SOLUTIONS OF THE EXAMPLES

IN

THE ELEMENTS

OF

STATICS AND DYNAMICS

BY

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FOR the following Solutions of the Examples in my *Elements of Statics and Dynamics* I am almost entirely indebted to a friend, to whom my best thanks are due. He has also carefully revised the whole of the proof-sheets.

I hope these solutions will be found useful to teachers and private students.

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ELEMENTS OF STATICS.—SOLUTIONS.

EXAMPLES. I. (Pages 15, 16.)

1. (i) $R=\sqrt{(24)^2+7^2}=\sqrt{625}=25.$

(ii) $Q=\sqrt{(14)^2-(13)^2}=\sqrt{27}=3\sqrt{3}.$

(iii) $R=\sqrt{7^2+8^2+2\,.\,7\,.\,8\cos 60^\circ}=\sqrt{169}=13.$

(iv) $R=\sqrt{5^2+9^2+2\,.\,5\,.\,9\cos 120^\circ}=\sqrt{61}.$

(v) $7^2=3^2+5^2+2\,.\,3\,.\,5\cos a,$

whence $\cos a=\frac{1}{2}$, *i.e.* $a=60^\circ$.

(vi) $\cos a=\sqrt{1-\sin^2 a}=\sqrt{1-\left(\frac{12}{13}\right)^2}=\pm\frac{5}{13};$

$$\therefore\ R=\sqrt{(13)^2+(14)^2\pm 2\,.\,13\,.\,14\,.\frac{5}{13}}=\sqrt{505},\ \text{or}\ 15.$$

(vii) $7^2=5^2+Q^2+2\,.\,5\,.\,Q\cos 60^\circ,$

whence $Q^2+5Q-24=0$, and so $Q=3$.

2. The resultant of two forces is greatest or least according as they act in the same straight line in the same direction or in opposite directions. Hence [cf. Art. 23], the greatest resultant of forces of 12 lbs. wt. and 8 lbs. wt. $=(12+8)$ lbs. wt. $=20$ lbs. wt.; and the least resultant $=(12-8)$ lbs. wt. $=4$ lbs. wt.

3. The forces of 3 lbs. wt. and 4 lbs. wt. act in the same straight line in opposite directions, and are, therefore, equivalent to a force of 1 lb. wt. in the direction of the force of 4 lbs. wt.; *i.e.* south. The forces of 5 lbs. wt. and 6 lbs. wt. also act in the same straight line in opposite directions, and are, therefore, equivalent to a force of 1 lb. wt. in the direction of the force of 6 lbs. wt., *i.e.* west. Hence the four forces are equivalent to two forces of 1 lb. wt. each, acting at right angles; therefore the resultant $=\sqrt{1^2+1^2}=\sqrt{2}$ lb. wt., and evidently acts along a line bisecting the angle between the lines of action of the forces of 4 lbs. wt. and 6 lbs. wt., *i.e.* in a direction south-west.

4. The resultant $= \sqrt{(84)^2+(187)^2} = \sqrt{42025} = 205$ lbs. wt.

5. Here $R = \sqrt{P^2 + (P\sqrt{2})^2 + 2 \cdot P \cdot P\sqrt{2}\cos 135^\circ} = P$ lbs. wt. at an angle $\tan^{-1}\dfrac{P\sqrt{2}\sin 135^\circ}{P + P\sqrt{2}\cos 135^\circ}$, *i.e.* $\tan^{-1}\infty$, *i.e.* at right angles to the direction of the first component.

6. If Q be the required force, we have

$$(2\sqrt{3})^2 = 2^2 + Q^2 + 2 \cdot 2 \cdot Q\cos 60^\circ,$$

whence $Q^2 + 2Q - 8 = 0$, and therefore $Q = 2$ lbs. wt.

7. If $\tan\alpha = \dfrac{12}{5}$, then $\cos\alpha = \dfrac{5}{13}$.

$$\therefore R = \sqrt{(13)^2 + (11)^2 + 2 \cdot 13 \cdot 11 \cdot \frac{5}{13}} = \sqrt{400} = 20 \text{ lbs. wt.}$$

8. If $\tan\alpha = \dfrac{4}{3}$, then $\cos\alpha = \dfrac{3}{5}$.

$$\therefore R = \sqrt{(10)^2 + 9^2 + 2 \cdot 10 \cdot 9 \cdot \frac{3}{5}} = \sqrt{289} = 17 \text{ lbs. wt.}$$

9. If the forces be each equal to P, α be the angle between them, and R be their resultant, we have

$$R^2 = 3P^2, \text{ so that } 3P^2 = P^2(2 + 2\cos\alpha),$$

whence $\cos\alpha = \dfrac{1}{2}$, *i.e.* $\alpha = 60^\circ$.

