MATRIX ANALYSIS FOR ELECTRICAL ENGINEERS

ALEXANDER VON WEISS

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FOR ELECTRICAL ENGINEERS

Alexander von Weiss

State Academy for Applied Technology, Nuremberg

Translated by EGON BRENNER

Associate Professor of Electrical Engineering
The City College of the City University of New York

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Translator's Preface

The job of translating and adopting the von Weiss book for this American edition is one that I tackled with considerable enthusiasm because I believe that the book fills a real need for many electrical engineers and engineering students. *Matrix analysis* and, more generally, *linear algebra* play an increasingly important part in engineering analysis and therefore a concise book that treats the subject thoroughly, yet at a mathematical level suited to the engineer's background, can be of considerable value. I believe that Dr. von Weiss has done an excellent job not only of selecting material but also of presentation. He has succeeded in merging real mathematics and practical application. The organization of his book is methodical without being pedantic, the numerical examples are worked in detail without inclusion of trivia, and the presentation is generally lucid. Except for the addition of some footnotes, the omission of German references, some minor modifications, and the shortening of Chapter 8, I have tried to be faithful to the original in this translation.

I am happy to acknowledge with gratitude the invaluable assistance rendered by Miss S. Silverstein in the preparation of the manuscript.

EGON BRENNER

New York, N.Y.

Author's Preface

In electrical engineering one often deals with linear relationships between voltages and currents. Matrix analysis is a useful tool for the solution of linear problems and can be used advantageously in electrical engineering. Thus it is the basic aim of this book to acquaint the electrical engineer with the basic concepts of matrix analysis and to convey insight into several applications of matrix methods to linear problems in engineering. The book is based on a series of lectures presented by the author to the German Engineering Society (V.D.E.) in the district North Bavaria. Intended to be an elementary introduction for electrical engineers, this book is tailored to the needs and background of such engineers.

Although the mathematical background required of the reader is elementary (as indicated by the level of the first few sections), the treatment permits practical applications to electrical engineering problems from the beginning. Where mathematical proofs would become burdensome, they have been omitted; the reader can always refer to the literature. Thus the reader is free from that which has little or no relevance to applications in electrical engineering, yet the presentation is mathematically and logically complete. About one third of the contents consists of numerical examples and exercises which are intended to acquaint the reader with the carrying out of actual operations. Applications are invariably chosen from electrical engineering; most problems involve network relationships. The choice of examples is based on the assumption that confidence in using a procedure is gained initially through the simplest examples worked out in complete detail.

It is of course understood that the potentialities of matrix analysis in electrical engineering are limited; when solving a particular example one must consider whether matrix techniques can be applied meaningfully. The solution of difficult problems naturally requires mathematical sophistication and physical insight; matrix analysis does not obviate these requirements.

The author hopes that this book will also serve to stimulate the reader to undertake further study in the interesting field of matrix analysis and linear algebra—may this book be a bridge leading to such study!

I owe special thanks to my colleague, Dr. Helmut Dietz, Nuremberg, who has scrutinized the manuscript with the eyes of a mathematician. I also thank him most cordially for his interest in bringing this book into being as well as for his numerous suggestions for the improvement of the presentation. Further I thank my colleague, Mr. Wilhelm Baumann, Bavaria, for his help in reading proof and for some sound advice. Finally I must thank the publishers for their constant willingness to accede to my requests as well as for their efforts to produce this book at a reasonable price.

A. von Weiss

Nuremberg

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Chapter 1

DETERMINANTS

In this chapter the concept of determinants is derived and explained. In addition, certain rules and theorems dealing with determinants are concisely reviewed.

