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Georges de Rham
Differentiable Manifolds

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Differentiable Manifolds

Forms, Currents, Harmonic Forms

Translated from the French
by F. R. Smith

Introduction to the English Edition
by S. S. Chern

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Préface à l'édition anglaise

Je tiens à remercier très sincèrement tous ceux qui ont rendu possible cette traduction de mon livre: d'abord Beno Eckmann, c'est grâce à lui que le projet a été réalisé; Shiing-Shen Chern qui a eu la grande amabilité d'écrire une Introduction; Springer-Verlag pour tous ses efforts et sa coopération; et le traducteur pour son travail consciencieux.

Voici quelques compléments historiques de nature personelle.

Attiré vers les Mathématiques dès 1924, c'est en 1928 et 1929 que j'ai fait ma thèse. En 1930, H. Lebesgue, à qui j'apportais mon manuscrit, m'a dirigé vers Elie Cartan, qui a bien voulu l'examiner et faire un rapport favorable. Ma thèse a paru en juin 1931, précédée de deux Notes aux Comptes Rendus en 1928 et 1929.

Peu après, S. Lefschetz écrit à H. Lebesgue "... vous nous rendriez grand service en suggérant à M. de Rham de nous envoyer quelques exemplaires de sa thèse. M. Hodge nous en a exposé la partie analytique ...".

Ce fut, semble-t-il, l'origine des travaux de Hodge publiés vers 1934–1936. L'énoncé de son théorème frappe par sa beauté et sa simplicité. Mais ses démonstrations m'ont paru très pénibles et trop difficiles. Ce qui m'a amené à reprendre le problème dans les travaux publiés en 1946. Pendant la guerre, les "Annals of Mathematics" ne parvenaient pas dans nos bibliothèques et j'ai ignoré le travail de H. Weyl auquel S. S. Chern fait heureusement allusion dans son Introduction. Je n'en ai eu connaissance que plus tard.

En 1950, à l'Institute for Advanced Study de Princeton, à la demande précisément de H. Weyl à qui je dois beaucoup pour l'intérêt qu'il m'a témoigné, j'ai fait une série d'exposés "Harmonic Integrals". De là, grâce à l'intérêt et l'amitié d'André Weil, est issu mon livre.

Lausanne, en mai 1984 — Georges de Rham

Introduction to the English Edition

William Hodge's theory of harmonic integrals was both bold and imaginative. In one step he found the key to the n-dimensional generalization of geometric function theory. His proof of the fundamental theorem contained a serious gap. This was filled in a masterful way by Hermann Weyl, using his earlier results on potential theory.

Professor de Rham's book is an introduction to differentiable manifolds. Its main objective seems to be the first detailed proof, different from Hodge-Weyl, of Hodge's fundamental theorem. It must have given him great pleasure in writing the book, for Hodge theory is a natural culmination of the de Rham theory.

In n-dimensional geometry a fundamental notion is the "duality" between chains and cochains, or domains of integration and the integrands. While the boundary operator is a global operator, the coboundary operator, i.e. exterior differentiation, is local. This makes cohomology theory a more convenient tool for analytical treatment and for applications. Poincaré recognized the importance of the multiple integrals and stated the main "theorems", while Elie Cartan developed the foundations of the exterior differential algebra and applied it to mechanics, differential systems, and differential geometry. The global theory was completed by de Rham's famous thesis in 1931. The thesis was long, because at that time topology was homology theory and the notion of cohomology did not exist.

A notion which includes both chains and cochains is that of a "current". This was introduced by de Rham and used effectively throughout the book. A zero-dimensional current is a distribution (in the sense of Laurent Schwartz), now a fundamental concept in mathematics.

There are now other proofs of Hodge's theorem. Perhaps the most natural approach is through pseudo-differential operators; cf. [5], [6]. The Milgram-Rosenbloom proof using the heat equation method is an idea with broad repercussions [3] Morrey, Eells, and Friedrichs gave a proof using a variational method [4].

