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edited by R. F. Streater

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Preface

The London Mathematical Society held a teaching conference at Bedford College in Regent's Park, from 23rd August until 11th September 1971, under the sponsorship of NATO. This book contains the lectures and some of the seminars.

The purpose of the conference was to interest mathematicians in some important problems in mathematical physics and to provide a coherent introduction to the subject at post-graduate level, for mathematicians and mathematical physicists.

Large areas of physics have been omitted: this was done deliberately. Many exciting fields of physics, such as astrophysics, are not yet ripe for axiomatization. Other fields, such as the application of group theory to symmetry, already have an extensive mathematical literature.

Our thanks are due to the seminar speakers. There has been no space to publish the talks given by J. Fabrey, J. G. Taylor, A. Jaffe, F. Constantinescu, C. J. Isham, R. F. Streater, F. Brownell, K. Nakagami, A. Wehrl, J. Combes, R. Hudson, R. Lavine, J. D. Roberts and B. Simon.

The students who helped behind the scenes, especially I. F. Wilde, V. Sakellariou and Mrs. D. Mathon, deserve mention. Above all, we must record our gratitude to Dr. P. D. F. Ion, the Treasurer who, with the help of his wife Heather, efficiently carried out the bulk of the work.

July 1972 R. F. STREATER

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Quantum Field Theory*

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INTRODUCTION

The purpose of this chapter is to give an introductory survey of an area which has been called "general (or "axiomatic") quantum field theory". An exposition of pertinent ideas and concepts will be given and, of course, some discussion of the motivation, aims and purpose of this approach.

To get some perspective let us note that quantum field theory itself is about as old as quantum mechanics. The essential formal structure and its application to the quantum theory of the electromagnetic field was developed in the years 1927–1929 and the most important applications to the interaction processes between matter and electromagnetic radiation were done by 1934 and recorded in the first edition of Heitler's book (1936).

The development of general (axiomatic) quantum field theory started about 20 years later. It is characterized by a much greater insistence on precision in the conceptual and mathematical description than had been customary. It may be argued that this emphasis on mathematical precision has sometimes been excessive and counter-productive. Its origin as a psychological necessity in the early fifties is easily understandable after the success of renormalization prescriptions had restored faith in the predictive power of quantum field theory without achieving a lucid formulation of its basic equations, thus leaving as a question how the field equations of quantum electrodynamics should be understood and whether these equations had any solutions. The method used (which led to the epiphet "axiomatic") was to desist for the time being from the consideration of a completely specified theory and to focus the attention first on the "general principles", the "postulates", the "framework", the

* Notes by R. Haag and I. F. Wilde.

"axioms" with the aim of testing their internal consistency or of finding ways to work out consequences which could be tested experimentally.

Accordingly one objective of this line of investigation has been to recognize the physical principles underlying a theory such as quantum electrodynamics in their simplest and most essential form (purified from the chance effects of historical development). The main concern here will be a description of the development of our understanding of these principles. They may be grouped into 4 main topics:

1. Mathematical structure of quantum physics.
2. Poincaré invariance.
3. Locality: Fields and local algebras.
4. States on interest: Energy-momentum spectrum and particle aspects.

1. MATHEMATICAL STRUCTURE OF QUANTUM PHYSICS

We shall make use here of two alternative and not entirely equivalent formulations. The first, described by von Neumann (1932), uses as the basic mathematical object a Hilbert space; the second, introduced by Segal (1947), will be called the algebraic formulation. In it the basic mathematical object is an abstract algebra (which, for technical reasons, is chosen to be a C^*-algebra†).

In both schemes the fundamental physical concepts may be taken to be the notions of "state" and "observable" as defined by von Neumann (1932), the states forming a convex set whose extremal points are called "pure states".

In the Hilbert space version an observable Q is mathematically represented by a self-adjoint operator acting on a Hilbert space $\mathscr{H}$. In the algebraic version it is, instead, a self-adjoint element of an abstract C^*-algebra. In both cases one has a definition of the spectral values of Q and the functions of Q. The spectral values are interpreted as the possible values which one may obtain in a measurement; a (real) function $F(Q)$ represents an observable measured by the same apparatus as Q, the only difference being that the scale of measured values is relabelled substituting the number $F_{(q)}$ for q. Note that in the C^*-algebra approach only the bounded observables are considered (observables with a bounded

† An abstract C^*-algebra (also called B^*-algebra) is a Banach *-algebra with a norm satisfying the condition

$$||A^*A|| = ||A||^2.$$

It should be noted that the norm of Hilbert space operators satisfies this condition and that, moreover, a norm with this property is often determined by the algebraic structure alone. Compare with the theory of the "minimal regular norm".

spectrum). This involves no additional physical assumption or restriction since, by a relabeling of the scale, we may always compress the spectrum into a bounded interval.

