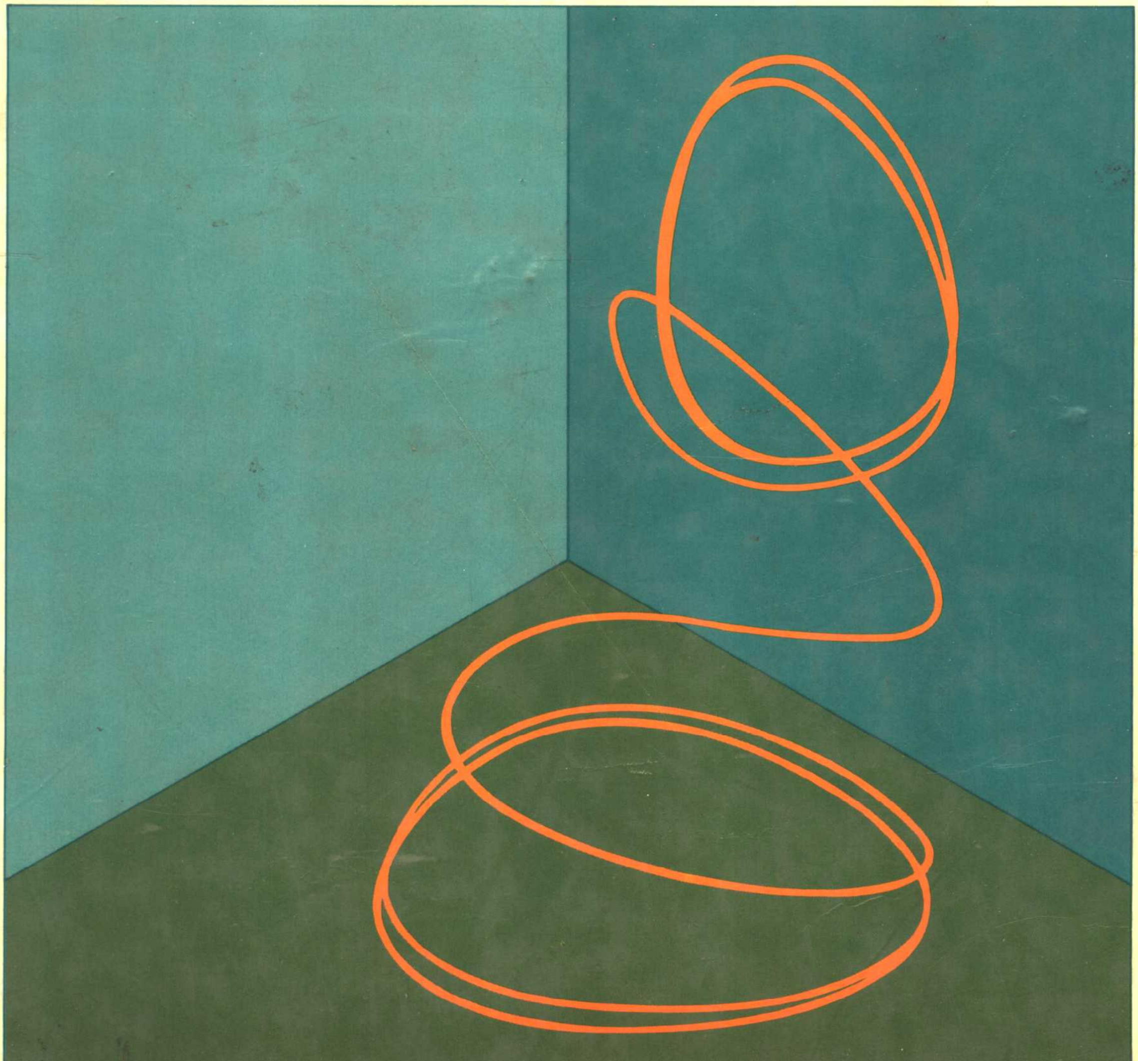
INTRODUCTION TO DIFFERENTIAL EQUATIONS

ODE, PDE, AND SERIES



RICHARD E. WILLIAMSON

Introduction to DIFFERENTIAL EQUATIONS

ODE, PDE, and Series

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The cover illustration depicts the solution trajectory of the Lotka-Volterra system

$$\dot{x} = x(3 - y)$$

$$\dot{y} = y(x + z - 3)$$

$$\dot{z} = z(2 - y),$$

for three species, with initial values $x(0) = .001$, $y(0) = 3$, $z(0) = 2$.

