Elements of Calculus

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CHARLES E. MERRILL PUBLISHING COMPANY, COLUMBUS, OHIO

A Bell & Howell Company

Standard Book Number 675-09411-9

Library of Congress Catalog Card Number 70-103620

1 2 3 4 5 6 7 8 9 — 76 75 74 73 72 71 70

Printed in the United States of America

Preface

Elements of Calculus is intended to provide students in the social, administrative, and biological sciences with a solid introduction to the calculus. It recognizes that college and university students in these traditionally non-mathematical areas often have insufficient background to compete with would-be engineers and physicists in the classroom, but nevertheless need a proper foundation for possible further study in such areas as probability, statistics, and linear algebra.

Assuming that beginning college students matriculate with varying degrees of mathematical preparation, we have included a summary of the basic topics necessary for the study of calculus. Thus the topics in Chapter 1 will be familiar to those students who have had three or four years of high school math. Some of the material in Chapters 2 and 3 may also fall into the category of review for such students.

Chapter 2 gives more attention to sequences than do many other texts. The notion of limit is fundamental to an understanding of the calculus, if not to its manipulative rules, and it seems to us that more exposure to sequences is a natural way to introduce the student to infinite processes.

Topics such as Taylor's Series, improper integrals, and multiple integrals are not usually found in a calculus book of this length. They are included here to prepare the student for further study in probability and statistics. It is also noteworthy that emphasis has been placed on applications in the social, administrative, and biological sciences wherever possible.

It should be possible to cover the topics included here in less time than is required for the traditional calculus course, although perhaps only a well-prepared class could complete all the topics in one semester.

This book has grown out of our more comprehensive text. *Elementary Statistics and Analysis* (Charles E. Merrill Publishing Company, 1967). Since many mathematics departments prefer a less rigorous treatment, we have adapted *Elements of Calculus* to meet this need. Many of the more rigorous proofs have been omitted, as well as the material having to do with probability and statisitics.

We would like to thank all of the students and teachers who have used this material in the past. Their suggestions and criticisms have been of great help to us in preparing the present text.

J. M. S.

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Contents

1

Preliminaries

1. Number System

Numbers of the form 1, 2, 3, . . . are called positive *integers*. Numbers obtained from these integers by addition, subtraction, multiplication and division (except by 0) are called *rational* numbers. Other numbers such as $\sqrt{5}$ and π are *irrational* numbers. The totality of rational and irrational numbers are called *real* numbers. This collection may also be thought of as the totality of infinite decimal expansions.

The operations of addition and multiplication obey the following rules. If a, b, c are any real numbers, then we have the

Associative Law: $(a + b) + c = a + (b + c)$, $(ab)c = a(bc)$;

Commutative Law: $a + b = b + a$, $ab = ba$;

Distributive Law: $a(b + c) = ab + ac$.

The symbolic statement "$a > 0$" is read "a greater than zero" and means that a is positive. $a < 0$ means that a is a negative number. "$a \geq 0$" means that either $a > 0$ or $a = 0$, with a similar statement for $a \leq 0$. If for two numbers a and b, $a - b > 0$, we write $a > b$.

The *absolute value* of a number b is written $|b|$ and is defined by

$$|b| = b \quad \text{if } b \geq 0,$$
$$|b| = -b \quad \text{if } b < 0.$$

Operations with inequalities are similar to those with equalities, as the next theorems show.

THEOREM 1.1. If a and b are real numbers, then

for any number k, $a < b$ implies that $a + k < b + k$,
for $k > 0$, $a < b$ implies that $a \cdot k < b \cdot k$,
for $k < 0$, $a < b$ implies that $a \cdot k > b \cdot k$.

THEOREM 1.2. If a, b, c are real numbers, then

$$a < b,\ b < c \text{ implies that } a < c.$$

THEOREM 1.3. For real numbers a and b

$$|a \cdot b| = |a| \cdot |b|,$$
$$|a + b| \leq |a| + |b|$$

PROBLEMS

Carry out the indicated arithmetic operations in the first twenty-five problems. a, b, c are numbers. Where a choice exists do not leave a square root in the denominator.