Hodge's theorem admits various extensions. The most important one is to cohomology theory with a coefficient sheaf, which was introduced by J. Leray and developed for the complex structure with great success by Henri Cartan and J-P. Serre [1], [6]. Its harmonic theory was first worked out by K. Kodaira [2]. When geometrical information is available, the harmonic theory allows the proof of "vanishing theorems" on cohomology groups, using an idea originated from S. Bochner. Such vanishing theorems are of great importance.

Modern developments in the general area of "elliptic operators on manifolds", such as the index theory and the spectral theory, have raced way beyond the content of this book. I believe, however, that in his enthusiasm for new results a mathematician will be well-advised to stop at this landmark, where he will have a lot to learn both on the mathematics and on the mathematical style.

Berkeley, February 1984 S. S. Chern

References

[1] Griffiths, P., Harris, J.: Principles of Algebraic Geometry. John Wiley 1978
[2] Kodaira, K.: On a differential-geometric method in the theory of analytic stacks. Proc. Nat. Acad. Sci., USA, vol. 29 (1953), 1268–1273
[3] Milgram, A.N., Rosenbloom, P.C.: Harmonic forms and heat conduction, I, II. Proc. Nat. Acad. Sci., USA, vol. 37 (1951), 180–184, 435–438
[4] Morrey, C.B.: Multiple Integrals in the Calculus of Variations. Grundlehren der math. Wiss. 130, Springer 1966
[5] Nirenberg, L.: Pseudo-differential operators, Global Analysis. Proc. Symp. Pure Math., vol. 16, Amer. Math. Soc., 149–167, 1970
[6] Wells, R.O.: Differential Analysis on Complex Manifolds. Prentice Hall, Inc. 1973; second edition, Graduate Texts in Mathematics, no. 65, Springer 1980

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Introduction

In this work, I have attempted to give a coherent exposition of the theory of differential forms on a manifold and harmonic forms on a Riemannian space.

The concept of a *current*, a notion so general that it includes as special cases both differential forms and chains, is the key to understanding how the homology properties of a manifold are immediately evident in the study of differential forms and of chains. The notion of *distribution*, introduced by L. Schwartz, motivated the precise definition adopted here. In our terminology, distributions are currents of degree zero, and a current can be considered as a differential form for which the coefficients are distributions.

The works of L. Schwartz, in particular his beautiful book on the Theory of Distributions, have been a very great asset in the elaboration of this work. The reader however will not need to be familiar with these. Leaving aside the applications of the theory, I have restricted myself to considering theorems which to me seem essential and I have tried to present simple and complete proofs of these, accessible to each reader having a minimum of mathematical background. Outside of topics contained in all degree programs, the knowledge of the most elementary notions of general topology and tensor calculus and also, for the final chapter, that of the Fredholm theorem, would in principle be adequate.

After the definition of differentiable manifolds, Chapter I establishes some results necessary for the sequel. In particular, the existence of partitions of unity and the theorem of Whitney concerning the problem of embedding a manifold in a Euclidean space are given.

Chapter II describes the elements of the theory of differential forms and differential chains together with the exterior differential calculus of E. Cartan and the generalised Stokes' formula. I have introduced and systematically used the notions of forms and chains of *even type* and *odd type*. These concepts allow the theory to be applied as well to non-orientable manifolds. Finally, the notion of *double form* prepares the way for the introduction of some other generalisations.

Chapter III is devoted to the definition and to the study of the general properties of currents. Certain properties of topological vector spaces which we require here are recalled along the way together with their proofs. I have introduced *double currents*, which generalise the *kernel distributions* of L. Schwartz, and the regularising operators which show in a precise way that a current can always be considered as the limit of a sequence of forms.

