TRANSFORM METHODS

With Applications To Engineering And Operations Research

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PREFACE

Laplace transforms and Z-transforms are essential mathematical equipment required in engineering, operations research, and the applied sciences. Concurrent with their usefulness is a level of sophistication which requires serious study beyond the stage of manipulative skills if one is to realize their full potential. Frequently a student acquires knowledge of transform methods as an aside in a course which is not devoted to transforms. Under such circumstances, the student may learn a set of recipes to solve certain standard problems but may be more apt to make serious errors than to benefit from the knowledge. This book was developed from material for an undergraduate course on transform methods which has been taught for a number of years at the Department of Industrial and Systems Engineering at the University of Florida. This course is taken by all undergraduate students, usually in their junior year, and by most entering graduate students. The objective of the course is primarily to give the student complete facility in using transforms, as well as a thorough understanding of the underlying mathematics, the reasoning, and the possible pitfalls. Applications are shown for motivation, but are not the primary emphasis. Our students have ample occasion to apply transforms later in their program, in such courses as: methods of operations research, inventory theory, reliability engineering, production control, and forecasting. In teaching our undergraduate course, we found that none of the available texts met our objectives. While there are many books which deal to some extent with the Laplace transform, they are either at a too abstract mathematical level, or their orientation is strongly towards electrical engineering and circuit theory. Books devoted to the Z-transform are scarce, and whatever coverage is given to this transform in books on related subjects is usually dominated by the sampled-data viewpoint of electrical engineers. There is, of course, a historical reason for this, in that electrical engineers were the first to make extensive use of both transform methods.

This book is written for junior or senior students of engineering, operations research, and applied mathematics. Prerequisites are the basic courses in differential and integral calculus that are usually covered during the freshman and sophomore years in engineering and science curricula. The organization of the material, and the development of methods and ideas from first principles, make the book suitable for self study and as a reference.

One of the purposes of the book is to provide a balanced treatment of both transform methods and to exploit the similarities between the methods. A basic treatment of complex variables without which transform methods cannot be understood is provided in Chapter 1. The structure of Chapters 2 and 3 for the Laplace transform is paralleled in Chapters 5 and 6 for the Z-transform. However, the motivation for using transforms given in Sections 2-1 and 2-2 and for the partial fraction expansion method given in Section 3-3 is not repeated in Chapters 5 and 6. For a study of Z-transforms alone, the student would cover Chapters 1 and 5 through 7. In the latter case, it is useful to also read sections 2-1, 2-2, and 3-3.

There are probably as many good pedagogical reasons for intermixing the theory and the applications as for keeping them separate. In this book I have purposely provided a separation of theory and applications. One reason for this is that I prefer to approach the material this way. In my years as student and practicing engineer I have been many times frustrated because I knew how to do it, but not why, and I believe strongly that the student should learn to walk before he can run. Another reason for the separation is that it provides greater flexibility and more ways in which the book can be used. For example, Chapters 1, 2, 3, 5, and 6 can be followed without the requirement that the student be familiar with the jargon of any particular engineering discipline. This sequence of chapters may be used in a course as it would typically be taught by a mathematics department, and may be open to anyone with background in calculus only. Chapters 4 and 7 cover applications from a variety of disciplines. Whereas the theory has been developed in depth and in great detail, the applications are broad. One objective here is to show the versatility of transform methods and the spectrum of problems that can be solved by them. A new dimension in the applications chapters is that they involve physical principles and modeling of natural phenomena. I have attempted to keep the discussion of these as elementary as possible in order to make it accessible to students not having backgrounds in the specialty. The sections in these chapters are self-contained and the instructor can therefore freely select and amplify applications in accordance with the specialty of his course and the emphasis he wishes to give it. Furthermore, particularly with students who have had some prior exposure to transforms, an instructor may wish to start out his course in the applications chapters, and then refer back to the operational rules and the theory as they are needed.

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CONTENTS

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INTRODUCTION TO COMPLEX VARIABLES

Functions in the transform domain of the Laplace transform or the Z-transform are, treated generally, functions of a complex variable. It is therefore important in the study and application of these transforms to have some knowledge of the theory of complex variables. The more advanced applications of complex variables to transform calculus are the contour integration in the complex plane and the theory of residues. Such advanced methods are outside the scope of this book, knowledge of the elements of complex variable theory, combined with some manipulative skills, will suffice. This chapter deals briefly with those necessary elements, but it assumes no prior knowledge of them. The reader who wishes to study the subject more thoroughly should consult a text devoted to complex variables such as Churchill [4].

