this book and the preparation of our final files for printing at the facilities of Oxford University Press. The Copy Editors and the Assistant Editor have carefully scrutinized a hard copy. The elimination of numerous typographical errors is due to their effort. American spelling came most naturally to all of us. We are grateful that our publisher allowed us to leave this orthography where it deviates from the British one and that, in addition, he permitted certain aberrations from the format standards of the series whose modifications would have upset our pagination.

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J.H.
K.H.H.
J.D.L.

Darmstadt and Baton Rouge,
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Introduction

This book focuses on a new aspect of the theory of Lie groups and Lie algebras, namely, the consideration of semigroups in Lie groups. The systematic development of a Lie theory of semigroups is motivated by their recent emergence in different contexts. Notably, they appeared at certain points in geometric control theory and in the theory of causal structures in mathematical physics. Beyond that, it is becoming increasingly clear that the broader perspective of considering not just the analytic subgroups of a Lie group, but the appropriate subsemigroups as well, leads to a fuller and richer theory of the original Lie group itself. Hence it is appropriate to consider this work as a new branch of Lie group theory, too.

Historically, the rudiments of a Lie theory of semigroups can be detected in Sophus Lie's own work. If the language had been available at the time, he could have expressed one of his basic results in this sentence: *The infinitesimal generators of a local semigroup of local differential transformations of some euclidean domain is a convex cone in a vector space.* However, in Lie's own diction, any family of transformations of a set which is closed under composition is called a group, irrespective of the presence of an identity or the existence of inverses. In fact, Lie attempted for a while to deduce the existence of an identity and the inverse from his other assumptions until the first concrete examples credited to Friedrich Engel showed the futility of such efforts. The word semigroup belongs to the vocabulary of the 20th century. There were some initiatives to deal with Lie semigroups such as the attempts by Einar Hille in the early nineteen-fifties which also made their way into the the book by Hille and Phillips, and the studies of Charles Loewner on certain types of subsemigroups of Lie groups extending into the nineteen-sixties. By and large, these efforts remained somewhat isolated and they were either aborted or ignored, or both.

It may appear surprising that further systematic investigations of semigroups in Lie groups were not undertaken. However, the technical obstacles are considerable, and incisive results of both generality and mathematical depth did not quickly appear on the horizon. Indeed the traditional tools of Lie theory were inadequate for dealing with the new theory. One needed to introduce the geometry of convex sets; certain techniques and ideas from geometric control theory also turned out to be quite useful. Additionally, specialized methods appropriate to the circumstances had to be introduced and developed. Only in very recent years has a significant body of results begun to emerge. A notable example is the investigation of invariant cones in Lie algebras due to KOSTANT, VINBERG, PANEITZ, and OL'SHANSKIĬ.

Besides having to cope with these technical obstacles and a historical scarcity of external stimuli, the Lie theory of semigroups often found itself in a no-man's land. Semigroup theorists have tended to regard subsemigroups of groups as a branch of group theory, while group theorists have concentrated on the subgroups of a group and paid scant attention to the subsemigroups. This has been a serious oversight. The Lie theory of semigroups is an interesting, rich, and applicable branch of study. This book is a first attempt to present a systematic Lie theory of semigroups. Numerous examples are also included. Apart from some background theory which we felt we should provide, its contents are of recent origin.

Although a strong motivation for this book is the development of a useful and applicable Lie theory of semigroups, major lines of applications will be deferred to later volumes. Nevertheless, let us briefly illustrate the emergence of semigroups in the context of geometric control. Let Ω denote a set of smooth vector fields on a manifold M; each vector field $X \in \Omega$ determines a local flow, say, $t \mapsto F_t(X)$ which associates with any $m \in M$ the unique largest solution $x: I \to M$, $I \subseteq \mathbf{R}$, $x(t) = F_t(X)\cdot m$ of the initial value problem