10. If P and Q be the required forces, we have

$$(\sqrt{10})^2 = P^2 + Q^2, \text{ and } (\sqrt{13})^2 = P^2 + Q^2 + 2PQ\cos 60^\circ,$$

i.e. $P^2 + Q^2 = 10$, and $P^2 + Q^2 + PQ = 13$.

Solving these equations, we have

$$P = 3 \text{ lbs. wt., and } Q = 1 \text{ lb. wt.}$$

11. (1) $P = P\sqrt{2(1+\cos\alpha)}$, *i.e.* $1 = 2 + 2\cos\alpha$;

whence $\cos\alpha = -\dfrac{1}{2}$, *i.e.* $\alpha = 120^\circ$.

(2) $\dfrac{P}{2} = P\sqrt{2(1+\cos\alpha)}$, *i.e.* $\dfrac{1}{4} = 2 + 2\cos\alpha$;

whence $\cos\alpha = -\dfrac{7}{8}$, *i.e.* $\alpha = \cos^{-1}\left(-\dfrac{7}{8}\right) = 151^\circ\, 3'$.

12. Here, if α be the required angle, we have

$$(\sqrt{A^2+B^2})^2=(A+B)^2+(A-B)^2+2(A+B)(A-B)\cos\alpha,$$

so that $$A^2+B^2=2(A^2+B^2)+2(A^2-B^2)\cos\alpha,$$

whence $$\cos\alpha=-\frac{A^2+B^2}{2(A^2-B^2)},\ \textit{i.e.}\ \alpha=\cos^{-1}\left(-\frac{1}{2}\frac{A^2+B^2}{A^2-B^2}\right).$$

13. Find the resultant (R) of the two given forces; let S be the third given force; the greatest resultant R and S can have is $R+S$ when they act in the same direction in the same straight line; *i.e.* S must act in the direction of R.

14. Take the figure of Art. 27.

(i) Make $OA=5$ ins., $\angle AOB=37°$, and cut off $OB=7\frac{1}{2}$ ins.; complete the parallelogram $OACB$; then OC is R.

[2 units of force = one inch.]

(ii) Make $OA=4\frac{1}{2}$ ins., $\angle AOB=134°$ and cut off $OB=3\frac{1}{2}$ ins.; complete the parallelogram $OACB$; then $OC=R$.

(iii) Make $OA=3\frac{1}{2}$ ins.; with centres O and A describe circles of radii 5 and $2\frac{1}{2}$ ins. to meet in C; complete the parallelogram $OACB$; then $\angle AOB=\alpha$.

(iv) Make $OA=3{\cdot}65$ ins. and $\angle AOB=65°$; draw AC parallel to OB; with centre O and radius $4{\cdot}35$ describe a circle to cut AC in C; complete the parallelogram $OACB$; then OB is Q.

EXAMPLES. II. (Pages 19, 20.)

1. The resolved parts are $10 \cos 30°$ and $10 \sin 30°$, respectively, *i.e.* $5\sqrt{3}$ lbs. wt. and 5 lbs. wt.

2. (1) $P \cos 45°$, *i.e.* $\frac{1}{2} P\sqrt{2}$. (2) $P \cos\left(\cos^{-1}\frac{12}{13}\right)$, *i.e.* $\frac{12}{13} P$.

3. The required force $= 100 \cos 60° = 50$ lbs. wt.

4. If the required forces be each equal to P, we have

$$(100)^2 = P^2 (2 + 2 \cos 60°);$$

whence $3P^2 = (100)^2$, and $P = \frac{100\sqrt{3}}{3} = 57{\cdot}735$ lbs. wt.

5. If x and y be the required forces respectively, we have

$$\frac{x}{\sin 45°} = \frac{y}{\sin 60°} = \frac{50}{\sin 105°} = \frac{50}{\cos 15°} = \frac{50}{\cos(45° - 30°)}$$

$$= \frac{100\sqrt{2}}{\sqrt{3}+1} = 50\sqrt{2}\,(\sqrt{3} - 1).$$

Hence $x = 50(\sqrt{3} - 1) = 36{\cdot}603$ lbs. wt.,

and $y = 25(\sqrt{18} - \sqrt{6}) = 44{\cdot}83$ lbs. wt., nearly.

6. If x and y be the required components respectively, we have

$$\frac{x}{\sin 45°} = \frac{y}{\sin 30°} = \frac{P}{\sin 75°} = \frac{P}{\sin(45° + 30°)} = \frac{P \times 2\sqrt{2}}{\sqrt{3}+1} = P\sqrt{2}\,(\sqrt{3} - 1).$$

Hence $x = P(\sqrt{3} - 1)$, and $y = \frac{P}{2}(\sqrt{6} - \sqrt{2})$.

7. The required force $= P\frac{\sin 45°}{\sin 60°} = P\sqrt{\frac{2}{3}}$.

8. If P and Q be the required forces respectively, we have

$$P = F \tan 60° = F\sqrt{3}, \text{ and } Q = F \sec 60° = 2F.$$

9. If a force F be resolved into two component forces P and Q, and P be at right angles to F and equal to it in magnitude, then the other angles are each 45°, and $Q=P\sqrt{2}=F\sqrt{2}$. Also the angle between the component forces is 135°.