1-1 Need for Complex Numbers

The system of *complex numbers* forms an extension of the system of *real numbers*. The need for this extension becomes apparent when one attempts to solve algebraic equations. The simplest example is that of the quadratic equation

$$z^2 + 2bz + c = 0 \tag{1-1}$$

where z is a variable and b and c are real coefficients. The values of z that satisfy this equation are known as the *roots* of the equation. A quadratic equation has exactly two roots. We denote the roots of (1-1) by z_1 and z_2.

The quadratic function of z can be written in *factored form* as

$$z^2 + 2bz + c = (z - z_1)(z - z_2)$$

Clearly, this function is 0 when $z = z_1$ or $z = z_2$. Solution of (1-1) yields

$$z_1 = -b + \sqrt{b^2 - c}, \qquad z_2 = -b - \sqrt{b^2 - c} \tag{1-2}$$

and these solutions are real numbers provided that $b^2 \geq c$ For $b^2 < c$, the radicands in (1-2) are negative, in this case (1-1) does not have a solution in the sense of real numbers. If we wish to solve equation (1-1) in all cases, then we must agree to the existence of a new mathematical object z that is not a real number. This new object is called a *complex number.*

1-2 Definitions

We can write a complex number z in terms of two real numbers x and y as

$$z = x + y\sqrt{-1}$$

but so far, this expression is meaningless, because $\sqrt{-1}$ has not yet been defined. However, we see that there is a correspondence between the complex number and two real numbers. This is used as the basis for the following axiomatic definition.

A complex number z is an ordered pair (x, y) of real numbers x and y, subject to certain rules of operation.

$$z = (x, y) \tag{1-3}$$

The complex number $(x, 0)$ is defined to be equal to the real number x:

$$x = (x, 0) \tag{1-4}$$

The system of real numbers is thus imbedded in the system of complex numbers. The rules of operation involve the equality, sum, and product of two complex numbers. Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. The rules are:

$$z_1 = z_2 \text{ if and only if } x_1 = x_2 \text{ and } y_1 = y_2 \tag{1-5}$$

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2) \tag{1-6}$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1) \tag{1-7}$$

Equations (1-3) through (1-7) form a complete definition of complex numbers. All other properties and rules of operation are logical consequences of these definitions.

1-3 Algebra of Complex Numbers

It is convenient to give names to the elements of the pair (x, y). Thus, x is called the *real part* of z and y the *imaginary part* of z. This is written as

$$x = \text{Re } z, \qquad y = \text{Im } z$$

By definition, a pair of the form $(x, 0)$ is a real number. A pair of the form $(0, y)$ is called a *pure imaginary number*. Application of the rule of addition (1-6) lets us write

$$(x, y) = (x, 0) + (0, y) \tag{1-8}$$

Hence every complex number is the sum of a real number and a pure imaginary number. The product of the real number a and the complex number (x, y) follows from (1-7) as

$$a(x, y) = (a, 0)(x, y) = (ax, ay)$$

and with this result, (1-8) is decomposed further into

$$(x, y) = x(1, 0) + y(0, 1) \tag{1-9}$$

An ordered pair of numbers is also called a two-dimensional *vector*. The reader who is familiar with vector algebra will recognize that (1-9) is analogous to representing a vector (x, y) as a combination of the unit vectors $(1, 0)$ and $(0, 1)$. In vector algebra, each unit vector is assigned a special symbol. Similarly, the complex number $(0, 1)$ is given the symbol i and is called the *imaginary unit*.

$$(0, 1) = i$$

No special symbol is necessary for $(1, 0)$, since it is the real number 1. Now (1-9) becomes

$$(x, y) = x + yi = x + iy \tag{1-10}$$

The nth power of the complex number z is written z^n and is defined as the product of n factors equal to z. By application of (1-7), we find that the powers of the imaginary unit are

$$i^2 = (0, 1)(0, 1) = (-1, 0) = -1$$
$$i^3 = (0, 1)(-1, 0) = (0, -1) = -i$$
$$i^4 = i^2 i^2 = (-1)(-1) = 1$$

and it follows by induction that

$$i^{2n} = (-1)^n, \quad n = 1, 2, 3, \ldots$$
$$i^{2n+1} = (-1)^n i, \quad n = 0, 1, 2, \ldots \tag{1-11}$$

With this result at hand, addition, subtraction, multiplication, and division are conveniently carried out using complex numbers in the form $x + iy$ and employing the rules of the algebra of real numbers. The symbol i is treated as if it were a real constant, and powers of i are reduced in accordance with (1-11). This procedure is justified by definitions (1-6) and (1-7).