$$(1) \qquad \dot{x}(t) = X\big(x(t)\big), \qquad x(0) = m.$$

In order to keep this illustration short let us assume that each $X \in \Omega$ is complete in the sense that (1) has a solution for all $t \geq 0$ and all $m \in M$. Then $F_t: M \to M$ is a smooth self-map of M for each $t \geq 0$. Now let us consider a function $c: [0, T] \to \Omega$, called a steering function, which is piecewise constant. Typically, we are thinking that such a function selects for each interval of constancy $[t_{k-1}, t_k[$ a vector field $X_k = c(t)$ with $t \in [t_{k-1}, t_k[$, and that each jump at time t_k, $k = 0, 1, \ldots, n$, $t_n = T$ represents a sudden switch which redirects the trajectory from one vector field X_k to the next vector field X_{k+1}. A solution of the initial value problem

$$(2) \qquad \dot{x}(t) = c(t)\big(x(t)\big), \qquad x(0) = m$$

is then a concatenation of solutions

$$\dot{x}_k(t) = X_k\big(x_k(t)\big), \quad x_k(t_{k-1}) = x_{k-1}(t_{k-1}) \qquad \text{for} \qquad t_{k-1} \leq t \leq t_k$$

with $t_0 = 0$, $t_n = T$ and $x_1(0) = m$. In the terms of the flows we have

$$(3) \quad x(t) = F_{t-t_{k-1}}(X_k)F_{t_{k-2}}(X_{k-1}) \cdots F_{t_1}(X_1)(m) \qquad \text{for} \qquad t_{k-1} \leq t < t_k,$$

and for $k = 1, \ldots, n-1$. A typical problem in systems theory is to determine the points of M which are attainable from a point $m \in M$ by traversing one of these trajectories obtained from the system Ω and all piecewise constant steering functions. This problem is then clearly tantamount to the question of which elements of M are in the orbit $S\cdot m$ of m where S is the semigroup generated (under composition) by all $F_t(X)$, $t \geq 0$, $X \in \Omega$.

If, in particular, M is the underlying manifold of a Lie group G and all vector fields $X \in \Omega$ are left-invariant, that is, if Ω is a subset of the Lie algebra $\mathfrak{g}$ of all left-invariant vector fields, then $F_t(X) = \exp t{\cdot}X$, and $S = S(\Omega)$ is the semigroup of all elements $\exp t_1{\cdot}X_1 \exp t_2{\cdot}X_2 \cdots \exp t_n{\cdot}X_n$ in G. Indeed, in several decisive parts of the general theory of semigroups in Lie groups, the framework of geometric control theory will organize our procedures.

The simplest special case, of course, is that of the the group $G = \mathbb{R}^n$, in which case we may also write $\Omega \subseteq \mathfrak{g} = \mathbb{R}^n$. Then S is the additive semigroup $\sum\{\mathbb{R}^+{\cdot}X\colon X \in \Omega\}$ which is stable under multiplication by non-negative scalars. The example of the group $\mathbb{R}^n$ and its subsemigroups demonstrates right away that a treatment of *all* subsemigroups is unreasonable. It is clearly those semigroups which are generated by the rays $\mathbb{R}^+{\cdot}x \subseteq S$ that are amenable to a general theory. Therefore, the idea of an infinitesimally generated semigroup in a Lie group will be crucial, and the whole theory will eventually have to concentrate on them.

The title of the book features the word *semigroup.* This word means different things to different people. For many a *functional analyst*, a semigroup is a strongly continuous one-parameter family of bounded operators on a Banach or a Hilbert space. Then semigroup theory is the description of the infinitesimal generation of these semigroups by unbounded closed operators and, as a branch of ergodic theory, the study of their behavior for large parameter values. For the *algebraist*, semigroup theory is a vast body of structure theory involving ideals, equivalence and order relations, idempotents, and generalized inverses, in short, a theory blending algebra with order. In *topological semigroup theory* a prevalent image is that of a compact semigroup, whose one outstanding feature is a minimal ideal full of idempotents.

None of these images is pertinent in the context of this book. While one-parameter semigroups do indeed play a crucial role here, they are only the raw material from which a distinctly multiparameter theory is built. We deal primarily, albeit not exclusively, with subsemigroups of Lie groups. In algebraic semigroup theory one has a whole subtheory characterizing semigroups which are embeddable in groups, but there is an inclination to consider those semigroups of little semigroup theoretical interest thereafter. And as far as compact semigroups are concerned, as soon as they are contained in a group, they are themselves compact groups: thus they instantaneously become the topic of classical group theory.