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Preface

We can trace the roots of most branches of mathematics to problems in geometry, in number theory, or in applied mathematics. For the study of differential equations, these roots lie primarily in geometry and in the application of mathematics to the physical world. Contact with these sources of inspiration has continued right up to the present and has been responsible for most of the subject's vitality ever since the time of Newton. This book depends very much on applied mathematics for motivation, but it is after all a mathematics text, so the fundamental mathematical ideas, including their geometric interpretation, appear before the applications. Encouraging a geometric point of view toward the subject is particularly important, because finding a neat solution formula is a relatively rare, though welcome, occurrence in practice, and it is very often the geometric properties of solutions that unify our view of them.

Students who use the book should have had the standard introduction to calculus in one dimension and to the elements of parametrized curves in two and three dimensions. It's necessary to know what a partial derivative is only for Chapters 7 and 8. The tradition of using an introduction to differential equations as an occasion for drill on integration technique has been somewhat slighted in this book by avoidance of exercises that are particularly demanding in that way; this is a personal preference, based on the idea that there are more important matters to emphasize in an introductory course. However, there are many routine exercises of the simpler sort. The essential ideas from linear algebra are included in the text, but the treatment is very much geared to differential equations, and the algebra topics are not treated in any greater generality than necessary.

The inclusion of numerical techniques is intended partly to take advantage of the availability of computers with graphics capability for displaying the geometry of solutions. Graphs of solutions are often difficult or impossible to draw using only the traditional methods of calculus, but simple numerical and graphical analysis can give very satisfying results, results that can be particularly satisfying in the case of nonlinear equations. There is no attempt here to discuss ways of making highly accurate approximations; accuracy enough to produce good pictures has been the primary aim, though the results are usually better than they need to be for that purpose. Furthermore, numerical approximations appear along with each type of equation rather than being lumped into a single chapter. In this way a geometric feeling for the subject can be maintained throughout the book without restriction to examples that happen to be analytically simple. Nevertheless, the book is designed so that anyone who chooses to ignore the numerical work completely will still have a coherent text.

Many students come to a first course in differential equations having had some experience with linear equations of order one and with second-order constant coefficient equations. In spite of this, it is not feasible to skip that material altogether in an introductory differential equations course, so there is the question of what new things to do at this stage. One response is to take up more applications, which is done in this book. An additional one is to introduce Wronskians and order reduction for second-order linear equations. Wronskians appear later in this book, after students have had a brief look at linear independence in at least two slightly different contexts. What is done instead is to develop the Green's functions for the second-order constant coefficient equations. The resulting formulas are easy to apply and even easy to remember. Differential operator notation is very useful here, and its introduction at this point is a help later when it comes to using it to solve simple linear systems. Using operators fairly systematically allays the suspicion some students get that, while manipulating operators is convenient, there is something slightly lowbrow about it.

Chapter 5 on general techniques for constant coefficient linear systems begins with the essential matrix algebra. The method described for computing e^{tA} is quite efficient whether it is carried out by hand or by symbolic calculator, its main limitation being that it requires a prior computation of the eigenvalues of A. Additional results on matrix algebra are collected at the end of the chapter.

The Picard existence theorem appears very late in the book, though questions about existence and uniqueness are raised, and answered concretely, when they come up naturally at earlier points.

The chapter on partial differential equations is fairly traditional except for the use, due to Euler, of exponential solutions to motivate the classification of second-order linear operators. The chapter on infinite series will be viewed by many people as a sort of appendix, so it is placed at the end of the book to be used at an instructor's discretion; it is designed to be used either for review or as an introduction to the topic for students who have not studied it systematically. An introduction to power series solutions of differential equations is at the end of the series chapter.

The routine exercises are intended to be used for technical drill and to underline general principles. Other exercises are there to broaden a student's perspective. An occasional exercise is starred to indicate that it is more difficult than most of the others. All of Chapter 7 and Section 4 of Chapter 9 are significantly more theoretical than the rest of the book.

While I was preparing the manuscript I had very valuable help at Dartmouth from Joe Bonin, Peter Bushell, Bob Christy, Nancy French, Peter Holland, Paul Latiolais, and Randy Weis, and I am very grateful to them. My thanks go also to Bob Sickles of Prentice-Hall for his sensitive approach to the entire project and to Paul Spencer for his careful work in seeing the book through production. Most of the book has been used in a preliminary version at Dartmouth and I want to thank the students who used it; their helpful comments have worked their way into the book at many points.