1-1 Basic Concepts and Rules

1-1a Concept of determinants. To arrive at the concept of determinants, consider a linear system consisting of two equations with two unknowns, x and y, and constant coefficients:

$$\begin{aligned} a_{11}x+a_{12}y &= c_1 \\ a_{21}x+a_{22}y &= c_2 \end{aligned} \tag{1-1}$$

Solving for x and y, one obtains

$$\begin{aligned} x(a_{11}a_{22}-a_{12}a_{21}) &= c_1a_{22}-a_{12}c_2 \\ y(a_{11}a_{22}-a_{12}a_{21}) &= a_{11}c_2-c_1a_{21} \end{aligned} \tag{1-1a}$$

The expression in the parentheses on the left side of Eq. (1-1a) is an entire, rational function of the coefficients† a_{ik}. Following Leibnitz, for this function one writes diagrammatically

$$a_{11}a_{22}-a_{12}a_{21} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = |a| \tag{1-2}$$

This square array corresponds to the array of the coefficients in Eq. (1-1) and is termed *determinant*. In the foregoing example the determinant consists of two *rows* (horizontal) and two *columns* (vertical).

†The subscript of a_{11} should be read as one-one, *not* eleven; for a_{12} read a-one-two not a-twelve, etc. The notation used is the "double subscript notation, the first subscript indicating the row, and the second the column position of the element in the determinant or matrix.

The right side of each of the equations (1-1a) can also be written as determinants. We denote these by Δ_x and Δ_y,

$$c_1 a_{22} - a_{12} c_1 = \begin{vmatrix} c_1 & a_{12} \\ c_2 & a_{22} \end{vmatrix} = \Delta_x$$

$$a_{11} c_2 - c_1 a_{21} = \begin{vmatrix} a_{11} & c_1 \\ a_{21} & c_2 \end{vmatrix} = \Delta_y$$

Consider now a system of three linear equations with constant coefficients a_{ik}, and three unknowns, x, y, and z,

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= c_1 \\ a_{21}x + a_{22}y + a_{23}z &= c_2 \\ a_{31}x + a_{32}y + a_{33}z &= c_3 \end{aligned} \tag{1-3}$$

and multiply in Eqs. (1-3)

the first line by $+(a_{22}a_{33} - a_{23}a_{32})$

the second line by $-(a_{12}a_{33} - a_{13}a_{32})$

the third line by $+(a_{12}a_{23} - a_{13}a_{22})$

If we add the three resulting equations, all terms involving y and z vanish and the result is

$$\begin{aligned} &[a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})]x \\ &= c_1(a_{22}a_{33} - a_{23}a_{32}) - c_2(a_{12}a_{33} - a_{13}a_{32}) + c_3(a_{12}a_{23} - a_{13}a_{22}) \end{aligned}$$

Defining the left side of the above equation as $|a|x$, one obtains by comparison with Eq. (1-2)

$$|a| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

or

$$|a| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = a_{11}\Delta_{11} - a_{12}\Delta_{12} + a_{13}\Delta_{13} \tag{1-4}$$

The factors multiplying the two-row two-column determinants are the coefficients of the first equation of the system (1-3). Therefore one defines $|a|$ as a three-row determinant

$$|a| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \tag{1-4a}$$

that is, $|a|$ is the array of coefficients in the equation system (1-3). One now finds that each of the three two-row determinants in the above development result if one deletes the first row together with one of the columns in the determinant (1-4a). Lines drawn to delete the appropriate row and column intersect at that element of the first row which is the multiplying factor (in Eq. 1-4) for the remaining two-row determinant.

A corresponding development can be applied to two-row determinants. It is in fact given in Eq. (1-2) if one sets $\Delta_{11} = a_{22}$ and $\Delta_{21} = a_{12}$ and if in addition a "one-row" determinant is one that is formed by a single element. One recognizes further that a two-row determinant consists of the sum of two one-row determinants; according to Eq. (1-2) this sum has $1 \cdot 2 = 2!$ product terms each consisting of two elements. Similarly, a three-row determinant consists of three two-row determinants and has $1 \cdot 2 \cdot 3 = 3!$ products, each consisting of three elements.

Generalizing the scheme of Eq. (1-4a) to n-columns and n-rows, one obtains an n-row determinant also termed a determinant of nth *order*:

$$\det(a) = |a| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots a_{2n} \\ \cdot & & \cdot \\ \cdot & & \\ a_{n1} & a_{n2} & a_{n3} \cdots a_{nn} \end{vmatrix} \tag{1-5}$$

In this scheme,† a_{ik} is the element in the ith row, kth column, and the ith row is:

$$a_{i1}, a_{i2}, \ldots, a_{in}$$

the kth column is:

$$\begin{matrix} a_{1k} \\ a_{2k} \\ \cdot \\ \cdot \\ \cdot \\ a_{nk} \end{matrix}$$

The elements a_{ii} form the *principal diagonal* (also referred to as "diagonal" for brevity); the other diagonal is termed *secondary* or *conjugate* diagonal of the determinant.