Example 1-1: Find the third power of the complex number $z = 1 + i$. Apply the binomial expansion to obtain

$$z^3 = (1 + i)^3 = 1 + 3i + 3i^2 + i^3$$

Next, substitute $i^2 = -1$, $i^3 = -i$.

$$z^3 = 1 + 3i - 3 - i$$

Finally, collect the real and imaginary terms.

$$z^3 = -2 + 2i$$

The operations of subtraction and division are the inverses of addition and multiplication. Subtraction of $z = x + iy$ is equivalent to addition of $-z = -x - iy$. Division by $z \neq 0$ is equivalent to multiplication by $z^{-1} = 1/z$. To determine the inverse of z we let $z^{-1} = u + iv$. It must hold that

$$zz^{-1} = (x + iy)(u + iv) = 1 + i0$$

Carrying out the multiplication and equating both real and imaginary parts, we obtain two simultaneous equations for u and v:

$$xu - yv = 1, \qquad yu + xv = 0$$

The solution of these equations is

$$u = \frac{x}{x^2 + y^2}, \qquad v = \frac{-y}{x^2 + y^2}$$

and thus we find that the inverse of z is

$$\frac{1}{z} = \frac{1}{x + iy} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2} \tag{1-12}$$

Note that the same result is obtained when both numerator and denominator are multiplied by $x - iy$:

$$\frac{1}{x + iy} = \frac{x - iy}{(x + iy)(x - iy)}$$

This multiplication has the effect of changing the complex denominator into a real denominator, because

$$(x + iy)(x - iy) = x^2 + y^2 \tag{1-13}$$

and the result (1-12) is thus at hand. The latter method is useful for carrying out the division of complex numbers.

Example 1-2: Find the complex number

$$z = \frac{(1 + i)(1 + 2i)}{2 - i}$$

First, perform the multiplication in the numerator.

$$z = \frac{1 + 2i + i + 2i^2}{2 - i} = \frac{-1 + 3i}{2 - i}$$

Next, multiply numerator and denominator by $2 + i$ to obtain

$$z = \frac{(-1 + 3i)(2 + i)}{(2 - i)(2 + i)} = \frac{-5 + 5i}{2^2 + 1^2} = -1 + i$$

We now summarize the rules of algebraic operations with complex numbers.

Addition $\quad (x_1 + iy_1) + (x_2 + iy_2) = x_1 + x_2 + i(y_1 + y_2)$

Subtraction $\quad (x_1 + iy_1) - (x_2 + iy_2) = x_1 - x_2 + i(y_1 - y_2)$

Multiplication $\quad (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$

Division $\quad \dfrac{x_1 + iy_1}{x_2 + iy_2} = \dfrac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\dfrac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}$

From the definitions, the reader can easily prove that the algebraic operations obey the commutative, associative, and distributive laws governing the operations with real numbers. These laws are:

Commutative Laws $\quad z_1 + z_2 = z_2 + z_1$

$\quad z_1z_2 = z_2z_1$

Associative Laws $\quad (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

$\quad (z_1z_2)z_3 = z_1(z_2z_3)$

Distributive Law $\quad z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

1-4 Complex Conjugates

The number $x - iy$ is called the *complex conjugate*, or briefly the *conjugate*, of the number $z = x + iy$ and is denoted by the symbol z^* (or sometimes $\bar{z}$). Thus,

$$\text{Re } z^* = \text{Re } z, \qquad \text{Im } z^* = -\text{Im } z$$

The designation *complex conjugates* is also used to refer to both z and z^*, as a pair. Pairs of complex conjugate numbers arise naturally as the roots of polynomial equations with real coefficients, as is seen in Eq. (1-2) for $b^2 < c$. If such an equation has a complex root, that root is always accompanied by a complex conjugate root. The product of z and z^* is always a real number. That property was employed in (1-13) to accomplish division of complex numbers. The following properties of complex numbers and their conjugates are easily deduced:

$$z + z^* = 2x = 2 \text{ Re } z$$

$$z - z^* = 2iy = 2i \text{ Im } z$$

$$zz^* = x^2 + y^2 = (\text{Re } z)^2 + (\text{Im } z)^2$$

$$(z_1 + z_2)^* = z_1^* + z_2^*$$

$$(z_1z_2)^* = z_1^*z_2^*$$

$$\left(\frac{1}{z}\right)^* = \frac{1}{z^*}$$

$$(z^*)^* = z$$

We note especially that the sum of two complex conjugate numbers is a real number and that the difference of two complex conjugate numbers is a pure imaginary number. The conjugate of the conjugate is the original number.

1-5 Geometric Representation

We have already mentioned that the ordered pair (x, y) is also a two-dimensional vector. It is therefore natural and very useful to give the complex number z a geometrical interpretation. This is accomplished by letting z be a point in the plane with coordinates x and y measured in a rectangular cartesian coordinate system. Figure 1-1 shows the complex number $(1, 2)$ as a point in the xy plane. When the xy plane is used to represent complex numbers, it is called the *complex plane*. Frequently, the complex plane is labeled by the letter used to represent the complex variable; thus we find the designations *z plane*, *s plane*, and others. In the complex plane, the real numbers are located on the x axis, therefore called the *real axis*. The pure imaginary numbers are located on the y axis, called the *imaginary axis*.