Richard E. Williamson

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1

First-Order Differential Equations

1 SOLUTIONS

1A General Formulas

In many applications of mathematics it is possible to establish an equation relating an unknown function $y = y(x)$ to one or more of its derivatives. Simple examples are

$$y'(x) + y(x) = x \quad \text{and} \quad y''(x) - y(x) = 0,$$

or alternatively,

$$\frac{dy}{dx} + y = x \quad \text{and} \quad \frac{d^2y}{dx^2} - y = 0,$$

where the equation is to hold for x in some interval $a < x < b$, perhaps $-\infty < x < \infty$. Such an equation is called a **differential equation.** In this chapter we will deal primarily with **first-order** differential equations, that is, those in which there occur derivatives of order one at most. Of the two previous equations, only the first is of first order; the other has order 2.

Equations are often meant to be solved, and differential equations are no exception. A **solution** of a differential equation is a function $y = y(x)$, which, when substituted along with its derivatives into the differential equation, satisfies the equation for all x in some specified interval that we will call the **domain** of the solution. In practice, there are sometimes elementary formulas for solutions, although by no means always. It is important to bear in mind that, while a strictly algebraic equation like $x^2 + 3x + 2 = 0$ has numbers for solutions, a differential equation has functions for solutions. In the following examples we will simply verify that certain formulas provide solutions; derivation of the solutions will be discussed later.

EXAMPLE 1 The differential equation

$$y' + y = x$$

has the solution $y = x - 1$ for $-\infty < x < \infty$. The reason is that $y' = 1$, so

$$y' + y = 1 + (x - 1) = x.$$

More generally, the same differential equation has the solution

$$y = x - 1 + ce^{-x}$$

for each choice of the constant c, for then

$$y' + y = (1 - ce^{-x}) + (x - 1 + ce^{-x}) = x.$$

Thus there are actually infinitely many distinct solutions, and the solution $y = x - 1$ can be obtained from the general formula above by setting $c = 0$. Figure 1(a) shows the graphs of the solutions corresponding to $c = 0$, $c = 0.15$, and $c = 0.25$. Notice that these graphs converge together since the term ce^{-x}, by which they differ, tends to 0 as $x \to +\infty$.

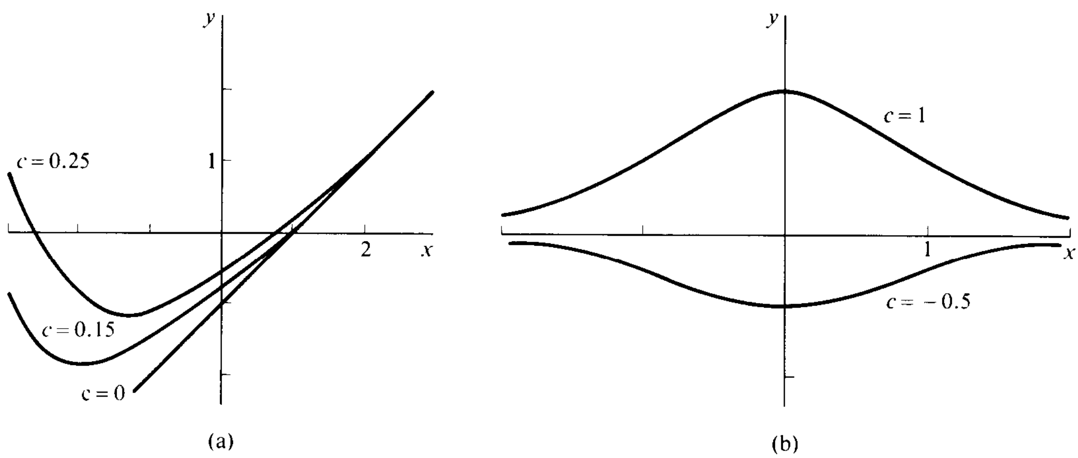


Figure 1

EXAMPLE 2 The differential equation

$$y' + xy = 0$$

has the solutions $y = ce^{-x^2/2}$, one for each value of c, since

$$y' + xy = (-cxe^{-x^2/2}) + x(ce^{-x^2/2}) = 0.$$

Figure 1(b) shows the graphs of the solutions corresponding to $c = -0.5$ and $c = 1$. For $c = 0$, we get $y = 0$ for $-\infty < x < \infty$, and the graph of this solution coincides with the x-axis.