10. Draw OB vertical and equal to 20 units of length, and OA horizontal and equal to 10 units of length. Complete the parallelogram $OABC$. Then OC represents the other force.

Clearly $OC=AB=\sqrt{(20)^2+(10)^2}=10\sqrt{5}=22{\cdot}36$ lbs. wt.

Also $\tan COB=\tan OBA=\frac{1}{2}$, so that the inclination to the vertical

$$=\tan^{-1}\frac{1}{2}=26°\,34'.$$

11. In Fig. Art. 34 make

$$OC=3\tfrac{1}{2}\text{ ins.},\ \angle COA=98°\ \text{and}\ \angle COB=40°.$$

EXAMPLES. III. (Pages 25, 26.)

1. If P, Q and R be the forces, we have, by Lami's Theorem,

(i) $P=Q=R$.

(ii) $\dfrac{P}{\sin 150°}=\dfrac{Q}{\sin 150°}=\dfrac{R}{\sin 60°}$;

hence $$P:Q:R=1:1:\sqrt{3}.$$

2. If P, Q and R be the forces, we have, by Lami's Theorem,

$$\frac{P}{\sin 120°}=\frac{Q}{\sin 150°}=\frac{R}{\sin 90°};$$

hence $$P:Q:R=\sqrt{3}:1:2.$$

3. Since $7P$ is equal to the resultant of $5P$ and $8P$, we have, if a be the required angle,

$$(7P)^2=(5P)^2+(8P)^2+2\,.\,5P\,.\,8P\cos a,$$

whence $$\cos a=-\frac{1}{2},\ \textit{i.e.}\ a=120°.$$

4. Draw a figure as in Art. 38, with $12P$ for P, $5P$ for Q, and $13P$ for R. The sides OL, LN and NO of the triangle OLN are proportional to 12, 5 and 13 respectively; and, since $(13)^2=(12)^2+5^2$, the angle OLN is therefore a right angle.

Again, $\tan LON=\frac{5}{12}=\cdot4166667$; therefore the angle $LON=22°\,37'$.

Hence the angle between the directions of the forces $5P$ and $12P$

$$=MOL=OLN=90°;$$

between the directions of the forces $12P$ and $13P$ the angle

$$=180°-22°\,37'=157°\,23',$$

and therefore, between the directions of the forces $13P$ and $5P$ the angle $=112°\,37'$.

5. Construct a triangle ABC with its sides CA, AB, and BC proportional to 2, 3 and 4 respectively; and with the side BC in the given direction. The forces $2P$ and $3P$ are parallel to CA and AB.

6. The force represented by BE is the resultant of forces represented by BD and DE, *i.e.* by $\frac{1}{2}BA$ and $\frac{1}{2}BC$; the force represented by DC is the resultant of forces represented by $\frac{1}{2}AC$ and $\frac{1}{2}BC$; but, by the triangle of forces, the forces represented by $\frac{1}{2}BA$ and $\frac{1}{2}AC$ have resultant represented by $\frac{1}{2}BC$; therefore the required resultant is represented in magnitude and direction by $\frac{3}{2}BC$.

7. By the triangle of forces, the resultant is $\lambda . AB$, acting at P parallel to AB, *i.e.* is constant in magnitude and direction.

8. The diagonals of $ABCD$ bisect each other in some point O, and the resultant of the attractions to A and C is proportional to $2 . PO$, $= 2\lambda . PO$ suppose, and is in the direction PO; so the resultant of the repulsions from B and D is proportional to $2 . OP$ $(= 2\lambda . PO)$ if the proportion be the same as for the attractions, and is in the direction OP. Hence P is in equilibrium independently of its position, *i.e.* wherever it is situated.

For Exs. 9—14 take the figure of Page 13 with $\angle AOC = \theta$.

9. Make $OA = 5$ inches (scale 5 lbs. = one inch) and $\angle AOC = 35°$. With centre A and radius 4 inches describe a circle to cut OC in C_1, C_2. Complete the parallelograms OAC_1B_1 and OAC_2B_2. Then OC_1, OC_2 give the two values of R, and AOB_1, AOB_2 the two values of α.

10. Draw $OA = 5$ ins. [scale 10 kilog. = one inch]; with centres O and A and radii 7 and 6 ins. describe circles to meet in C. Complete the parallelogram $OACB$. Then AOB and AOC are the required angles α and θ.

11. Draw $OA = 3$ inches and $AOB = 130°$: with centre O and radius 4 ins. draw a circle to cut AC, parallel to OB, in C. Complete the parallelogram $OACB$; then $OB = Q$ and $\angle AOC = \theta$.