Using the concepts of state and observables† as understood by von Neumann (1932) the questions to be answered by the theory can be cast in the form: what is the expectation value of the observable Q in the state ω.‡

One difference between the descriptions of von Neumann (1932) and Segal (1947) is now the following. In the first, pure states are conventionally represented by unit vectors (more precisely by rays) in the Hilbert space and the expectation value of the observable Q in the state described by the vector Ψ is given by

$$\omega_\Psi(Q) = (\Psi, Q\Psi). \tag{1.1}$$

Impure states are described by density matrices. A density matrix ρ is a positive operator of trace class, with trace 1. The expectation value of Q in such a state is given by

$$\omega_\rho(Q) = \text{Trace}\,(\rho\, Q). \tag{1.2}$$

The special case (1.1) results if ρ is a one-dimensional projection on the vector Ψ. In the algebraic approach (Segal 1947) on the other hand, one says that giving a state ω is synonymous with giving the expectation values of all observables in this state. This means that a "state" is considered to be a normalized, positive, linear form over the algebra $\mathfrak{A}$. In other words, a state ω assigns to each $A \in \mathfrak{A}$ a number $\omega(A)$ (in general a complex number) such that

$$\begin{aligned}
&(1)\ \omega(1) = 1 && \text{normalization}\\
&(2)\ \omega(A^*A) > 0 && \text{positivity (implying also } \omega(A^*) = \omega(A))\\
&(3)\ \omega(\lambda_1 A_1 + \lambda_2 A_2) = \lambda_1\,\omega(A_1) + \lambda_2\,\omega(A_2) && \text{(linearity)}
\end{aligned} \tag{1.3}$$

Conversely, any expectation functional over $\mathfrak{A}$, i.e. any assignment of numbers $\omega(A)$ satisfying the above three conditions, will be considered as a possible state.

The relation between the schemes of von Neumann and Segal is seen when one considers an irreducible representation π of $\mathfrak{A}$ by operators on a

† The elements of the algebra also appear in another rôle. Each element represents an "operation" in the sense of Haag and Kastler (1964). This interpretation is actually somewhat better suited to quantum field theory, but we shall not use it here.

‡ The probability of finding a Q-value within some interval can also be obtained as the expectation value of $F(Q)$ where F is the characteristic function of the interval.

Hilbert space $\mathscr{H}$. We restrict the discussion here to the most important case where $\mathfrak{A}$ is simple so that every representation is faithful. Let $\pi(A)$ denote the representor of the element $A \in \mathfrak{A}$. The set $\pi(\mathfrak{A})$ is then a concrete operator algebra isomorphic to the abstract algebra $\mathfrak{A}$ and the operator norm $\|\pi(A)\|$ equals $\|A\|$. If Ψ is any vector of unit length then the functional

$$\omega_\Psi(A) = (\Psi, \pi(A)\,\Psi) \tag{1.4}$$

is a normalized, positive linear form over $\mathfrak{A}$ i.e. a "state". The irreducibility of the representation π implies that this state is pure and that $\omega_\Psi = \omega_\Phi$ only if $\Phi = e^{i\delta}\Psi$. In other words, each ray in $\mathscr{H}$ determines a distinct pure state over $\mathfrak{A}$. We shall call the family of states of the form (1.4) as Ψ runs through $\mathscr{H}$ the family of vector states of the representation π†.

The essential difference between the schemes of von Neumann and Segal is therefore the following. In general the algebra will allow many inequivalent irreducible representations and will therefore possess many distinct families of pure states, each family being the set of vector states occurring in *one* equivalence class of irreducible representations.

It may be important or useful to consider several of these families of states side by side. In this case the von Neumann scheme is obviously too narrow. Indeed the need for a generalization of this scheme has also been suggested directly by physical arguments within the Hilbert space formulation. In Wick *et al.* (1952) it was pointed out that no coherent superposition of two state vectors describing states with respectively integer and half-integer spin is possible and that, more generally, one has to expect that the states should be grouped into sectors so that the matrix elements of all observables between state vectors belonging to different sectors vanish ("superselection rules"). We may note that the algebraic formulation leads in a very natural way to this possibility of superselection rules, the above mentioned families of states corresponding to superselection sectors.