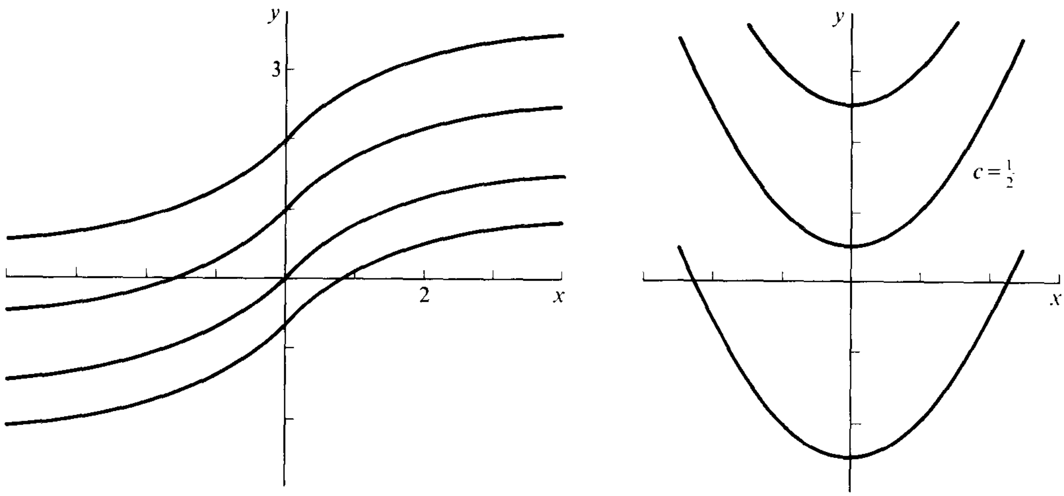
EXAMPLE 3 Suppose $f(x)$ defines a continuous function on some interval $a \leq x \leq b$. The differential equation

$$y' = f(x)$$

then has the solution

$$y = \int_a^x f(t)\,dt + c$$

for each choice of the constant c. In this example, the role of the constant c distinguishes the various solutions from one another by making a constant vertical shift in the graph. Figure 2(a) shows some examples for the case in



(a) (b)

Figure 2

which $f(x) = 1/(1 + x^2)$. Choosing $a = 0$, we have

$$y = \int_0^x \frac{dt}{1 + t^2} + c$$
$$= \arctan x + c.$$

It might be that $f(x)$ was some complicated function that did not have an indefinite integral in terms of an elementary formula. We could still find an approximation to the values of a solution by numerical integration over successive intervals of the form $a \leq t \leq x$. Quite good results can be obtained this way using Simpson's rule, for example. While this may sound like a lot of work, recall that studying and computing a function like arctan x for the first time is also a lot of work. On the other hand, arctan x is "well known" and has many nice properties, so a solution that can be expressed in terms of it is often to be preferred over a strictly numerical computation.

1B Initial Values

A typical first-order differential equation has infinitely many solutions on some interval, and each of these solutions has a graph passing through infinitely many points. To distinguish one solution from another in a convenient way, we can pick a point x_0 in the common domain of several solutions. Then let $y_0 = y(x_0)$, where $y(x)$ is some particular solution, and label that solution with the pair (x_0, y_0). Geometrically, what we have proposed is to pick a point (x_0, y_0) on the given solution curve and then use that point to specify the solution. The choice (x_0, y_0) is called an **initial condition** and is usually expressed in the form $y(x_0) = y_0$. The question as to when such a condition specifies one and only one solution is taken up in Chapter 7; in the meantime we will concentrate on examples for which the choice of a point determines a unique solution curve containing the point.

EXAMPLE 4 The very simple differential equation

$$y' = x$$

has solutions

$$y = \int_0^x t\,dt + c$$

$$= \tfrac{1}{2}x^2 + c.$$

To pass through the point $(x_0, y_0) = (-1, 1)$, the solution must satisfy

$$1 = \tfrac{1}{2}(-1)^2 + c.$$

Then $c = \frac{1}{2}$, and the desired solution is $y = \frac{1}{2}x^2 + \frac{1}{2}$; this and two other solutions are shown in Fig. 2(b).

EXAMPLE 5 The differential equation

$$\frac{dy}{dt} = -ty$$

has solutions

$$y = ce^{-t^2/2}$$

To pass through the point $(t_0, y_0) = (0, 2)$, a solution must satisfy

$$2 = ce^0 = c.$$

Thus $c = 2$, and the particular solution selected by the condition $(t_0, y_0) = (0, 2)$ is $y = 2e^{-t^2/2}$.