A helpful preliminary idea of the type of semigroup which shall occupy us in this book is that of a closed convex cone in $\mathbb{R}^n$. In fact, the theory of such cones is basic and thus needs much initial attention. Thus in the kind of Lie theory we have to deal with, *geometry of convex sets* is added to *linear algebra, calculus, global analysis, and topology.*

To highlight by comparison and contrast the main concerns of this book, let us recall the basic components of the theory of Lie groups. Traditional Lie group theory deals, firstly, with the *infinitesimal* structure theory of Lie groups and their subgroups. The basic tool is linear algebra applied to Lie algebras. Secondly, it deals with the *local* structure theory by means of the exponential function in which an amazing wealth of information is encoded. This approach to Lie theory uses analysis on open sets in $\mathbb{R}^n$, that is, calculus of several real variables. Finally, one has to deal with the *global* structure of Lie groups by means of global differential geometry, analysis on manifolds, and algebraic topology. The structure of a Lie group is uniquely determined by two data, one infinitesimal, one global: its Lie algebra and its universal covering homomorphism whose kernel is the fundamental group.

These features of classical Lie group theory roughly correspond to the lines of its historic development: Sophus Lie, its creator, developed the idea of infinitesimal transformations of a local group of transformations and thus the concept of infinitesimal generators of a (local) Lie group, and he invented for their analysis the type of algebra which now bears his name. The tools of analysis available to him in the later decades of the nineteenth century allowed him to develop an infinitesimal and local theory constrained to open domains of euclidean space. He was able to inspect examples of global groups. Indeed the groups of geometry provided an ample supply even at that time. A systematic treatment of the global theory, however, required the tools of topology and global differential geometry that soon became available through the work of Henri Poincaré, Georg Frobenius, Elie Cartan, Hermann Weyl, Heinz Hopf, and numerous other mathematicians.

Even more distinctly than in the case of groups, the Lie theory of semigroups falls into at least three parts:

1) *the infinitesimal theory,*
2) *the local theory,* and
3) *the global theory.*

The *infinitesimal theory* deals with those subsets of Lie algebras which are the exact infinitesimal generating sets of (local) subsemigroups of Lie groups. The tools belong to classical Lie algebra theory *and* to the theory of convex bodies and cones. The *local theory* has the task of characterizing local infinitesimally generated semigroups in a Lie group and must lead to the Fundamental Theorem in the sense of Lie. Historically, the direction of constructing, for a given Lie algebra, a (local) group with the given algebra as tangent set at the origin was hard. The corresponding task is much harder in the case of semigroups. Finally, the *global theory*, perhaps the least developed portion of the Lie theory of semigroups at this time, is concerned with the structure of infinitesimally generated subsemigroups of Lie groups and, in particular, with the global variant of the Fundamental Theorem: If a set of infinitesimal generators is given which is already known to be the tangent set of a local semigroup, is it always the tangent set of a (global) subsemigroup of a Lie group? Since one discovers very quickly that the answer to this question is negative, it converts immediately to the hard question: Which local infinitesimal generating sets are global? More accurately: Given a subset W in the Lie algebra $\mathfrak{g}$ of a Lie group G such that W is the precise set of tangent vectors at the origin of some local subsemigroup in G, what are necessary and sufficient conditions that there is a (global!) subsemigroup S in G whose set $\mathbf{L}(S)$ of infinitesimal generators is exactly W?

These outlines were drawn following the contours of classical Lie group theory. Yet in developing a Lie theory of semigroups, one recognizes very quickly that the analogy with Lie group theory does not carry very far at all. This may account for the apparent fact that most previous attempts at a Lie theory of semigroups were abandoned sooner or later.

However, there are more pieces to this puzzle. Up to this point we have considered subsemigroups of Lie groups as the proper territory of a Lie semigroup theory. But in looking back at classical vistas of Lie *group* theory, we find other views on a possible Lie *semigroup* theory just as natural: Given a topological semigroup, say, on a manifold with or without boundary, introduce a suitable differentiable structure and study the objects so obtained in the abstract! Clarify to which extent the semigroups arising in this fashion can be embedded into Lie groups—at least locally in the vicinity of an identity! Even on the historical plane, this viewpoint is natural because it is close to Sophus Lie's original vantage point. As a consequence we have to face a fourth aspect which we might call

4) *the abstract Lie semigroup and embedding theory.*

We shall address this issue, too, and find that our original attitude is justified. Any reasonably defined Lie semigroup can be embedded into a Lie group at least locally on a neighborhood of the identity. This is reassuring. Yet many interesting problems remain open in the entire theory.