In Chapter IV, currents are used to define and study the homology groups of a manifold. We will find there complete proofs of theorems which provide a relationship between differential forms and chains from the homological viewpoint in a way analogous to the Poincaré duality theorem in a differentiable manifold.

Chapter V deals with the principles of the theory of harmonic differential forms in a Riemann space. First, the Hodge theorem is established using the integral equations method which in the case of compact spaces leads to the most complete results. The lemmas upon which this method rests are proved in detail. So too are all the properties of the geodesic distance which occur there. The method of orthogonal projection in a Hilbert space which allows us to proceed further in the non-compact case is considered next, and the decomposition theorem of Kodaira is generalised by the introduction of currents. We deduce from these formulas providing us with an integral representation of the Kronecker index of two chains. Next, by the method of E. E. Levi, we prove that the harmonic differential forms are analytic in an analytic Riemannian space. In particular, it follows that such a form cannot have a zero of infinite order unless it is identically zero. We make note that by virtue of a theorem of N. Aronsjan, A. Krzywicki and J. Szarski, this is moreover valid for harmonic forms in a C^{∞} Riemannian space. Finally we conclude with an interesting theorem of A. Andreotti and E. Vesentini which concerns square summable harmonic forms on complete, non-compact Riemannian spaces.

Chapter I. Notions About Manifolds

§ 1. The Notion of a Manifold and a Differentiable Structure

An n dimensional *manifold* is a separable topological space, each point of which has a neighbourhood homeomorphic to an open n dimensional ball. Moreover we shall always suppose that this space admits a *countable base* of open sets, that is, there exist a countable sequence of open sets such that any open set may be expressed as a union of sets of the sequence.

On an n dimensional manifold V a *differentiable structure of infinite order*, or, more briefly, a C^∞ structure, is defined by prescribing at each point x of V a class of real valued functions which are said to be C^∞ at x such that the following axiom is satisfied.

Axiom for C^∞ Structures. *For each point u of V there exists an open neighbourhood U of u and n real valued functions defined on U, $x_1(x), \ldots, x_n(x)$, such that :*

(a) The map $x \to (x_1(x), \ldots, x_n(x))$ of U into $\mathbb{R}^n$ is a homeomorphism of U onto a connected open subset of $\mathbb{R}^n$ such that each function f defined on U or on a subset of U can be expressed with the aid of $x_1, \ldots, x_n$,

$$f(x) = f(x_1(x), \ldots, x_n(x));$$

(b) The function $f(x)$ is C^∞ at a point of U if and only if there exists an open neighbourhood W of this point, contained in U, such that $f(x)$ is defined on W and such that $f(x_1, \ldots, x_n)$ has continuous derivatives of all orders for values of the variables $x_1, \ldots, x_n$ corresponding to points of W.

A function is said to be C^∞ if it is C^∞ at each point of V and C^∞ on a subset A of V if it is C^∞ at each point of A. It follows from (b) that the set of all points where a function is C^∞ is an open set.

Every system of n functions $x_1, \ldots, x_n$ defined in an open set U and having the properties (a) and (b) is called *a system of local coordinates in U*. Our axiom ensures the existence of such a system in an open neighbourhood of each point.

From (b), these coordinates are C^∞ functions in U. If $f_1, \ldots, f_n$ is another system of local coordinates in U, the functions $f_1, \ldots, f_n$ are C^∞ functions for which the Jacobian with respect to $x_1, \ldots, x_n$ is non-zero in U. On the other hand, by virtue of the classical implicit function theorem, if $f_1, \ldots, f_n$ are C^∞ functions on U for which the Jacobian with respect to $x_1, \ldots, x_n$ is $\neq 0$ at a point u' of U, there exists a neighbourhood U' of u' in which $f_1, \ldots, f_n$ are local coordinates.

In practice, a C^∞ structure is most often defined by an open covering $\{U_i\}$ of V with a system of local coordinates in each U_i. On the intersection $U_i \cap U_j$, the coordinates of one system must be infinitely differentiable functions of the coordinates of the other, in accordance with (b).