EXAMPLE 6 When a drug is administered to a human body it has at time t a certain concentration $c(t)$ in the body's fluids. Careful measurements show that this concentration typically decreases, as time increases, according to an exponential law, $c(t) = c_0e^{-at}$, a constant, $a > 0$, where c_0 is the concentration at $t = 0$: $c(0) = c_0$. Since $c'(t) = -ac_0e^{-at} = -ac(t)$, it follows that $c(t)$ satisfies the first-order differential equation

$$\frac{dc}{dt} = -ac.$$

In words, the differential equation says that the rate at which the concentration decreases over time is proportional to the concentration at any time, the constant of proportionality being $a > 0$; the minus sign accounts for decreasing concentration as opposed to increasing. If another dose of the same size as the initial dose is administered at time $t_1 > 0$, the concentration in the body jumps to the sum of the two concentrations:

$$c_0 + c_0e^{-at_1} = c_0(1 + e^{-at_1}).$$

After n equal doses, spaced at equal time intervals t_1, the concentration is expressible as a geometric series:

$$c(nt_1) = c_0(1 + e^{-at_1} + \cdots + e^{-(n-1)at_1}) = c_0\frac{1 - e^{-nat_1}}{1 - e^{at_1}}.$$

Exercise 8 asks for a derivation of this last equation and some conclusions that can be drawn from it.

The previous example shows that it may be of some interest to start with a formula for a family of solutions to a differential equation and then find a differential equation that is satisfied by all solutions in the family. In principle, this is just a matter of differentiating the given solution formula with respect to the chosen independent variable and then algebraically eliminating the constant.

EXAMPLE 7 The formula $x + cy = 0$ describes a family of lines through the origin of the (x, y) plane, each line with a different slope $-1/c$. Differentiating with respect to x gives

$$1 + c\frac{dy}{dx} = 0.$$

But $c = -x/y$, so the differential equation is

$$1 + \left(\frac{-x}{y}\right)\frac{dy}{dx} = 0 \quad \text{or} \quad x\frac{dy}{dx} = y.$$

EXERCISES

1. For each of the following differential equations, check to see whether the given function is a solution on the specified interval.

(a) $y' + y = 0$; $y = ce^{-x}$, $\quad -\infty < x < \infty$, c const.

(b) $y' = 2y$; $y = ce^{2x}$, $\quad -\infty < x < \infty$, c const.

(c) $\dfrac{dy}{dt} + 2ty = 0$; $y = ce^{-t^2}$, $\quad -\infty < t < \infty$, b const.

(d) $y' = y + x$; $y = ce^{-x} - x - 1$, $\quad -\infty < x < \infty$, c const.

(e) $x\dfrac{du}{dx} = 1$; $u = ln(cx)$, $\quad 0 < x$, c const.

(f) $\dfrac{dy}{dx} = 1 + y^2$; $y = \tan x$, $\quad \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$; $y = \tan(x - c)$, $\dfrac{-\pi}{2} + c < x < \dfrac{\pi}{2} + c$, c const.

(g) $\dfrac{dz}{dx} = 2\sqrt{|z|}$; $z = x|x|$, $\quad -\infty < x < \infty$.

(h) $\dfrac{dy}{dx} = 3x^2y + 1$; $y = e^{x^3}\displaystyle\int_0^x e^{-t^3}\,dt + ce^{x^3}$, $\quad -\infty < x < \infty$, c const.

(i) $\dfrac{dy}{dx} = \sqrt{1 + x^3}$; $y = \displaystyle\int_0^x \sqrt{1 + t^3}\,dt$, $\quad -1 < x < \infty$.

(j) $\dfrac{dy}{dx} = 2x\sqrt{1 + x^6}$; $y = \displaystyle\int_0^{x^2} \sqrt{1 + t^3}\,dt$ $\quad -\infty < x < \infty$.

(k) $\dfrac{d^2y}{dt^2} + 4y = 0$; $y = c_1, \cos 2t + c_2 \sin 2t$, $\quad -\infty < t < \infty$, c_1, c_2 const.

(l) $\dfrac{d^2y}{dt^2} - 4y = 0$; $y = c_1e^{2t} + c_2e^{-2t}$, $\quad -\infty < t < \infty$, c_1, c_2 const.

(m) $\dfrac{dy}{dt} = \sqrt{y}$; $y = 0$, $\quad -\infty < t < \infty$.

(n) $\dfrac{dy}{dt} = \sqrt{y}$; $y = \begin{cases} 0, & t < 0 \\ \dfrac{t^2}{4}, & t \geq 0. \end{cases}$

2. (a) The differential equation

$$\left(\frac{dx}{dt}\right)^2 + x^2 = 0$$

has only one real-valued solution on a given interval $a < t < b$; explain why. What is the unique solution?