Let us now look at the lay-out of the book and highlight some of its results. We begin with a fundamental fact which was, in a way, known to Sophus Lie, which is explicitly and clearly stated in Loewner's work, and appears in some form in a variety of contexts where semigroups in Lie groups have been considered. Let us consider a subsemigroup S of a Lie group G and its exponential function $\exp\colon \mathbf{L}(G) \to G$. In order to skip technicalities—which eventually we shall have to face squarely—we shall assume for now that S is closed. We define

$$\mathbf{L}(S) = \{X \in \mathbf{L}(G) \colon \exp \mathbb{R}^+ \cdot X \subseteq S\}.$$

Then the set $W = \mathbf{L}(S)$ is topologically closed; it is stable under addition and is closed under multiplication by non-negative scalars in the finite dimensional real vector space $\mathbf{L}(S)$. We shall call such sets *cones* or, more frequently, *wedges*. Indeed, W will contain a largest vector subspace $W \cap -W$ called the *edge*, which in general is not zero and plays a crucial role in the overall theory; this is one reason why we prefer the terminology of "wedge" (another is that the word "cone" sometimes refers to not necessarily convex objects). But in the literature the terminology "cone" is so prevalent that we have decided to use the two terms synonymously. Those wedges, whose edge is zero, will be called *pointed cones*.

We have to prepare adequate background information on wedges. Chapter I serves this purpose. We deal with the structure theory of wedges in two ways: Firstly in terms of duality, secondly in terms of geometry. If W is a wedge in a finite dimensional real vector space L then *the dual wedge* W^* is the set of all functionals ω in the dual $\widehat{L}$ of L satisfying $\langle\omega, x\rangle \geq 0$ for all $x \in W$. Frequently we can realize the dual wedge in L itself; this happens as soon as we are given, through natural circumstances, a nondegenerate bilinear symmetric form B on L (for instance, a scalar product, or a Cartan-Killing form) in which case we have $W^* = \{y \in L:\ B(x, y) \geq 0 \text{ for all } x \in W\}$. We are particularly interested in wedges with interior points; this means that the dual is a pointed cone. The geometry of such wedges is determined by the structure of their boundary. A helpful concept is that of a face. A special type of face is particularly suited for duality, namely, the concept of an *exposed face.* We shall analyse this concept in terms of duality in great detail; at this point it suffices to understand its geometrical meaning. A support hyperplane of a wedge with inner points is a hyperplane meeting the wedge non-trivially and bounding a closed half-space containing the wedge. An exposed face is the intersection of a support hyperplane with the wedge (or the whole wedge). A non-zero point on the wedge is an *exposed point* if it lies on a one-dimensional exposed face. Unfortunately, if W has a non-zero edge then W has no exposed points. We need focus on the next best object, namely, those points $x \in W$ for which $x + (W \cap -W)$ is an exposed face. These points are called E^1-points. If W is pointed, then the E^1-points are exactly the exposed points. Of even greater importance are the so-called C^1-points. A point x is a *C^1-point* of a wedge with inner points if there is one and only one support hyperplane of the wedge containing x. In an *arbitrary* wedge, a point is called a C^1-point if it is a C^1-point of W in the vector space $W - W$ in which W does have inner points. There is a close correspondence between the C^1-points of W and the E^1-points of W^* which is encapsuled in the so-called Transgression Theorem (I.2.35).

Two types of wedges are particularly familiar: polyhedral and Lorentzian ones. A wedge is *polyhedral* if it is the intersection of finitely many closed half-spaces; it is *Lorentzian* if it is one half of the solid double cone defined by a Lorentzian form. A boundary point of a polyhedral cone is either a C^1-point or a E^1-point or neither of the two; each non-zero boundary point of a Lorentzian cone is both a C^1- and an E^1-point. This is the starting point of a small theory of *round* cones which we shall develop because we need it later in the infinitesimal Lie theory of semigroups.