In the space $\mathbb{R}^n$, there exists a canonical C^∞ structure, given by the system of coordinates $x_1, \ldots, x_n$ defining $\mathbb{R}^n$; the C^∞ functions are infinitely differentiable functions of $x_1, \ldots, x_n$.

A function defined on a manifold with a C^∞ structure is said to be C^r, r being an integer $\geqq 0$, if, when expressed in terms of the local coordinates, it has continuous derivatives up to and including order r. A C^0 function is simply a continuous function.

A C^∞ *manifold* is a manifold with a C^∞ structure. As only C^∞ manifolds are to be considered in the following, we shall call them simply *manifolds*[1].

We may define analogously the notions of differentiable structure of order r and of real analytic structure: it is sufficient to modify condition (b) of the above axiom by demanding only that $f(x_1, \ldots, x_n)$ has continuous derivatives up to order r or that it is real analytic. Thus, we obtain the notions of C^r manifold and of real analytic manifold.

§2. Partition of Unity. Functions on Product Spaces[2]

Let $f(x)$ be a function equal to $\exp(-x^{-2})$ for $x>0$ and to 0 for $x \leqq 0$, and let

$$g(x) = \frac{\int_{-\infty}^{x} f(t) f(1-t)\, dt}{\int_{-\infty}^{\infty} f(t) f(1-t)\, dt}.$$

The function $h(x) = g(x+1) - g(x)$ is C^∞, $\geqq 0$ and zero outside of the interval $(-1, 1)$, and we have

$$\sum_{j=-\infty}^{\infty} h(x-j) = 1,$$

as is easily verified. By successively replacing x with $\frac{x_1}{\varepsilon}, \ldots, \frac{x_n}{\varepsilon}$ and multiplying together the relations obtained, it follows that

$$\sum_{j_1, \ldots, j_n} h\left(\frac{x_1}{\varepsilon} - j_1\right) h\left(\frac{x_2}{\varepsilon} - j_2\right) \ldots h\left(\frac{x_n}{\varepsilon} - j_n\right) = 1.$$

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[1] This definition of differentiable manifold is equivalent to that of Whitney, and, in this form, very near to that given by Chevalley (Whitney [1], Chevalley [1], Chapter III).

[2] For this §, c.f., Dieudonné [1] and Schwartz [2], p. 23.

Denoting the system of n integer indices $j_1,\ldots,j_n$ by the positive integers $j=j(j_1,\ldots,j_n)$, the point $(x_1,\ldots,x_n)$ of $\mathbb{R}^n$ by x and putting

$$\phi_j(x)=h\left(\frac{x_1}{\varepsilon}-j_1\right)\ldots h\left(\frac{x_n}{\varepsilon}-j_n\right),$$

this relation may be written as

$$\sum_j \phi_j(x)=1. \tag{1}$$

The function $\phi_j(x)$ is C^∞, $\geqq 0$ and vanishes outside the cube of side length 2ε and centre at the point $(j_1\varepsilon,\ldots,j_n\varepsilon)$, i.e., outside of the cube defined by the relations $|x_i-j_i\varepsilon|\leqq\varepsilon$ $(i=1,\ldots,n)$.

The *support* of a continuous function is defined to be the smallest *closed* set outside of which the function vanishes. The support of $\phi_j(x)$ is the cube described above.

Let A be a compact set in $\mathbb{R}^n$ and B an open set containing A. If ε is sufficiently small, the length of the diagonal of the cube of side length 2ε is less than the distance of A to the frontier of B, and therefore, if such a cube meets A, it is contained in B. Let us consider then the functions $\phi_j(x)$ with support meeting A. There is a finite number of them and their sum is a C^∞ function with compact support in B, its values lying entirely between 0 and 1 (the endpoints are not excluded) and equal to 1 on A. Thus, we have proved the following proposition.