Expanding the nth order determinant in the manner indicated above, one obtains

$$\det(a) = a_{11}\Delta_{11} - a_{12}\Delta_{12} + a_{13}\Delta_{13} + \quad - \quad \ldots + (-1)^{1+n}\Delta_{1n} \tag{1-5a}$$

where the n determinants Δ_{1k} are obtained by deleting row 1 and column k in det (a), as indicated above, and are of order $(n-1)$. Each of these determinants can be expanded further into $n-1$ determinants of order $n-2$ and so on. Finally one obtains a sum of $n!$ product terms each

† The notation det (a) in place of $|a|$ is generally more convenient because no confusion with absolute value signs is possible.

consisting of n elements, that is a *numerical value*. From the foregoing discussion one concludes as follows:

> A determinant is a square array of numbers with n columns and n rows (n^2 elements). The *value* of the determinant is determined as a sum of $n!$ terms and is formed by expansion of the elements according to certain rules.

1-1b Expansion theorem, minors. The expansion of the determinant in the example of Eq. (1-4) is the expansion "about the first row." Expansions can be made about any other row or about any column. One can easily verify that expansion about any row yields the same result. For example, expansion about row 2 of the third-order determinant, Eq. (1-4a), gives

$$\det(a) = -a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

or

$$\det(a) = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} = -a_{21}\Delta_{21} + a_{22}\Delta_{22} - a_{23}\Delta_{23}$$

Using row 3, one obtains

$$\det(a) = a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

or

$$\det(a) = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} = a_{31}\Delta_{31} - a_{32}\Delta_{32} + a_{33}\Delta_{33}$$

Correspondingly, the above determinant can also be expanded about a column. Using, for example, the second column.

$$\det(a) = -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

or

$$\det(a) = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} = -a_{12}\Delta_{12} + a_{22}\Delta_{22} - a_{32}\Delta_{32}$$

In general, an nth order determinant can be expanded about the ith row:

$$\det(a) = \sum_{m=1}^{n} a_{im}A_{im} = \sum_{m=1}^{n} a_{im}(-1)^{i+m}\Delta_{im} \tag{1-6}$$

or about the kth column:

$$\det(a) = \sum_{m=1}^{n} a_{mk}A_{mk} = \sum_{m=1}^{n} a_{mk}(-1)^{m+k}\Delta_{mk} \tag{1-7}$$

The n determinants Δ_{im} (or Δ_{mk}) are of order $(n-1)$ and are termed *minors or subdeterminants.*

Laplace's Expansion Theorem. An nth order determinant can be expanded about any row or column into n minors; each minor will be of order $(n-1)$.

From the above one recognizes:

The minor Δ_{ik} of the element a_{ik} is found by deleting row i and column k in the original determinant. The minor is taken with positive sign when $(i+k)$ is an even number, and with negative sign when $(i+k)$ is an odd number.

The element a_{ik} is at the intersection of the lines deleting row i and column k; the signs assigned to the minors are distributed in a checkerboard pattern:

$$\begin{vmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & & & & \end{vmatrix}$$

The minor together with the correct sign [that is, multiplied by $(-1)^{i+k}$] is termed *cofactor* A_{ik} of the element a_{ik}.

For the third-order determinant of Eq. (1-4) the minor Δ_{12} is obtained by deleting the first row and the second column. Since $1+2 = 3$ is odd,

$$A_{12} = -\Delta_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

Similarly the minor Δ_{22} is obtained by deleting the second row and the second column, also $2+2 = 4$, an even number, and

$$A_{22} = \Delta_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

1-1c Miscellaneous theorems and rules. Without proof we now cite important rules and theorems for calculation with determinants.

RULE 1 The value of a determinant does not change when all rows are interchanged with their corresponding columns. Thus rows and columns play the same role in determinants.