12. Draw $OA = 6$ ins., $AOB = 70°$ and $AOC = 40°$; through A draw AC parallel to OB; then AC is Q and OC is R.

13. Draw $OA=6$ ins., $\angle AOC=50^\circ$ and make $OC=4$ inches. Join AC and complete the parallelogram $OACB$. Then OB is Q and AOB is α.

14. Draw $OA=4$ ins., $\angle AOB=55^\circ$ and draw AC parallel to OB. With centre O and radius 5 ins. draw a circle to cut AC in C. Then $\angle AOC$ is θ and AC is Q.

15. Let OC be the direction of the boat's length; make $\angle AOC=20^\circ$ and $OA=5$ ins. [Scale 1 cwt. $=1$ inch.]

On the other side of OC from OA take OB such that

$$COB=180^\circ-40^\circ=140^\circ.$$

Draw AC parallel to OB to meet OC in C; complete the parallelogram $OACB$; then on the given scale OC is the resultant force and OB the resultant reaction of the water.

EXAMPLES. IV. (Pages 26—28.)

1. Take the second figure in Art. 27 and we have $P=80$, $\alpha=120^\circ$, and $\angle COB=90^\circ$. Let Q be the required force.

Since $OB=BC\sin OCB=BC\sin COA=BC\sin 30^\circ=\frac{1}{2}BC$,

therefore $$Q=\frac{1}{2}P=40.$$

2. Let $2P$ and P be the given forces, α be the angle between them, and R be their resultant. Then $R=2P$, and we have

$$(2P)^2=(2P)^2+P^2+2\,.\,2P\,.\,P\cos\alpha,$$

whence $$\cos\alpha=-\frac{1}{4},\ i.e.\ \alpha=\cos^{-1}\left(-\frac{1}{4}\right),\ i.e.\ 104^\circ\,29'.$$

3. The resultant is always nearer to the greater force. Take the figure in Art. 38, with $P=3$ lbs. wt., the $\angle LOM=90°$, and the $\angle LOR=150°$; let Q and R be the required forces. Then the $\angle ROM=120°$, and we have

$$\frac{Q}{\sin 150°}=\frac{R}{\sin 90°}=\frac{3}{\sin 120°}, \text{ i.e. } 2Q=R=\frac{6}{\sqrt{3}},$$

$$\therefore\ Q=\sqrt{3} \text{ lb. wt., and } R=2\sqrt{3} \text{ lbs. wt.}$$

4. Take the second figure in Art. 27, with 30 lbs. wt. for P, $\varpi=\frac{5}{3}\times 90°=150°$, and OC perpendicular to OB; Q and R being required. Then we have

$$R^2=(30)^2+Q^2+2\,.\,30\,.\,Q\cos 150°,$$

i.e. $$R^2=(30)^2+Q^2-30\sqrt{3}Q;$$

also, since $BC=OA$, and the angle BOC is a right angle, we have

$$R^2=(30)^2-Q^2.$$

$$\therefore\ (30)^2+Q^2-30\sqrt{3}Q=(30)^2-Q^2.$$

$$\therefore\ Q=15\sqrt{3} \text{ lbs. wt.}$$

Also, $$R^2=(30)^2-Q^2=(15)^2\,[2^2-(\sqrt{3})^2]=(15)^2,$$

so that $$R=15 \text{ lbs. wt.}$$

Otherwise thus:—

$$OB=BC\cos OBC, \text{ i.e. } Q=30\cos 30°=15\sqrt{3} \text{ lbs. wt.}$$

Also, $$OC=BC\cos OCB, \text{ i.e. } R=30\cos 60°=15 \text{ lbs. wt.}$$

5. Let $3P$ and $5P$ be the forces, and nP be their resultant. Take the second figure in Art. 27, with $3P$ for Q, nP for R, $5P$ for P, and OC at right angles to OB. Then, since $BC^2=OB^2+OC^2$, we have

$$(5P)^2=(3P)^2+(nP)^2,$$

$$\therefore\ 25=9+n^2, \text{ i.e. } n^2=16, \text{ and } n=4,$$

Hence $$5P:nP=5:4.$$

6. If P and Q be the forces, and R their resultant be perpendicular to Q, we have

$$P+Q=18\ldots(1) \qquad R=12\ldots(2)$$

and $$P^2-Q^2=R^2, \text{ i.e. } (P+Q)(P-Q)=R^2\ldots(3).$$

Substituting from (1) and (2) in (3), we have

$$18(P-Q)=144, \text{ i.e. } P-Q=8\ldots(4).$$

From (1) and (4), we have $P=13$, and $Q=5$.

7. Let the forces P and Q be represented by OA and OB respectively; complete the parallelogram $OACB$ with the diagonal OC (which represents R) equal to OA. Produce OA to D, making $AD=OA$, and complete the parallelogram $ODEB$; then OE represents the new resultant. Also $CE=CB=CO$; hence the angle BOE is a right angle, being an angle in a semicircle, and therefore OE is at right angles to OB.