Lemma. *If A is a compact set in $\mathbb{R}^n$ and B is an open set containing A, there exists a C^∞ function with compact support in B, with values lying between 0 and 1 and equal to 1 on A.*

This lemma will be used to prove the following theorem.

Theorem 1. *For any open covering $\{U_i\}$ of a manifold V, where i runs through any set of indices, it is possible to find a collection of functions ϕ_j, where j runs through a finite or countably infinite set of indices, satisfying the following conditions:*

1) $\phi_j\geqq 0$, $\quad \sum_j \phi_j=1$;

2) *ϕ_j is C^∞ and has compact support contained in one of the U_i;*

3) *Each point of V has a neighbourhood which meets only a finite number of the supports of the ϕ_j.*

If condition 1) is satisfied, the expression $\sum_j \phi_j$ is called a *partition of unity.* We express condition 2) by saying that this partition is subordinate to the covering U_i and condition 3) by saying that it is locally finite. By virtue of the Borel-Lebesgue theorem, this condition 3) is equivalent to the following:

3′) *Every compact set meets the supports of only a finite number of the ϕ_j.*

The collection of the ϕ_j will be finite depending on whether V is compact or not.

In order to prove this theorem, let us first show that there exists an open covering $\{G_i\}$ of V having the following properties:

(a) It is subordinate to the covering $\{U_i\}$, that is, the closure $\bar{G}_i$ of each G_i is contained in one of the U_j.

(b) It is locally finite, that is, a compact set can meet only a finite number of the G_i;

(c) The closure $\bar{G}_i$ of each G_i is contained in the domain of a coordinate system.

Since it is supposed that the manifold has a countable open basis, we know that from every open covering, a finite or countable covering is able to be extracted. For each point of V, we can find an open set U' which contains it and which has compact closure contained in one of the U_i and in the domain of a coordinate system; consequently, there exists a finite or countable covering $\{U_i'\}$ formed by such sets and this covering possesses properties (a) and (c). If it is not locally finite, such a one can be constructed by shrinking the U_i' in the following way.

Consider the sequence of compact sets $K_1, K_2, \ldots$ and the sequence of integers $j_1, j_2, \ldots$ defined recurrently by putting

$$K_1 = \overline{U_1'}, \quad j_1 = 1;$$

j_m = the smallest integer $> j_{m-1}$ such that $K_{m-1} \subset \bigcup_{j=1}^{j_m} U_j'$, $K_m = \bigcup_{j=1}^{j_m} \overline{U_j'}$. K_{m-1} is contained in the interior of K_m and any compact set is contained in K_m whenever m is sufficiently large. We then define the G by putting

$$G_i = U_i' \text{ if } i \leqq j_2,$$
$$G_i = U_i' \cap \mathscr{C} K_{m-1} \text{ if } j_m < i \leqq j_{m+1} \text{ with } m > 1,$$

where the symbol $\mathscr{C}$ denotes the complement in V.

In order to ensure that the G_i form a covering of V, it suffices to verify the relation

$$\bigcup_{i=1}^{j_{m+1}} G_i = \bigcup_{i=1}^{j_{m+1}} U_i'.$$

This relation is exact for $m=1$, as $G_i = U'$ if $i \leqq j_2$. Proceeding inductively, let us suppose that

$$\bigcup_{i=1}^{j_m} G_i = \bigcup_{i=1}^{j_m} U_i' = A.$$

The required relation then follows immediately from the fact that, as $K_{m-1} \subset A$ we have $G_i \cup A = U_i' \cup A$ for $j_m < i \leqq j_{m+1}$.

This covering $\{G_i\}$ has properties (a) and (c) because $G_i \subset U'_i$, and it also has property (b), that is, it is locally finite, since each compact set is contained in K_m for m sufficiently large and G_i does not meet K_m if $i > j_{m+1}$.