There are several results in the first Chapter which are applied later. Some of them are of independent interest. The first is a classical theorem of MAZUR'S saying that *the set of C^1-points $C^1(W)$ of a convex closed set W with inner points in a* separable *Banach space is a dense G_δ in the boundary ∂W*. This result is non-trivial even in the case of finite dimensional vector spaces. Since the C^1-points play a central role, we give a complete proof of the Density Theorem. In the finite dimensional situation this implies a dual result due to STRASZEWICZ which says that *a finite dimensional cone W is the closed additive span of $E^1(W)$*, the set of its $E^1(W)$-points.

A further tool of crucial importance is a theorem on ordinary differential equations due to BONY and BREZIS. It deals with the invariance of closed sets under flows. For a brief discussion let A denote a closed subset of a finite dimensional vector space and let U be an open subset containing A. Let X be a vector field on U satisfying a local Lipschitz condition. Then X defines a local flow $(t,u) \mapsto F_t(u)$ on U via $F_t(u) = x(t)$, where x is a solution of the initial value problem $\dot{x}(t) = X(x(t))$, $x(0) = u$. We say that A is invariant under the flow F if $F_t(a) \in A$ for all $a \in A$ and all $t \geq 0$ such that $F_t(a)$ is defined. The point is that the invariance of A under F can be expressed in terms of X and the geometry of A. For this purpose we need a definition.

1. Definition. A *subtangent vector* of a subset W of a topological vector space L at a point $w \in L$ is a vector x such that there are elements w_n and numbers r_n such that

(i) $\lim w_n = w$, $w_n \in W$,

(ii) $0 \leq r_n \in \mathbb{R}$,

(iii) $x = \lim r_n(w_n - w)$.

The set of all subtangent vectors of W at w will be denoted $L_w(W)$. If $w = 0$, we shall write $L(W)$ instead of $L_0(W)$. We shall call x a *tangent vector of* W *at* w if both x and $-x$ are subtangent vectors. The set of tangent vectors of W at w, denoted $T_w(W)$, therefore is $L_w(W) \cap -L_w(W)$.

(It is no problem to define a subtangent vector x of a subset W of a differentiable manifold M at a point $w \in M$. Under such circumstanceas, x is an element of the tangent space $T(M)_w$ of M at w.)

2. Theorem. *If A is a closed subset of a finite dimensional vector space L and U an open subset containing A, and if X is a vector field on U satisfying a local Lipschitz condition, then A is invariant under the local flow defined by X if and only if*

$$X(a) \in L_a(A) \qquad \text{for all} \qquad a \in A.$$

■

(The theorem, by the way, remains intact for closed subsets and vector fields on differentiable manifolds.)

Our primary applications of this theorem concerns wedges in finite dimensional vector spaces and their invariance under linear flows. In fact we shall prove the following Invariance Theorem for Wedges:

3. Theorem. *Let W be a generating wedge in a finite dimensional vector space L and $X: L \to L$ a linear map. Then the following conditions are equivalent:*

(1) $e^{t \cdot X} W \subseteq W$ *for all* $t \in \mathbb{R}^+$ *(respectively, for all* $T \in \mathbb{R}$*).*

(2) $X(w) \in L_w(W)$ *(respectively,* $X(w) \in T_w(W)$*) for all* $w \in W$.

(3) $X(c) \in L_c(W)$ *(respectively,* $X(c) \in T_c(W)$*) for all* $c \in C^1(W)$.

(4) $X(e) \in L_e(W)$ *(respectively,* $X(e) \in T_e(W)$*) for all* $e \in E^1(W)$. ■

The equivalence of (1) and (2) is a rather immediate consequence of the Bony–Brezis Theorem; the equivalence of (3) with these conditions requires the Mazur Density Theorem—but that is not enough; the proof further requires a result which we shall call the Confinement Theorem which says that a flow confined in a wedge by the tangent hyperplanes in all C^1-points cannot seep out through the corners. A duality argument will establish the equivalence of (4) with the other conditions.

The reading of Chapter I will not demand many prerequisites. We have some cause to formulate the theory of wedges without restriction of the dimension as far as this generality can be sustained painlessly. *However*, for a first reading little is lost to the reader who prefers to restrict attention to the finite dimensional case. In this situation, most of the material is elementary, yet not trivial. For the Mazur Density Theorem, some background in functional analysis is required such as familiarity with Baire category arguments. The Bony–Brezis Theorem demands some knowledge on ordinary differential equations. Given all of this, however, the first chapter is self-contained.