1. $(a + b)(a - b)$.
2. $(3a + 4b)(3a - 4b)$.
3. $2[(a + b) - (3a + 2b)] - 4(a - 2b)$.
4. $(a - b)(-3a - 4b)$.
5. $(a + b)(a + b)(a + b)(a + b)$.
6. $\dfrac{1}{a + b} - \dfrac{1}{a - b}$.
7. $\dfrac{3}{3a + b} + \dfrac{2}{5b - 2a}$.
8. $\dfrac{2a + 3b}{7a - 4b} - \dfrac{6a - 5b}{a - 9b}$.
9. $(a - b)\dfrac{2a + 1}{3a - 2b} + (b + 4a)\dfrac{5b - 2}{3a + 2b}$.
10. $1 + \dfrac{a}{1 + [a/(1 + a/b)]}$.
11. $(1 - a)(1 + a + a^2 + a^3 + a^4)$.
12. $(100a + 10b + c)^2$.
13. $(a + b - c)(a - b + c)$.
14. $a[b - \{c - a(b - c[a + 1])\}]$.

15. $\dfrac{a}{b+c} + \dfrac{b}{a+c} + \dfrac{c}{a+b}$.

16. $(3a + 2\sqrt{b})(4a - \sqrt{b})$.

17. $(2a - \sqrt{b})(3a - 2\sqrt{b}) \cdot (5a - 3\sqrt{b})$.

18. $(1 + \sqrt{a})(1 + \sqrt{a}) \cdot (1 + \sqrt{a})(1 + \sqrt{a})$.

19. $\dfrac{a + 2\sqrt{b}}{a - \sqrt{b}} \dfrac{a + \sqrt{b}}{a + \sqrt{b}}$

20. $(\sqrt{a + \sqrt{b}})(\sqrt{a - \sqrt{b}})$.

21. $\dfrac{1}{3a + 4\sqrt{b}} \dfrac{3a - 4\sqrt{b}}{3a - 4\sqrt{b}}$.

22. $(\sqrt{a} - \sqrt{b})^2$.

23. $(\sqrt[3]{a} + \sqrt[3]{b})^3$.

24. $\sqrt{a^3} - (\sqrt{a})^3$.

25. $\sqrt[3]{a^2} - (\sqrt[3]{a})^2$.

26. Write as repeating decimals: $\frac{1}{9}, \frac{2}{19}, \frac{168367}{1111000}$.

27. Find the rational numbers that are equal to the given repeating decimals.

$$2.636363\ldots, \qquad 0.8128128128\ldots, \qquad 5.642131313\ldots$$

28. Is the inequality $4 + \sqrt{10} < 3 + \sqrt{17}$ true or false?
[*Hint*: Assume that it is true and square both sides. Is the inequality preserved? Rearrange and square again to eliminate the radical.]

29. Is the inequality $\sqrt{69} + \sqrt{26} < \sqrt{78} + \sqrt{21}$ true or false?

30. Is the inequality $\sqrt{879} + \sqrt{134} < \sqrt{630} + \sqrt{260}$ true or false?

31. If n is any even integer, then it may be expressed in the form $2k$, where k is an integer.

(a) How can an odd integer be expressed?
(b) How can an integer that is divisible by four be expressed?
(c) The integer n, when divided by four, has a remainder of three. Express n in a way to illustrate this property.

32. If n is an integer, when is n^2 even? when odd?

33. Using Problem 31, show that the assumption that there is a fraction a/b such that $a^2/b^2 = 2$ leads to a contradiction.

34. Illustrate Theorem 1.3 for various choices of a and b.

35. Find the values of a such that $|a| = 5$.

36. Find the values of b such that $|b| < 5$.

2. Solution of Equations and Inequalities

An *identity* is an equation in x that is true for all (possibly a few exceptions) real values of x. Examples are

$$x^2 - 4 = (x - 2)(x + 2) \quad \text{and} \quad \frac{1}{\sqrt{x} + 3} = \frac{\sqrt{x} - 3}{x - 9} \text{ for } x \neq 9.$$

Equations involving x that may be manipulated into the form

$$ax + b = 0, a \neq 0$$

are called *linear* or *first-degree* equations. The solution from this form is immediate, i.e., $x = -b/a$. A linear inequality

$$ax + b < 0, a \neq 0$$

may be solved for x using Theorem 1.1 to give

$$x < -\frac{b}{a} \text{ if } a > 0 \quad \text{or} \quad x > -\frac{b}{a} \text{ if } a < 0.$$

Quadratic or *second-degree* equations have the form

$$ax^2 + bx + c = 0, a \neq 0.$$

The solutions are given by the quadratic formula in the next theorem.

THEOREM 2.1. The *quadratic* equation $ax^2 + bx + c = 0$ for $a \neq 0$ has either two, one, or no solutions (in the real number system). If $(b^2 - 4ac)$ *is positive*, the *two solutions* are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

If $(b^2 - 4ac)$ *is zero*, the *one solution* is

$$x = \frac{-b}{2a}.$$

If $(b^2 - 4ac)$ *is negative*, there are *no solutions*.