We can also find another open covering $\{H_i\}$ such that $\bar{H}_i \subset G_i$, and, from the lemma, there is a C^∞ function ψ_i with values lying between 0 and 1, with support in G_i and which is equal to 1 on H_i. The sum $\psi = \sum_i \psi_i$ is $\geqq 1$ everywhere and the function $\phi_j = \frac{\psi_j}{\psi}$ is C^∞, its support is contained in G_j and we have

$$\phi_j \geqq 0, \qquad \sum_j \phi_j = 1,$$

so that all the conditions of Theorem 1 are verified. □

This theorem implies the following proposition, which in the case where $V = \mathbb{R}^n$ reduces to the lemma proved above.

Corollary 1. *If A is a compact set in V and B is an open set containing A, there exists a C^∞ function which has compact support in B and which has its range of values lying between 0 and 1 and is equal to 1 on A.*

In fact, if $1 = \sum \phi_j$ is a partition of unity subordinate to the covering formed by B and the complement of A and satisfying the conditions of Theorem 1, the sum of the ϕ_j which have support meeting A is a function which has all the required properties. □

Corollary 2.[3] *Let $\{U_i\}$ be an open covering of V. There exists a collection of functions ϕ_i, where i runs through the same index set, such that*

1) $\phi_i \geqq 0, \quad \sum_i \phi_i = 1$;

2) *ϕ_i is C^∞ and its support is contained in U_i;*

3) *Each point of V has a neighbourhood which meets only a finite number of the supports of the ϕ_i.*

We note that Corollary 2, which easily follows from Theorem 1, contains Corollary 1 as a particular case: in fact, it is sufficient to consider the covering formed by the two sets $U_1 = \mathscr{C}A$ and $U_2 = B$. Also, it implies Theorem 1 in the case where the U_i are assumed to be relatively compact. Condition 1) of Corollary 2 implies of course that the number of the ϕ_i which are not identically zero is finite or countable.

Theorem 2. *Let U and W be open bounded intervals in $\mathbb{R}^l$ and $\mathbb{R}^m$ and let y, z and $x = (y, z)$ be variable points in $\mathbb{R}^l$, $\mathbb{R}^m$ and $\mathbb{R}^n = \mathbb{R}^l \times \mathbb{R}^m (n = m + l)$, respectively. Each C^∞ function of x, $\phi(x)$, with support in $U \times W$, can be approximated by a sequence of functions $\sigma_k(x)$ $(k = 1, 2, 3, \ldots)$, each of which is the sum of a finite number of products $\phi_1(y)\phi_2(z)$ of a C^∞ function of y, $\phi_1(y)$, with*

[3] This result was communicated to me by L. Schwartz.

support in U by a C^∞ function of z, $\phi_2(z)$, with support in W, so that as $k\to\infty$, $\sigma_k(x)$ tends uniformly to $\phi(x)$ and each derivative of $\sigma_k(x)$ tends uniformly to the corresponding derivative of $\phi(x)$.

This theorem will be used to prove another more general theorem (Theorem 6, §7) of which this is a particular case. For the proof, since the coordinates of x are $x_1,\dots,x_n$, with $x_1,\dots,x_l$ being the coordinates of y and $x_{l+1},\dots,x_n$ those of z, we can suppose that $U\times W$ is the cube $0<x_i<1$, $i=1,\dots,n$.

Let $f(x)$ be the function obtained by differentiating $\phi(x)$ p times with respect to each variable,

$$f(x)=\frac{\partial^{np}\phi}{\partial x_1^p\dots\partial x_n^p}(x).$$

We use the partition of unity in $\mathbb{R}^n$ given by formula (1), noting that each function $\phi_j(x)$ is the product of a function of y by a function of z. As the support of $f(x)$ is contained in that of $\phi(x)$, which is a compact set in $U\times W$, we can choose ε sufficiently small so that each cube of side length 2ε which meets this support will be contained in $U\times W$, and so that the oscillation of $f(x)$ on such a cube will be less than a predetermined positive number η.