The second and third chapter are devoted to *the infinitesimal Lie theory of semigroups*: they deal with those wedges in finite dimensional Lie algebras which arise as tangent sets of semigroups and local semigroups in Lie groups.

Let S denote again a closed subsemigroup of a finite dimensional real Lie group G and set $W = \mathbf{L}(S)$. It is not hard to verify that $\mathbf{L}(S) = L_0(\exp^{-1}(S))$. We have observed above that W is a wedge in the Lie algebra $L = \mathbf{L}(G)$. More generally, if U is an open neighborhood of the identity in G and $S \subseteq U$ a subset satisfying $SS \cap U \subseteq S$, then the set $W = L_0(\exp^{-1}(S))$ of subtangent vectors of the pull-back of S under the exponential function is always a wedge. But how does such a wedge relate to the *Lie algebra* structure of L? The answer is not part of the classical repertory. It was discovered independently by OL'SHANSKIĬ and by HOFMANN and LAWSON, that *every subtangent wedge W of a local semigroup in a Lie algebra satisfies*

$$(*) \qquad e^{\operatorname{ad} x} W = W \qquad \textit{for all} \qquad x \in W \cap -W,$$

where $\operatorname{ad} x : L \to L$ as usual is the inner derivation of L given by $(\operatorname{ad} x)(y) = [x, y]$. Recall, in this context, that every derivation D of L gives rise to an automorphsm e^D of the Lie algebra L. It is clear that every pointed cone trivially satisfies condition $(*)$, and that this condition implies that the edge $W \cap -W$ of the wedge is a Lie subalgebra.

All of this, once understood, is comparatively easy to establish. It is much harder to accomplish Sophus Lie's Fundamental Theorem for a local theory of semigroups by showing that, conversely, *if a wedge W in the Lie algebra $\mathbf{L}(G)$ of a Lie group satisfies $(*)$, then there is an open neighborhood U of the identity in G and a subset $S \subseteq U$ with $SS \cap U \subseteq S$ such that $W = L_0(\exp^{-1} S)$.* This result is the core of the entire *local Lie theory of semigroups* and was established by the authors. A whole chapter is devoted to a proof of this fact, namely, Chapter IV.

However, this carries us beyond the infinitesimal theory, but it amply justifies the terminology of calling *Lie wedge* any wedge in a Lie algebra satisfying condition $(*)$ above.

Condition $(*)$ has a drawback. An infinitesimal equivalent of the semigroup property should be expressed in terms of the Lie bracket alone and not with the aid of a convergent power series. An immediate corollary of Theorem 3 is the key to such a reformulation.

4. Corollary. *If W is a wedge in a finite dimensional Lie algebra L, then for any element $y \in L$, the following conditions are equivalent:*

(1) $e^{\operatorname{ad} y} W = W$.

(2) $[w, y] \in T_w(W)$ *for all* $w \in W$.

(3) $[c, y] \in T_c(W)$ *for all* $c \in C^1(W)$.

(4) $[e, y] \in T_e(W)$ *for all* $e \in E^1(W)$. ∎

This allows us to conclude

5. Theorem. (The Characterization Theorem for Lie wedges) *For a wedge W in a finite dimensional Lie algebra L, the following conditions are equivalent:*

(1) *W is a Lie wedge.*

(2) $[w, W \cap -W] \subseteq T_w(W)$ *for all* $w \in W$.

(3) $[c, W \cap -W] \subseteq T_c(W)$ *for all* $c \in C^1(W)$.

(4) $[e, W \cap -W] \subseteq T_e(W)$ *for all* $e \in E^1(W)$. ∎

Let us pause to inspect condition (2) for the elements $w \in W \cap -W$ of the edge. For each such element, $T_w(W) = W \cap -W$. Thus we note once more that the edge of a Lie wedge is a Lie subalgebra. In particular if W happens to be a vector space—which is the case precisely when $W = -W = W \cap -W$—then Theorem 5 expresses nothing else but the fact that a vector space is a Lie wedge if and only if it is a Lie subalgebra.