8. Let the forces P and Q be represented by OA and OB respectively; complete the parallelogram $OACB$; the diagonal OC represents the resultant $\sqrt{3}Q$, and the $\angle AOC=30°$. We have

$$\frac{\sin OAC}{OC}=\frac{\sin COA}{AC}, \text{ i.e. } \frac{\sin OAC}{\sqrt{3}Q}=\frac{\sin 30°}{Q},$$

so that $\sin OAC=\frac{\sqrt{3}}{2}$, *i.e.* the $\angle OAC=60°$ or $120°$.

If the $\angle OAC=60°$, then $OA=2 \, . \, AC$, *i.e.* $P=2Q$.

If the $\angle OAC=120°$, then the $\angle ACO=30°$, *i.e.* $P=Q$.

9. Since the direction of the resultant is unaltered when the first force becomes $4P$ and the second force becomes $P+12$ lbs. wt., the ratios of the components in the two cases must be the same. Hence

$$\frac{2P}{P}=\frac{4P}{P+12}.$$

$$\therefore P+12=2P, \text{ and } P=12 \text{ lbs. wt.}$$

10. We have $(2m+1)^2(P^2+Q^2)=P^2+Q^2+2PQ\cos\theta$,

and $(2m-1)^2(P^2+R^2)=P^2+Q^2+2PQ\sin\theta$;

$\therefore (P^2+Q^2)(4m^2+4m)=2PQ\cos\theta$,

and $(P^2+Q^2)(4m^2-4m)=2PQ\sin\theta$;

$$\therefore \tan\theta=\frac{4m^2-4m}{4m^2+4m}=\frac{m-1}{m+1}.$$

11. Let a be the angle between P and Q; then we have

$R^2=P^2+Q^2+2PQ\cos a$ ....(1)

$4R^2=P^2+4Q^2+4PQ\cos a$ ....(2)

and $4R^2=P^2+Q^2+2PQ\cos(180°-a)$,

i.e. $4R^2=P^2+Q^2-2PQ\cos a$ ....(3).

From (1) and (2), $2R^2=2Q^2-P^2$;

" (2) " (3), $12R^2=3P^2+6Q^2$, *i.e.* $4R^2=2Q^2+P^2$.

Hence, by addition, $6R^2=4Q^2$, *i.e.* $3R^2=2Q^2$;

and, by subtraction, $2R^2=2P^2$, *i.e.* $R^2=P^2$;

$$\therefore \frac{P^2}{2}=\frac{Q^2}{3}=\frac{R^2}{2}, \text{ i.e. } P:Q:R=\sqrt{2}:\sqrt{3}:\sqrt{2}.$$

12. Let the forces P and Q be represented by OA and OB respectively; complete the parallelogram $OACB$; the diagonal OC represents the resultant R. Produce OA to D, making $AD=OC$, and complete the parallelogram $ODEB$; join OE; then OE represents the resultant of $(P+R)$ and Q; and the $\angle COD=\theta$. Now $CE=AD=OC$; therefore the $\angle CEO=$ the $\angle COE$. But the $\angle CEO=$ the $\angle EOD$, since OD is parallel to CE; therefore the $\angle COE=$ the $\angle EOD=\frac{\theta}{2}$. Q. E. D.

13. If AB represent the force P, which is turned through an angle α and then represented by AD, the system which was in equilibrium has had the force represented by AB taken away from it and the force represented by AD added to it. Hence, if BA be produced to C, so that $AC=AB$, the system is now equivalent to forces represented by AD and AC, which have a resultant (R, say) represented by AE, a diagonal of the parallelogram of which AC and AD are adjacent sides; and since $AD=AC$, therefore AE bisects the $\angle CAD$, so that the $\angle BAE=\frac{\pi}{2}+\frac{\alpha}{2}$. Hence, if α become $\alpha+\beta$, the $\angle BAE$ will become $\frac{\pi}{2}+\frac{\alpha+\beta}{2}$; *i.e.* if AD be turned through a further angle β, R turns through a further angle $\frac{\alpha+\beta}{2}+\frac{\pi}{2}$ *i.e.* $\frac{\beta}{2}$, so that the inclination of R alters by half the amount that that of P does.

14. If OP be the line of action of the given force, and Q and R be the given points, the components will be equal if their directions are equally inclined to OP. Hence, if RM be perpendicular to OP and be produced to S so that $SM=MR$, then SQ or QS meets OP in T so that QT and RT are equally inclined to OP; and therefore if TP represent the given force in magnitude and UV be drawn through the centre of TP perpendicular to TP and meeting TR and TQ in U and V the required components are TU and TV.