THEOREM 2.2. If $b^2 - 4ac$ is not negative, then the quadratic expression $ax^2 + bx + c$ can be factored into

$$a\left[x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right]\left[x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right].$$

The solution of quadratic inequalities breaks up into several cases. Let r_1, r_2 with $r_1 \leq r_2$ be the two solutions of the quadratic equation in the above theorems.

THEOREM 2.3. The values of x that satisfy

$$ax^2 + bx + c \leq 0$$

are given in the table.

	$b^2 - 4ac \geq 0$	$b^2 - 4ac < 0$
$a > 0$	$r_1 \leq x \leq r_2$	No values of x
$a < 0$	$x \leq r_1 \leq$ or $x \geq r_2$	All values of x

PROBLEMS

In the first thirteen problems solve for x, y and z.

1. $3x^2 - 4x + 7 = 0.$

2. $x^2 + 2x - 8 = 0.$

3. $4x^2 - 12x + 9 = 0.$

4. $x^2 - 0.9x - 13.12 = 0.$

5. $5x^2 - 3x - 2 = 0.$

6. $3x + 5y = 7$
$2x - 3y = 4.$

7. $5x - 6y = 9$
$10x - 12y = 7.$

8. $x + 2y - 3z = 6$
$4x - y + z = 3$
$5x + 3y + 2z = -7.$

9. $-x - y + 2z = 3$
$2x - 7y - z = 4$
$y + z = 5.$

10. $x + z = 1$
$y + z = 1$
$x + y = 1.$

11. $x + 2y - 3z = 2$
$3x - 4y + z = 1$
$5x - 5z = 5.$

12. $y = x^2 - 4x + 2$
$2x + y - 6 = 0.$

13. $x^2 + y^2 = 8$
$x + y = 1.$

14. Determine k so that the following two equations admit only one solution.

$$x^2 - 6x + 3 - y = 0$$
$$kx + y - 2 = 0.$$

15. Factor $3x^2 - 16x - 35.$

16. Factor $8x^2 - 20x - 27.$

17. Factor $x^2 + 3x - 5.$

In the following problems, solve for x.

18. $3x + 6 < 7x - 2.$

19. $-2x - 3 \geq 3x + 1.$

20. $x > -2$ and $x > 1.5.$

21. $x > 2$ and $x \leq -3.$

22. $(x - 2)^2 < 3.$

23. $x^2 - 3x - 5 < 0.$

24. $x^2 - 6x + 9 \leq 9.$

25. $x^2 + 4x > 0.$

26. $(x - 2)(x - 3)(x - 4) > 0.$

27. $(x - 2)(x - 3)(x - 4)(x - 5) < 0.$

28. $|2x - 7| = 4.$

29. $|x - 7| < 2.$

30. $|3x + 8| \geq 3.$

3. Set Operations and Enumeration

A collection of objects is called a *set*. Much of the time the objects or elements of the set will be numbers or points, but this is not necessarily so.

Operations with sets are fundamental in mathematics and the basic definitions or rules are given in the next few paragraphs.

The set that contains all of the elements under consideration is called the *space* and will be denoted by $\mathcal{E}$. The empty set, that is, the set that contains no elements, will be called the *null* set and denoted by $\emptyset$.

If the points of a set A are also in a set B, we say that A is *included* in B, and write $A \subset B$ or $B \supset A$, the latter relation being read as "*B includes A*."

If A and B consist of the same points, we have $A \subset B$ and $B \subset A$, and we write $A = B$.

EXAMPLE 3.1. Consider some specific sets in $\mathcal{E}$, whose elements are e_i.

A is the set consisting of e_1, e_2, e_3, e_4
B: e_2, e_3
C: e_5, e_6
D: $e_2, e_4, e_6, \cdots, e_{2n}, \cdots$
H is the set consisting of all e_i that are not in A.

Then for these sets

$$\emptyset \subset B \subset A \subset \mathcal{E} \quad \text{or} \quad \mathcal{E} \supset A \supset B \supset \emptyset$$

and

$$\emptyset \subset C \subset H \subset \mathcal{E} \quad \text{or} \quad \mathcal{E} \supset H \supset C \supset \emptyset$$

where we adopt the convention that the empty set is included in any set.

Given any set S there is another set, called its *complement*, that consists of all of the points that are not in S. The complement is denoted by S'. Using the sets in Example 3.1,

$A' = H$
$H' = A$
C' is the set whose points are $e_1, e_2, e_3, e_4, e_7, e_8, \cdots$
D' is the set: $e_1, e_3, e_5, \cdots, e_{2n-1}, \cdots$
$\emptyset' = \mathcal{E}$
$\mathcal{E}' = \emptyset$.