The Lie wedge condition $(*)$ is of the type of an invariance condition which suggest the concept of an invariant wedge which will engage much of our energy in this book.

6. Definition. A wedge W in a Lie algebra L is *invariant* if

$$(**) \qquad e^{\operatorname{ad} x} W = W \qquad \text{for all} \qquad x \in L.$$

From Corollary 4 we obtain

7. Theorem. (The Characterization Theorem for Invariant Wedges—Elementary Version) *For a wedge W in a Lie algebra L, the following conditions are equivalent:*

(1) *W is invariant.*

(2) $[w, L] \subseteq T_w(W)$ *for all* $w \in W$.

(3) $[c, L] \subseteq T_c(W)$ *for all* $c \in C^1(W)$.

(4) $[e, L] \subseteq T_e(W)$ *for all* $e \in E^1(W)$. ∎

If, in condition (2) we consider once again only elements $w \in W \cap -W$ in the edge, we find that

$$[W \cap -W, L] \subseteq W \cap -W,$$

that is, that the edge of an invariant wedge is always an ideal. More trivially, $(**)$ directly implies that $W - W$ is an ideal, too. In particular, a vector space is an invariant wedge if and only if it is an ideal. Thus in a very immediate sense, Lie wedges generalize subalgebras, invariant wedges generalize ideals.

One of the familiar properties of Lie group theory is that local Lie subgroups of a Lie group are ruled smoothly by local one parameter subsemigroups. Sometimes this is expressed as the theorem of the "existence of canonical coordinates of the first kind". It is one of the unpleasant surprises that even in the simplest examples, nice infinitesimally generated local subsemigroups in Lie groups fail to be ruled by local one-parameter semigroups. Such examples exist in 3-dimensional Lie groups such as the Heisenberg group, the group $\mathbb{R}^2 \rtimes \mathrm{SO}(2)$ of euclidean motions of the plane, in $\mathrm{Sl}(2,\mathbb{R})$ and in $\mathrm{SO}(3)$. We shall discuss such examples explicitly and in detail in various parts of the book. The one-parameter subsemigroups of a closed subsemigroup S of G are $t \mapsto \exp t{\cdot}X$ with $X \in \mathbf{L}(S)$. Even if S is algebraically generated by $\exp \mathbf{L}(S)$ and $\mathbf{L}(S)$ is the exact set of subtangent vectors of $\exp^{-1}(S)$ at 0, in general there are arbitrarily small elements of the form $\exp X_1 \cdots \exp X_n$ with $X_1, \dots, X_n \in \mathbf{L}(S)$ which *cannot* be written in the form $\exp Y$ with some $Y \in \mathbf{L}(S)$. If S happens to be a group, this is always the case. This deficiency in the Lie theory of semigroups is a fact of life we have to live with whether we like it or not.

However, if W is an invariant wedge in the Lie algebra $\mathbf{L}(G)$, then there is always an open neighborhood B of 0 in $\mathbf{L}(G)$ which is mapped homeomorphically onto an identity neighborhood U in G under the exponential function such that $(\exp(B\cap W))^2 \cap U \subseteq \exp(B\cap W)$. In other words, *invariant wedges always define local subsemigroups which are ruled smoothly by one-parameter subsemigroups.* However, not every subsemigroup which is locally ruled by one parameter subsemigroups is invariant. Hence the hunt is on for those Lie wedges which belong to local semigroups behaving more like local groups. The Campbell–Hausdorff multiplication which is give near 0 by $X * Y = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X[X,Y]] + \frac{1}{12}[Y,[Y,X]] + \cdots$ in terms of a universally defined infinite series in Lie monomials allows us a definition in terms the Lie algebra L:

8. Definition. A wedge W in a finite dimensional real Lie algebra L is a *Lie semialgebra* if there is an open neighborhood B of 0 such that the series for $X * Y$ converges for all $(X,Y) \in B \times B$ and such that

$$(\dagger) \qquad (B \cap W) * (B \cap W) \subseteq W.$$

There is minute variance in terminology among the authors here. Because the idea was first introduced by Hofmann, in his papers Lawson has called a Lie semialgebra also a *Hofmann wedge.* In this book we shall use the term Lie semialgebra.