RULE 2 Exchanging two rows or two columns changes the sign of the determinant.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix}$$

From the expansion about a row or column, we have rule 3:

RULE 3 Multiplication of a determinant by a scalar factor p is identical to multiplying all elements of *one* row *or* of *one* column by p, for example:

$$p \det(a) = p \cdot \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} pa_1 & pb_1 & pc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & pb_1 & c_1 \\ a_2 & pb_2 & c_2 \\ a_3 & pb_3 & c_3 \end{vmatrix}$$

RULE 4 If two rows are identical or if two columns are identical, then the value of the determinant is zero.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = 0, \quad \begin{vmatrix} a_1 & na_1 & c_1 \\ a_2 & na_2 & c_2 \\ a_3 & na_3 & c_3 \end{vmatrix} = n \cdot \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = 0$$

RULE 5 If all the elements in a row or column are zero, then the value of the determinant is zero

$$\begin{vmatrix} a_1 & 0 & c_1 \\ a_2 & 0 & c_2 \\ a_3 & 0 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

RULE 6 If the elements in one row or in one column are each the sum of the same number of terms, then the determinant can be evaluated as the sum of the same number of determinants.

$$\begin{vmatrix} a_1+p_1+q_1 & b_1 & c_1 \\ a_2+p_2+q_2 & b_2 & c_2 \\ a_3+p_3+q_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} p_1 & b_1 & c_1 \\ p_2 & b_2 & c_2 \\ p_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} q_1 & b_1 & c_1 \\ q_2 & b_2 & c_2 \\ q_3 & b_3 & c_3 \end{vmatrix}$$

Using rule 6 together with rule 4 yields rule 7:

RULE 7 The value of a determinant remains unchanged when one adds to any column (row) another column (row) which has been multiplied by any scalar factor p.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+pb_1 & b_1 & c_1 \\ a_2+pb_2 & b_2 & c_2 \\ a_3+pb_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2+pa_1 & b_2+pb_1 & c_2+pc_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

In evaluation of determinants rule 7 together with the expansion about a row or column is often used (see Sec. 1-2).

1-1d Cramer's rule. If in Eq. (1-1) the determinant of the coefficients is not zero, that is, if

$$\det(a) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$$

then, from Eq. (1-2)

$$x = \frac{1}{\det(a)} \cdot \begin{vmatrix} \overset{\downarrow}{c_1} & a_{12} \\ c_2 & a_{22} \end{vmatrix} = \frac{\Delta_x}{\det(a)}$$

and

$$y = \frac{1}{\det(a)} \cdot \begin{vmatrix} a_{11} & \overset{\downarrow}{c_1} \\ a_{21} & c_2 \end{vmatrix} = \frac{\Delta_y}{\det(a)}$$

Corresponding results are obtained for n equations with n unknowns. Given the system of linear equations with constant coefficients

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= y_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= y_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n &= y_n \end{aligned} \tag{1-8}$$

for which the determinant of the coefficients is assumed not zero:

$$\det(a) = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{12} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} \neq 0$$

then the values $x_1, x_2, \ldots, x_n$ are determined uniquely by *Cramer's rule*:

$$x_k = \frac{\Delta_k}{\det(a)} \tag{1-9}$$

The determinant Δ_k is obtained by replacing the column of coefficients $a_{1k}, \ldots, a_{nk}$ in the determinant of coefficients by the values $y_1, \ldots, y_n$, respectively. Cramer's rule, illustrated subsequently, permits solution of a set of linear equations in elegant fashion when the determinant of the coefficients does not vanish. The practical application of this rule by hand is limited to cases where reasonably low-order determinants occur since evaluation of fifth—or higher—order determinants becomes unduly tedious and time consuming.

If in Eq. (1-8) all values $y_1 = y_2 = \ldots = y_n = 0$, then the system of equations is homogeneous. In this case, according to rule 5 the numerators in Eq. (1-9), Δ_k all vanish. Hence if $\det(a) \neq 0$, only the trivial solutions $x_1 = x_2 = \ldots x_n = 0$ result. Nontrivial solutions are possible only when $\det(a) = 0$; as shown in Chap. 3, Sec. 3-2, this case results in an infinite number of solutions.