Given any two sets R and S in $\mathcal{E}$ the *union* of R and S is the set of points that are in either R or S or both. The union is denoted by $R \cup S$. The *intersection* of R and S is the set of points that are in both R and S. It is denoted by $R \cap S$.

For the sets in Example 3.1 we have

$A \cup C$: $e_1, e_2, e_3, e_4, e_5, e_6$ $\qquad A \cap C = \emptyset$
$A \cup B$: $e_1, e_2, e_3, e_4 = A$ $\qquad A \cap B = B$
$A \cup H = \mathcal{E}$ $\qquad A \cap H = \emptyset$

and for any set R

$$R \cup R' = \mathcal{E} \quad \text{and} \quad R \cap R' = \emptyset.$$

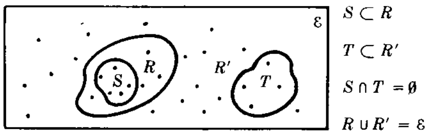

Figure 3.1

Some suggestive geometric diagrams for the preceding definitions are given in Figures 3.1 and 3.2.

Although the definitions of the operations of set union and intersection are given in terms of two sets, the meaning of expressions such as

$$(R \cap S) \cup (T \cap V)$$

should cause no difficulty. In words, the preceding set will consist of those points that are either in both R and S or in both T and V.

EXAMPLE 3.2. Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution: Let L and M denote the sets on the left and right side respectively. We will show that $L \subset M$ and $M \subset L$, which gives the desired equality.

If a point u is in L then u is in both A and $(B \cup C)$. This gives u in A and either B or C, which says that u is in either $(A \cap B)$ or in $(A \cap C)$. But this gives u in M so that $L \subset M$.

If a point u is in M, then u is in either $(A \cap B)$ or $(A \cap C)$. If u is in $(A \cap B)$, then it is in both A and B so that it is in both A and $(B \cup C)$. Thus u is in L. If u is in $(A \cap C)$ the same argument gives u in L. Thus $M \subset L$.

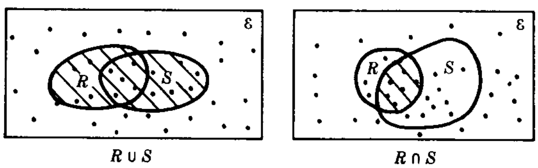

Figure 3.2

A problem that has many applications in probability is that of counting or enumerating the elements of a set that has been constructed in some special way. The interesting problems, rather obviously, are those where the number of elements is large.

THEOREM 3.1. Let A be a set with n elements and B be a set with m elements. Define C to be the set of ordered pairs of elements, the first from A, the second from B. Then C has nm elements.

EXAMPLE 3.3. If there are five roads from A to B and three roads from B to C, in how many ways can one travel from A to C?

Solution: (5)(3) = 15. If the roads from A to B are labeled as x_i, and those from B to C as y_i, the following list contains all possible paths.

$$\begin{array}{ccccc} x_1y_1 & x_2y_1 & x_3y_1 & x_4y_1 & x_5y_1 \\ x_1y_2 & x_2y_2 & x_3y_2 & x_4x_2 & x_5y_2 \\ x_1y_3 & x_2y_3 & x_3y_3 & x_4y_3 & x_5y_3 \end{array}$$

An important modification of the preceding theorem arises when the second element is chosen from the remaining elements of A rather than from a different set B. The number of such ordered pairs is then $n(n-1)$.

EXAMPLE 3.4. In how many ways can two cards be drawn from a deck of 52 distinct cards?

Solutions: The first card can be drawn in 52 ways. After that the second can be drawn in 51 ways. Hence the desired number is 52(51) = 2652.

If k elements are selected from A in an ordered way we call the arrangement a *permutation* of the n objects taken k at a time.

THEOREM 3.2. The number of permutations of n objects taken k at a time is

$$n(n-1)(n-2)\cdots(n-k+1).$$

An important special case arises when $k = n$, i.e., the number of arrangements or permutations of n objects. The symbol $n!$ read *n factorial*, is defined by

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1, \quad 0! = 1.$$

Thus the number of permutations of n objects taken k at a time can also be written as $n!/(n-k)!$.

EXAMPLE 3.5. Find the number of different three-card draws from a deck of 52 playing cards.

Solution:

$$\frac{52!}{(52-3)!} = \frac{52!}{49!} = 52(51)(50) = 132600.$$

EXAMPLE 3.6. Find the number of different ways that four persons selected from ten persons could be seated in four particular chairs.

Solution:

$$\frac{10!}{6!} = 10(9)(8)(7) = 5040.$$