The discussion of this point may be summarized in a very crude fashion by saying that the algebraic version of the theory (of Segal, Haag and Kastler) is the mathematical development of the original Heisenberg–Born–Jordan approach whereas the Hilbert space version of von Neumann is the development of Schrödinger's approach. In quantum *mechanics* (with a finite number of degrees of freedom) von Neumann's uniqueness theorem for irreducible representations of the canonical commutation relations means that both approaches lead to identical schemes. In the case of quantum

† If one has to consider also impure states, for instance in statistical mechanics, one is led to the family of "normal states of the representation π", i.e. those of the form $\omega_0(A) = \mathrm{Tr}(\rho\pi(A))$ where ρ runs through all density matrices.

field theory (where we are dealing with infinitely many degrees of freedom) the relevant algebras are such that they allow (uncountably) many inequivalent irreducible representations. Hence the algebraic scheme appears at first sight to be much less restrictive. One may, however, ask: are all the (uncountably many) equivalence classes of irreducible representations of $\mathfrak{A}$ really needed? How can we understand then that in elementary particle physics one believes that one has to deal with a denumerable set of superselection sectors only?

The answer is somewhat tricky. On the one hand we can ask: Suppose a state ω is experimentally prepared and we want to find out by monitoring experiments into which family this state belongs: i.e. we want to "measure" its superselection quantum numbers. One must admit that this is impossible because, first of all, any actual course of measurement will determine only the expectation values of a finite number of observables A_i $(i = 1, \ldots N)$ and these only with a finite accuracy. Thus, we obtain a set of numbers a_i and errors ε_i and the knowledge that the unknown state ω has the properties

$$|\omega(A_i) - a_i| < \varepsilon_i \qquad i = 1, \ldots N. \tag{1.5}$$

It is true that we may choose N as large as we like and, possibly, can reduce the ε_i to smaller and smaller values. Still, in any given effort we can only determine the information (1.5) for some N, and some ε_i. This information characterizes, however, precisely what is called mathematically a weak*-neighbourhood in state space. Thus we do not determine ω but only know that it lies in a certain weak*-neighbourhood. If this consideration is combined with a theorem by Fell (1960) which, when specialized to the case of pure states over a simple algebra, says: "the pure states of one family are weakly*-dense in the set of all pure states" then we realize that the distinction between different families (superselection sectors) is too fine to be discernable in a realistic experiment. What then about the physical significance of superselection rules? What one actually does is to idealize the situation, adding for convenience information which one does not have and does not need in principle. Thus in elementary particle physics one uses the idealization that far away from the centre of our laboratory there shall be no matter; all states considered shall look like the vacuum state in far away observations. With this idealization we single out a small subset of states as the "states of interest" and the remaining superselection rules within this subset become significant.

A typical example of such a superselection quantum number is the total charge or the total baryon number. In principle this cannot be measured because it would necessitate an infinitely extended apparatus.

But we can measure the baryon number in the relevant part of the laboratory and, if we pretend that there is no matter outside of this region, then this number is identical with the total baryon number. This example illustrates perhaps both Fell's theorem (how to approximate weakly states in one sector by states in another sector) and the reason for the significance of certain superselection quantum numbers.

As examples of algebras and representations let us briefly consider the case of canonical commutation relations (CCR). Formally, for one degree of freedom, by this is meant a pair of hermitian operators p and q representing momentum and position, respectively, which satisfy the Heisenberg relation

$$[q,p] \equiv qp - pq = i. \tag{1.6}$$

Immediately we see that (1.6) implies that at least one of the pair p and q is an unbounded operator. For, suppose p and q are both bounded. Then

$$p^n q - qp^n = \sum_{\substack{l,\, m \geqslant 0 \\ l+m=n-1}} p^l [p,q] p^m = -inp^{n-1}. \tag{1.7}$$

But

$$\|p^n q - qp^n\| \leqslant 2\|p^n\| \, \|q\|$$

and so by (1.7),

$$n \|p^{n-1}\| \leqslant 2 \|p^n\| \, \|q\| \leqslant 2 \|p^{n-1}\| \, \|p\| \, \|q\|. \tag{1.8}$$

This implies that

$$n < 2 \|p\| \, \|q\| \tag{1.9}$$

for all integers n, which is impossible if p and q are bounded.

A formulation formally equivalent to the Heisenberg relation is the Weyl relation:

$$e^{ips} e^{iqt} = e^{ist} e^{iqt} e^{ips} \tag{1.10}$$

for real s and t.

This is technically more convenient since the operators in consideration are now bounded. We take this as our definition.