Let c_j be the centre of the cubic support of ϕ_j. Put

$$P(x)=\sum_j f(c_j)\phi_j(x).$$

This sum reduces to a finite number of terms, since $f(c_j)=0$ whenever c_j is outside of $U\times W$. Furthermore, since $|f(c_j)-f(x)|<\eta$ when x belongs to the support of $\phi_j(x)$, we have

$$|f(c_j)-f(x)|\phi_j(x)\leqq\eta\phi_j(x),$$

and the results follows, since $P(x)-f(x)=\sum_j(f(c_j)-f(x))\phi_j(x)$,

(2) $$|P(x)-f(x)|\leqq\eta.$$

Note that $P(x)$ is the sum of a finite number of products of a C^∞ function of y with support in U by a C^∞ function of z with support in W.

Let $Q(x)$ be the function obtained by integrating $P(x)$ p times consecutively with respect to each variable starting from the origin,

$$Q(0)=0,\qquad \frac{\partial^{np}Q}{\partial x_1^p\dots\partial x_n^p}(x)=P(x).$$

The inequality (2) implies, for each derivative operator D whose order with respect to each variable does not exceed p,

(3) $$|DQ(x)-D\phi(x)|\leqq\eta\ \text{ for }\ x\in U\times W.$$

The function $Q(x)$ is, like $P(x)$, a sum of a finite number of products of a C^∞ function of x by a C^∞ function of y, but in general its support is not compact and the inequality above does not hold everywhere outside of $U \times W$.

We can find two compact sets A and B, contained in U and W respectively, such that $A \times B$ contains the support of $\phi(x)$, and, by the lemma, there exists a C^∞ function, $L_1(y)$, with support in U, with values between 0 and 1 and equal to 1 on A, and a C^∞ function, $L_2(x)$, with support in W, with values between 0 and 1 and equal to 1 on B. Let $L(x) = L_1(x) L_2(x)$ and

$$\sigma(x) = L(x) Q(x).$$

The function $\sigma(x)$ is the sum of a finite number of products of a C^∞ function of y with support in U by a C^∞ function of z with support in W. As $L(x) = 1$ on the support of $\phi(x)$, we have $\phi(x) = L(x)\phi(x)$ and

$$\sigma(x) - \phi(x) = L(x)(Q(x) - \phi(x)).$$

Denote by M the maximum of the modulus of the derivatives $DL(x)$ of $L(x)$ with the orders of differentiation with respect to each variable not exceeding p (here $L(x)$ is understood as the zero order derivative). By the differentiation rule, each of the derivatives $D(\sigma(x) - \phi(x))$ is the sum of at most 2^{np} products of a derivative of $L(x)$ by a derivative of $Q(x) - \phi(x)$, the orders of the derivatives with respect to each variable never exceeding p. Thus it follows from (3) that we have

$$|D\sigma(x) - D\phi(x)| \leqq 2^{np} M\eta$$

at each point $x \in U \times W$. But since $\sigma(x) - \phi(x)$ vanishes at each point outside of $U \times W$, these inequalities are valid for all x.

Denote by $\sigma_k(x)$ the function $\sigma(x)$ obtained by choosing $p = k$ and $\eta = (k 2^{nk} M)^{-1}$. The above inequality shows that the difference $\sigma_k(x) - \phi(x)$ has absolute value less than $\frac{1}{k}$ for each of its derivatives of order $\leqq k$. Thus, as $k \to \infty$, $\sigma_k(x)$ tends to $\phi(x)$ in the required manner and this completes the proof. □

§3. Maps and Imbeddings of Manifolds

Let V and W be two manifolds. A map μ of W into V, $\mu y = x$, $y \in W$, $x \in V$, is said to be C^r if the local coordinates of $x = \mu y$ are C^r functions of the local coordinates of y.