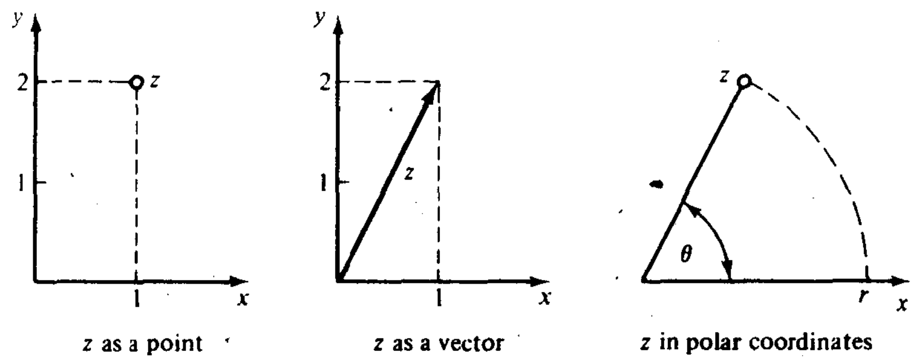

Figure 1-1. Representations in the complex plane

In another useful geometrical interpretation, z is a vector from the origin $(0, 0)$ to the point (x, y). Vectors having the same length and the same direction are defined to be equal. Thus, by translation, z is also the vector from an arbitrary point (a, b) to the point $(a + x, b + y)$. Figure 1-1 shows the vector representation of the number $(1, 2)$. The rules for addition and subtraction of complex numbers are the same as the rules for addition and subtraction of vectors. Sums and differences therefore follow the parallelogram law of vector combination, as shown in Fig. 1-2. In accordance with the two interpretations, it is common to refer to the complex number z freely as the *point* z or the *vector* z.

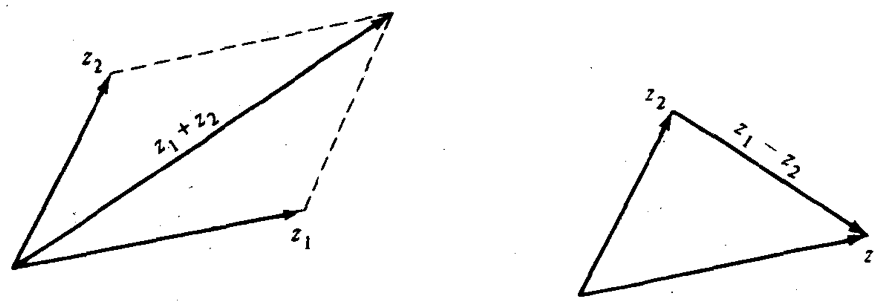

Figure 1-2. Addition and subtraction of complex numbers

1-6 Polar Form

The point z in the complex plane has the polar coordinates (r, θ). In these coordinates, $r \geq 0$ is the distance from the origin to the point z, or the length of the vector z, and θ is the angle between the positive x axis and the vector z, measured in the positive (counterclockwise) direction (see Fig. 1-1). The relationship between the polar coordinates and the cartesian coordinates is

$$x = r \cos \theta, \qquad y = r \sin \theta \tag{1-14}$$

Substituting this in $z = x + iy$, we obtain the *polar form* of the complex number z,

$$z = r(\cos \theta + i \sin \theta) \tag{1-15}$$

The distance r is the *absolute value*, also called the *modulus*, of z, and is denoted $|z|$. The angle θ is called the *argument* of z and is written arg z. Note that this angle can be represented in infinitely many ways. This follows from (1-15) and the periodicity of the sine and cosine functions:

$$\cos(\theta \pm 2k\pi) = \cos \theta, \qquad k = 0, 1, 2, \ldots$$
$$\sin(\theta \pm 2k\pi) = \sin \theta, \qquad k = 0, 1, 2, \ldots$$

Thus, whenever the angle θ in (1-15) is increased or decreased by an integral multiple of 2π, the same number z is obtained. In order to avoid difficulties, we decide beforehand upon some interval of length 2π from which θ is to be taken, say $0 \leq \theta < 2\pi$ or $-\pi < \theta \leq \pi$. That interval is called the *principal range* of arg z, and values in that interval are the *principal values* of arg z, written Arg z. To each vector z then corresponds a unique principal value Arg z, and the relation between arg z and Arg z is given by arg $z =$ Arg $z \pm 2k\pi$, $k = 0, 1, 2, \ldots$.

The polar coordinates r and θ expressed in terms of x and y are

$$r = |z| = \sqrt{x^2 + y^2} \tag{1-16}$$

and, if $r \neq 0$

$$\theta = \text{Arg } z = \arctan \frac{y}{x} \tag{1-17}$$