Definition 1.1. A representation of the CCR for one degree of freedom, in the Weyl form, is a pair of maps V and U from the real line, $\mathbf{R}$, into unitary operators on a Hilbert space $\mathscr{H}$, such that

(i) $V(\cdot)$ and $U(\cdot)$ are strongly continuous unitary representations of $\mathbf{R}$,

(ii) $V(s)U(t) = e^{ist} U(t)V(s)$. (1.11)

The strong continuity allows us to recover p and q by Stone's theorem:

$$V(s) = e^{isp}, \quad U(t) = e^{itq}. \tag{1.12}$$

At first sight, one might expect there to be many such representations. That this is not so is the content of the Stone–von Neumann uniqueness theorem†, which says that if the representation is irreducible (i.e. if the only operators commuting with the $V(\cdot)$ and $U(\cdot)$ are multiples of the identity) then, up to unitary equivalence, V, U and $\mathscr{H}$ are unique.

So under the conditions above, we are always essentially in the Schrödinger representation, which is given by

$$\begin{aligned} V(s) &: f(x) \to f(x+s) \\ U(t) &: g(x) \to e^{itx} g(x) \end{aligned} \tag{1.13}$$

for $s, t \in \mathbf{R}$, and $f, g \in = L^2(\mathbf{R}, dx)$. Here, p and q are represented by $-i(d/dx)$ and multiplication by x, respectively.

Let us now relate the CCR to a C^*-algebra! Consider the objects

$$W(s;t) = V(s)\,U(t)$$

as abstract elements for which (corresponding to (1.11)) a multiplication law and a *-operator are defined by

$$W(s_2;t_2)\,W(s_1;t_1) = e^{-is_1t_2}\,W(s_1+s_2;t_1+t_2) \tag{1.14}$$

$$W(s;t)^* = e^{-ist}\,W(-s;-t). \tag{1.15}$$

Considering now the set of finite linear combinations of such $W(\cdot;\cdot)$ with complex coefficients, i.e. objects of the form

$$A = \Sigma\, C_k\, W(s_k;t_k)$$

we have a *-algebra. We may equip this algebra with a C^*-norm by assigning as $\|A\|$ the operator norm of the representor of A in the Schrödinger representation. Completing the set $\{A\}$ in this norm topology we obtain a C^*-algebra $\mathfrak{A}$.

It is perhaps important to realize that a representation of this C^*-algebra is not necessarily a representation of the CCR in the sense of the definition 1.1. The following amusing example gives an irreducible representation of $\mathfrak{A}$ in a non-separable Hilbert space and such that the representors of $W(s;t)$ are not strongly continuous in the parameter t.

† See the lectures of B. Simon.

We let K be the Hilbert space obtained from the inner product

$$(f,g) = \lim_{a\to\infty} \frac{1}{2a} \int_{-a}^{a} \bar{f}(x)\, g(x)\, dx \tag{1.13}$$

on functions on $\mathbf{R}$. Clearly, any $L^2(\mathbf{R}, dx)$ functions has zero norm, here. For any $f \in K$, and $s, t \in \mathbf{R}$, we define $V(s)$ and $U(t)$ exactly as in (1.12). Evidently, the Weyl relations (1.10) hold. However, $f(x) = e^{ix}$ is a unit vector in K, and is orthogonal to $U(t)f$ whenever t is non-zero. Thus

$$\|U(t)f - f\| = \sqrt{2} \tag{1.14}$$

for any non-zero t. It is also clear that K is non-separable—for example, $\{e^{ikx}, x \in \mathbf{R}\}$ is an uncountable family of pairwise orthogonal (normalizable) vectors.

Suppose now that we have a system with a countable number of degrees of freedom. That is to say, we are concerned with a countable family of independent p's and q's: $\{p_k, q_k; k = 1, 2 \ldots\}$. To each degree of freedom, we have the Schrödinger representation with the associated $V_k(\cdot)$ and $U_k(\cdot)$; i.e. we have a collection of triples $\{V_k, U_k, \mathscr{H}_k\}$ and abstract C^*-algebras $\mathfrak{A}_k$.

Let $S = \{i_1 \ldots i_n\}$ be any finite set of integers. Then we define $\mathfrak{A}(S)$ to be the C^*-algebra resulting from the tensor product of the C^*-algebras $\mathfrak{A}_k$ with $k \in S$. That is to say, we consider first finite linear combinations of products of the w_k with $k \in S$ with the obvious definition of multiplication and *-operation. Then we assign a norm and complete it. The norm in question is obtained again as the operator norm in a representation, namely the tensor product of Schrödinger representations of the $\mathfrak{A}_k$.