In $\mathbb{R}^n$, a set is said to be *of measure zero* (in the sense of Lebesgue) if there is a finite or countably infinite covering by balls for which the sum of the volumes is arbitrarily small. The finite or countable union of sets of measure zero is again of measure zero, and the complement of a set of measure zero is everywhere dense.

If μ is a C^1 map from $\mathbb{R}^n$ to itself, the image μA of each set $A \subset \mathbb{R}^n$ of measure zero is of measure zero. In fact, given a bounded domain D, as μ satisfies a

Lipschitz condition, there exists a positive number a such that $|\mu x-\mu y|<a|x-y|$ for all points x and y in D, where $|x-y|$ denotes the distance between the points x and y. Thus, the image of a ball of radius r, contained in D, is contained in a ball of radius $\leqq ar$, and so, we deduce that the image of each bounded set of measure zero is a set of measure zero. This is also true even for each unbounded set of measure zero because such a set is the countable union of bounded sets of measure zero.

It follows from this that the sets of measure zero in $\mathbb{R}^n$ form an invariant class under C^1 homeomorphisms. Thus we say that *a set A contained in an n dimensional manifold V is of measure zero* if it is the finite or countable union of sets A_i, each of which is contained in a domain for which there exists a C^1 homeomorphism into $\mathbb{R}^n$ mapping A_i to a set of measure zero in $\mathbb{R}^n$.

Recalling some properties of sets of measure zero in $\mathbb{R}^n$, we thus obtain the following theorem.

Theorem 3. *Let μ be a C^1 map of an m dimensional manifold W into an n dimensional manifold V. If $m=n$, and if A is of measure zero in W, μA is of measure zero in V. If $m<n$, μW is of measure zero in V, and its complement is therefore everywhere dense in V.*

We remark that the proof for the case $m<n$ follows from the case $m=n$, since the map μ may be extended to a C^1 map of $W\times\mathbb{R}^{n-m}$ into V, $\mu_1(y,z)=\mu y$, and then, μW is the image under μ_1 of the set formed by all the pairs (y,z_0), for some fixed $z_0\in\mathbb{R}^{n-m}$, and this set has measure zero in $W\times\mathbb{R}^{n-m}$.

Let us suppose again that the dimensions of W and of V are equal. A point of W is called a *critical point of the map μ of W into V* if the Jacobian of the local coordinates of $x=\mu y$ with respect to the local coordinates of y vanishes at this point.

Theorem 4 (A. Sard)[4]. *Let V and W be two manifolds of the same dimension and let μ be a C^1 map of W into V. The image μE of the set E of critical points of μ is a set of measure zero in V.*

This theorem is an immediate consequence of the following proposition which we shall prove: if μ is a C^1 map of $\mathbb{R}^n$ to $\mathbb{R}^n$ and if E is the set of critical points of μ lying in the cube C: $(0\leqq x_i\leqq 1, i=1,2,\dots,n)$, μE is of measure zero.

There is a Lipschitz constant a, which we can fix whenever an upper bound of the modulus of the first derivatives of the coordinates of μx with respect to the coordinates of x in C is known, such that $|\mu x-\mu y|\leqq a|x-y|$ for all points x and y of C. Further, as these derivatives are continuous functions, there exists a function $b(r)$, defined and >0 for $r>0$ and tending to zero with r, such that, if we denote the linear map tangent to μ at x by M_x, $|\mu y-M_x y|\leqq|y-x|b(|y-x|)$.

If $x\in E$, M_x has zero determinant and, when we vary y only, $M_x y$ remains in an $(n-1)$ dimensional plane Π. If, in addition $|y-x|\leqq\varepsilon$, we have $|\mu x-\mu y|\leqq a\varepsilon$, and the distance of μy to Π remains $\leqq\varepsilon b(\varepsilon)$. Thus, the point μy stays in the

[4] Sard [1].