15. If A and B be the given points, and the forces P and Q along AC and BC meet in C, their resultant passes through C; and if the directions of the two forces be turned round A and B through equal angles CAD and CBD, the resultant will now pass through D, and meet its former direction in some point E. Also, since the $\angle CAD=$ the $\angle CBD$, a circle would go round $BACD$; and since the $\angle ACB=$ the $\angle ADB$, the resultant is unaltered in magnitude and must make the same angle ADE with AD as the angle ACE with AC, *i.e.* the $\angle ADE=$ the $\angle ACE$; therefore a circle would go round $ACDE$. Hence, since there is but one circumscribing circle to the triangle ACD, E lies on the circle round ACB. Also the forces P and Q being given and the angle ACB always the same the direction of the resultant divides ACB into angles the ratio of whose series is known. Hence the angle BCE is known and therefore E is a fixed point.

16. By Art. 42, Cor., the resultant of the forces represented by PA and PB must pass through F, the middle point of AB; also it passes through P and through C; hence P must lie on the straight line CF.

17. If P be the given force represented by AB acting at A, on a definite scale, then A and B are fixed; and if the other components be Q (which is invariable) represented by AC, and R, then, completing the parallelogram $ACBD$, R is represented by AD; also $BD=AC$, so that DB is constant; hence the locus of D is a definite circle with B as centre.

18. Let D, E and F be the middle points respectively of the sides BC, CA and AB of the triangle ABC, and P be any point. Then forces represented by PB and PC have a resultant represented by $2PD$; forces represented by PC and PA have a resultant represented by $2PE$; and forces represented by PA and PB have resultant represented by $2PF$. Hence the system $2PA$, $2PB$ and $2PC$ is equivalent to the system $2PD$, $2PE$ and $2PF$; *i.e.* the system PA, PB and PC is equivalent to the system PD, PE and PF.

19. Let $ABCD$ be the quadrilateral, and P be the required point. Join AC and BD, and bisect them in E and F, respectively. Then forces represented by PA and PC have resultant represented by $2PE$; and forces represented by PB and PD have resultant represented by $2PF$; hence, for equilibrium, PE and PF must lie in one straight line and be equal and opposite; therefore P is at the middle point of EF.

20. Through B draw a line parallel to AC to meet CD in L. Then forces represented by AB and BC are equivalent, by the triangle of forces, to a force represented by AC acting at B or at L.

The resultant of the first three forces is therefore the resultant of two forces acting at L represented respectively by AC and CD, *i.e.* by the triangle of forces, is represented by a force at L equal and parallel to AD.

Finally this force and the fourth force are equivalent to a force represented by $2AD$ acting at the middle point of DL.

21. By the polygon of forces, the force represented by AB is equivalent to forces represented by AH, HF and FB; the force represented by DC is equivalent to forces represented by DH, HF and FC; but the forces represented by AH and FB neutralise the forces represented by DH and FC, respectively; hence the resultant is parallel to HF and equal to $2HF$.

22. The force represented by EG is equivalent to forces represented by EA, AD and DG; and the force represented by HF is equivalent to forces represented by HD, DC and CF. Also, the force represented by EG is equivalent to forces represented by EB, BC and

CG; and the force represented by HF is equivalent to forces represented by HA, AB and BF. Hence the system $2EG$ and $2HF$ is equivalent to the system AD and DC, and AB and BC, *i.e.* to AC and AC; and, therefore, the resultant of forces represented by EG and HF is represented by AC.

23. Through O the centre of the circle draw OC and OD perpendicular respectively to A_1PA_3 and A_2PA_4.

Then, since C is the middle point of A_1A_3, we have

$$PA_1 - PA_3 = PC + CA_1 - (CA_3 - PC) = 2PC.$$

Similarly $$PA_2 - PA_4 = 2PD.$$

The resultant of the four forces is therefore the resultant of $2PC$ and $2PD$. *i.e.* is represented by $4PE$, where CD meets PO in E.

Now, if a be the common inclination of A_1A_3 and A_2A_4 to the line PO, we have

$$PE = PC \cos a = PO \cos^2 a.$$

Hence the resultant is independent of the radius of the circle.

EXAMPLES. V. (Pages 33–35.)

1. Here $$X = 1 + 2\cos 60^\circ = 1 + 1 = 2.$$

$$Y = \sqrt{3} + 2\sin 60^\circ = \sqrt{3} + \sqrt{3} = 2\sqrt{3};$$

$$\therefore\ F = \sqrt{X^2 + Y^2} = \sqrt{4 + 12} = 4 \text{ lbs. wt.},$$

and $$\tan\theta = \frac{2\sqrt{3}}{2} = \sqrt{3} = \tan 60^\circ, \text{ i.e. } \theta = 60^\circ,$$

so that the resultant is a force of 4 lbs. wt. in the direction AQ.