If $S_1 \subset S_2$, we can consider $\mathfrak{A}(S_1)$ as a subalgebra of $\mathfrak{A}(S_2)$;

$$\mathfrak{A}(S_1) \subset \mathfrak{A}(S_2) \quad \text{if} \quad S_1 \subset S_2. \tag{1.15}$$

For any S_1 and S_2 we can consider $\mathfrak{A}(S_1)$ and $\mathfrak{A}(S_2)$ as subalgebras of $\mathfrak{A}(S_1 \cup S_2)$. If $S_1 \cap S_2 = \varnothing$, it is clear that $\mathfrak{A}(S_1)$ and $\mathfrak{A}(S_2)$ commute. In fact, $\mathfrak{A}(S_1 \cup S_2) = \mathfrak{A}(S_1) \otimes \mathfrak{A}(S_2)$ in this case.

If I is an infinite subset of integer, we define $\mathfrak{A}(I)$ to be the inductive limit of the $\mathfrak{A}(S)$, with $S \in I$:

$$\mathfrak{A}(I) = \overline{\bigcup_{S \subset I} \mathfrak{A}(S)}. \tag{1.16}$$

If $I = \mathbb{Z}$, we write $\mathfrak{A}$ instead of $\mathfrak{A}(\mathbb{Z})$.

If ω_1 and ω_2 are states on $\mathfrak{A}(S_1)$ and $\mathfrak{A}(S_2)$ respectively, and $S_1 \cap S_2 = \varnothing$, then we can define a state ω on $(S_1 \cup S_2)$ by extension of

$$\omega(AB) = \omega_1(A)\, \omega_2(B) \tag{1.17}$$

for

$$A \in \mathfrak{A}(S_1); \quad B \in \mathfrak{A}(S_2).$$

ω is called a product extension of ω_1 and ω_2. Clearly

$$\omega \restriction \mathfrak{A}(S_i) = \omega_i, \quad i = 1, 2.$$

Definition 1.2. Let ω and σ be any two states on $\mathfrak{A}$, and let $\{S_n\}$ be a sequence of finite subsets of integers such that $S_n \subset S_{n+1}$ and $\cup_n S_n = \mathbb{Z}$. Let $S_n' = \mathbb{Z} - S_n$, and $\omega_n' = \omega \restriction \mathfrak{A}(S_n')$; $\sigma_n' = \sigma \restriction \mathfrak{A}(S_n')$. Then we say that ω and σ have the same asymptotic tail if

$$\|\omega_n' - \sigma_n'\| \underset{n\to\infty}{\to} 0.$$

We can exhibit now one reason for the occurance of many inequivalent irreducible representations of the algebra of a system with infinitely many degrees of freedom. It is not difficult to show that in an irreducible representation of $\mathfrak{A}$, all normal states have the same asymptotic tail. On the other hand, let $S_1, S_2, \ldots$ be a partition of $\mathbb{Z}$ into disjoint finite subsets, and let $\{\omega_k\}$ be a sequence of states on the $\{\mathfrak{A}(S_k)\}$. As we vary the sequences $\{\omega_k\}$, we obtain various product states on $\mathfrak{A}$. Thus we can construct states with very different asymptotic tails, i.e. states belonging to different families, to different equivalent classes of representations.

2. POINCARE INVARIANCE

We recall that a Poincaré transformation (a, Λ) is given by

$$x_\mu \to x_\mu' = \sum_\nu \Lambda_{\mu\nu} x_\nu + a_\mu \tag{2.1}$$

where $\mu, \nu = 0, 1, 2, 3$ and $\Lambda_{\mu\nu}$ is a real 4×4 matrix satisfying $\Lambda^T G \Lambda = G$, where G is diagonal with entries $(1, -1, -1, -1)$, and a_μ is real. Then $L = (a, \Lambda)$ does not change the Lorentz distance between any two space-time points:

$$(x' - y')^2 = (x - y)^2. \tag{2.2}$$

In the Hilbert space version one assumes that the connected part of the Poincaré group is represented by unitary operators, $U(L)$. The action of $U(L)$ on a state vector Ψ is interpreted as giving a state which is specified by the same physical set-up as Ψ but with the preparing apparatus shifted by L.

An operator undergoes the transformation

$$A \to A' = U(L) A U^{-1}(L). \tag{2.3}$$

Then $(\Psi', A'\Psi') = (\Psi, A\Psi)$ and so the expectation values are unaltered as we expect.