2. Taking the force of 5 lbs. wt. in the direction OX, the force of 3 lbs. wt. in the direction OY, and the force of 4 lbs. wt. in the direction bisecting the angle XOY, we have

$$X = 5 + 4\cos 45^\circ = 5 + 2\sqrt{2},$$

and $$Y = 3 + 4\sin 45^\circ = 3 + 2\sqrt{2};$$

$$\therefore\ F = \sqrt{X^2 + Y^2} = \sqrt{50 + 32\sqrt{2}} = 9{\cdot}76 \text{ lbs. wt.},$$

and $$\tan\theta = \frac{3 + 2\sqrt{2}}{5 + 2\sqrt{2}} = \frac{7 + 4\sqrt{2}}{17},$$

$$\text{i.e. } \theta = \tan^{-1}\frac{7 + 4\sqrt{2}}{17} = 36^\circ\, 40'.$$

3. Let the three forces be represented by the equal straight lines OB, OC and OA, so that the $\angle BOA$ = the $\angle COA = 60^\circ$. The resultant of P in the direction OB and P in the direction $OC = 2P\cos 60^\circ = P$, in the direction OA; therefore the required resultant is $2P$ in the direction OA.

4. Taking the force $13P$ in the direction OX, the force $10P$ in the second quadrant in the direction at 120° to OX, and the force $5P$ in the third quadrant in the direction at 120° to the directions of forces $13P$ and $10P$, we have

$$X = 13P - 10P\cos 60° - 5P\cos 60° = \frac{11}{2}P,$$

and $$Y = 10P\sin 60° - 5P\sin 60° = \frac{5\sqrt{3}}{2}P;$$

$$\therefore\ F = \sqrt{X^2 + Y^2} = \frac{P}{2}\sqrt{121 + 75} = 7P;$$

also, if θ be the inclination of the direction of the resultant with the third force, $13P$, we have

$$\cos\theta = \frac{11}{2}P \div 7P = \frac{11}{14},\ i.e.\ \theta = \cos^{-1}\frac{11}{14} = 38°\,13',\ \text{nearly.}$$

5. Forces represented by $2P$, $2P$ and $2P$ in the given directions are in equilibrium [cf. the first figure in Art. 36], and may be removed, leaving forces represented by P and $2P$ acting at an angle of 120°. Hence, by Art. 27, we have

$$R = \sqrt{(2P)^2 + P^2 + 2 \cdot 2P \cdot P\cos 120°} = \sqrt{5P^2 - 2P^2} = P\sqrt{3}.$$

Again, since $(P\sqrt{3})^2 = (2P)^2 - P^2$, the $\angle ACB$ (Fig. Art. 36) is a right angle. Hence the $\angle CAB = 30°$, the $\angle CBA$ being 60°.

6. Through O draw the two fixed lines OX and OY perpendicular to BC and AB respectively. Let the force P_3 act along OY, and the force P_4 along OX, P_1 acting along OA in the second quadrant and P_2 along OB in the first quadrant. Let $P_1 = 4P$, $P_2 = 6P$, $P_3 = 5P$, and $P_4 = P$. Then we have

$$X = P_4 + P_2\cos 45° + 0 - P_1\cos 45°$$

$$= P\left(1 + \frac{6}{\sqrt{2}} - \frac{4}{\sqrt{2}}\right) = P(1 + \sqrt{2}),$$

and $$Y = 0 + P_2\sin 45° + P_3 + P_1\sin 45°$$

$$= P\left(\frac{6}{\sqrt{2}} + 5 + \frac{4}{\sqrt{2}}\right) = 5P(1 + \sqrt{2});$$

$$\therefore\ F = \sqrt{X^2 + Y^2} = P(1 + \sqrt{2})\sqrt{26} = P \times 12{\cdot}31,$$

i.e. the resultant is proportional to 12·31, and if θ be the angle its direction makes with OX, i.e. with AB,

$$\tan\theta = \frac{Y}{X} = 5,$$

i.e. $$\theta = \tan^{-1}5,\ i.e.\ 78°\,41'.$$

7. Here we have

$$X=1+6\cos 45^\circ=1+3\sqrt{2},$$
$$Y=9+6\sin 45^\circ=9+3\sqrt{2};$$
$$\therefore\ F=\sqrt{X^2+Y^2}=\sqrt{118+60\sqrt{2}}=14\cdot24 \text{ lbs. wt.}$$

8. Draw a figure similar to that on p. 31. Let the two forces of 4 lbs. wt. act in the directions OX and OB at 60° to each other, the force of 1 lb. wt. in the direction OC at 60° to OB, and the force of 3 lbs. wt. in the direction OX'. The force of 4 lbs. wt. in the direction OX and the force of 3 lbs. wt. in the direction OX' have resultant 1 lb. wt. in the direction OX; forces of 1 lb. wt. in the direction OX and 1 lb. wt. in the direction OC have resultant 1 lb. wt. in the direction OB [since they act at 120°, and, therefore, have resultant $=2\cos 60^\circ=1$]; therefore the resultant of the four given forces is 5 lbs. wt. in the direction OB. Hence, for equilibrium, the required force is 5 lbs. wt. in the direction opposite to OB, *i.e.* opposite to the direction of the second force.

9. If the angles between P and Q, Q and R, and R and $S=a$, the angle between P and $S=3a=108^\circ$, *i.e.* $a=36^\circ$; and the forces being all equal, their resultant clearly acts in the direction bisecting the angle between Q and R. If each force be equal to P, the resultant of P and $S=2P\cos 54^\circ$; and the resultant of Q and $R=2P\cos 18^\circ$; hence the required resultant

$$=2P(\cos 54^\circ+\cos 18^\circ)=4P\cos 36^\circ\cos 18^\circ=\frac{P}{4}(\sqrt{5}+1)\sqrt{10+2\sqrt{5}}.$$

10. If $ABCDEF$ be the hexagon, and the given forces respectively act at A in the directions AB, AC, AD, AE and AF, the resultant obviously acts in the direction AD, and

$$=5+2\sqrt{3}\cos 30^\circ+2\,.\,2\cos 60^\circ=10 \text{ lbs. wt.}$$

11. Let $ABCDEF$ be the hexagon, and let the given forces respectively act at A in the directions AB, AC, AD, AE and AF. Take AB and AE coinciding with the fixed lines OX and OY. Then we have

$$X=2+3\cos 30^\circ+4\cos 60^\circ-6\cos 60^\circ=1+\frac{3\sqrt{3}}{2},$$

and $$Y=5+3\sin 30^\circ+4\sin 60^\circ+6\sin 60^\circ=\frac{13}{2}+5\sqrt{3};$$

$$\therefore\ F=\sqrt{X^2+Y^2}=\sqrt{125+68\sqrt{3}}=15\cdot58 \text{ lbs. wt.};$$

and $$\tan\theta=\frac{13+10\sqrt{3}}{2+3\sqrt{3}}=\frac{64+19\sqrt{3}}{23}=4\cdot213,$$

i.e. $$\theta=\tan^{-1}4\cdot213=76^\circ 39'$$

12. Let $ABCDE$ be the pentagon, and the forces of 7, 1, 1 and 3 lbs. wt. act along AB, AC, AD and AE respectively. Let XOX' and YOY' be the two fixed lines as on Page 31—OX coinciding with AB.

Then the $\angle ABC = \frac{1}{5}(5\pi - 2\pi) = \frac{3}{5}\pi$, *i.e.* the $\angle BAC = \frac{1}{2}\cdot\frac{2}{5}\pi = \frac{\pi}{5} = 36°$;

the $\angle DAB = \frac{2}{5}\pi = 72°$, and the $\angle EAB = \frac{3}{5}\pi = 108°$.

Hence we have

$$X = 7 + 1 \,.\, \cos 36° + 1 \,.\, \cos 72° - 3 \cos 72°$$

$$= 7 + \frac{\sqrt{5}+1}{4} - 2 \,.\, \frac{\sqrt{5}-1}{4} = \frac{31-\sqrt{5}}{4},$$

$$Y = 1 \,.\, \sin 36° + 1 \,.\, \sin 72° + 3 \,.\, \sin 72°$$

$$= \frac{\sqrt{10-2\sqrt{5}}}{4} + 4 \,.\, \frac{\sqrt{10+2\sqrt{5}}}{4},$$

$$\therefore\ F = \sqrt{X^2+Y^2} = \frac{1}{4}\sqrt{1136} = \sqrt{71} \text{ lbs. wt.}$$

13. If the equal forces P act on the angular point A of the octagon $ABCDEFGH$, in the directions AB, AC, AD, AE, AF, AG and AH, their resultant acts in the direction AE, by symmetry, and

$$= 2P\cos 67\tfrac{1}{2}° + 2P\cos 45° + 2P\cos 22\tfrac{1}{2}° + P$$

$$= P[\sqrt{2-\sqrt{2}} + \sqrt{2} + \sqrt{2+\sqrt{2}} + 1]$$

$$= P[\cdot 765 + 1\cdot 414 + 1\cdot 847 + 1]$$

$$= P \times 5\cdot 027.$$

14. Taking OX as the fixed line, we have

$$X = 11\cos 18°\,18' + 7\cos 74°\,50' - 8\cos 49°\,40'$$

$$= 10\cdot 4436805 + 1\cdot 8313939 - 5\cdot 1778672$$

$$= 7\cdot 0972072;$$

$$Y = 11\sin 18°\,18' + 7\sin 74°\,50' + 8\sin 49°\,40'$$

$$= 3\cdot 4539175 + 6\cdot 7561823 + 6\cdot 0983352$$

$$= 16\cdot 3084350;$$

whence $F = \sqrt{X^2+Y^2} = 17\cdot 79$ lbs. wt.,

and $\tan\theta = \frac{Y}{X}$,

so that $L\tan\theta = 10 + \log Y - \log X$, whence $\theta = 66°\,29'$.