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COLLECTED WORKS ON THE PROBLEMS OF FLUID MECHANICS

VOLUME I

CONTENTS

The Flow Pattern of a Supersonic Projectile

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Summary

A complete first approximation is given of the supersonic flow past any slender, axisymmetrical body-wake combination, whose meridian section may have discontinuities in slope. A hypothesis, which is amply substantiated, is made that the failures of linearized theory for a description of the flow pattern may be corrected by replacing the approximate characteristics in that theory by the exact characteristics (or at least by a sufficiently good approximation to the exact ones); the ideas involved are described in the introduction (Part 1) since they are of general application, and the mathematical details for the projectile are furnished in Parts 2 and 3. To complete the description of the flow, the shocks, which occur in regions where the characteristics would otherwise form a limit line and the solution cease to be single-valued, are determined (Part 4) from the simple geometrical property that, to a first order in its strength, a shock bisects the angle between the characteristics on each side of it. This condition proves extremely powerful in the mathematical analysis, and it has additional value in that it also gives a simple qualitative picture of how the shocks occur and fit into the pattern. The general theory is applied in Part 5 to the typical example of the 5–10 calibre ogival headed bullet with a suitable wake. In Part 6, the value of the pressure at any point of the fluid is determined, and, in Part 7, the way in which the drag is related to the rate of increase of the energy of the fluid is investigated. The latter leads to an interesting new expression for the von Karman drag in terms of a function which is fundamental to the whole theory. Finally, in the Appendix, the corresponding theory for the two-dimensional steady and one-dimensional unsteady flows is set out since it gives some new information on these topics. Probably the most important result obtained in the Appendix is in the problem of a piston which oscillates periodically. It is found that, at a large distance from the piston, the strengths of the shocks depend only on the properties of the fluid, the distance from the piston and the period of the oscillation; they are independent of the particular piston motion.

1. *Introduction*

A mathematical theory is given of the disturbance produced in the surrounding air by a projectile moving with supersonic speed. A solution of the flow

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inside the wake is not obtained; its mean boundary is assumed to be known and it is the flow past the given body-wake combination which is described. The theory is first written out assuming that the body[1] is axisymmetrical, slender and pointed at the nose (with the Mach number sufficiently in excess of 1 for the front shock to be attached), although discontinuities in the slope of the meridian section are allowed; then a *complete* first approximation to the whole flow pattern is given. If some of these conditions are not satisfied, the description is not complete but, as will be explained later, many important results can be taken over to these cases without modification.

Valuable information of the shape of the wake and a picture of the shocks which occur in the flow are provided by photographs of bullets in flight; the basic flow pattern is sketched in Figure 1. At the base of the projectile there is a

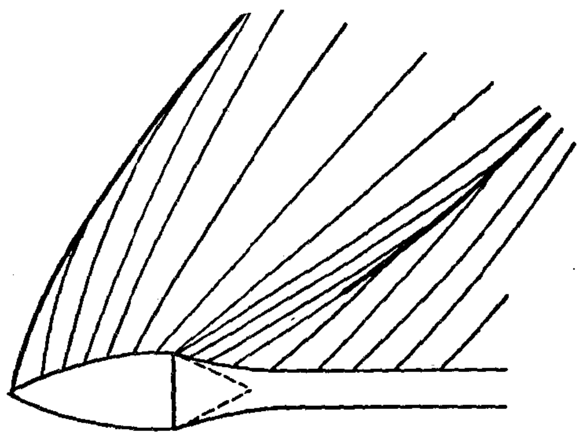

FIGURE 1

roughly conical dead air region at a lower pressure so that the stream expands sharply round the base; a typical value for the angle through which it turns is 12°. But then, the stream is recompressed as the boundary layer between it and the dead air region thickens to form the turbulent wake of roughly constant cross-section. Fluctuations of the wake boundary due to the turbulence inside will not affect the main features of the flow and are ignored. Turning now to the flow pattern, there is an attached front shock which curves round towards the undisturbed Mach direction and weakens as the distance from the body increases; ahead of it the flow is undisturbed. At the rear, there will always be a second shock (this would be true even if the wake were taken to be of constant cross-section equal to the base of the body). A rear shock is immediately more difficult to deal with than the front one since there is a non-uniform state on *both* sides of it, but in addition, further complications arise in that more than one shock may be formed, particularly if there are discontinuities in the slope of the meridian section (although not only in such cases). Thus, there may be a shock system at the rear consisting of several shocks, ultimately running together to

[1]The term "body" will henceforth be used to include the wake.

form the main shock which decays in a similar way to the front one. The prediction and description of such a complex shock system forms the most fascinating part of the theory.

The corresponding problem of two dimensional supersonic flow past an aerofoil was solved by Friedrichs [1]. The theory is considerably easier since a solution of the exact equations of the isentropic flow is already known, i.e. the so-called "simple wave", and it is only necessary to fit on the shocks in a suitable manner. Moreover the determination of the shocks themselves is less complicated since Friedrichs assumes that there are just single shocks attached to the leading and trailing edges, respectively, and that the flow behind the rear shock is undisturbed. If the aerofoil section has no compressive discontinuities in slope and is pointed at each end, this is a good approximation to the truth[2]; hence discussion of the formation of a shock *inside* a disturbed region is avoided in this case. However, it is more important in the analogous unsteady problems of one dimensional plane waves which Friedrichs also treats, and in these problems he does consider shock formation inside a wave, but is only able to obtain the details of the shock near the point of formation; he cannot go on to describe its ultimate decay, for example. Now, these problems can be solved by the methods of this paper and so, although its main objective is the projectile problem, the corresponding theory for two-dimensional flow (and the analogous one dimensional wave problems) in which the flow pattern for *any* thin aerofoil section is described, is set out briefly in the Appendix. Of course this theory would only give the first approximation whereas the Friedrichs theory, with a certain modification which is described in the next paragraph, is correct to a second order; thus the present method loses some accuracy but penetrates further.

The question of the accuracy of Friedrichs' theory raises a very important point. That theory and also the theory which will be described for the axisymmetrical problem, use solutions of the isentropic equations of motion to describe the flow and then the occurrence and positions of curved shocks are determined from them. This procedure has been criticised on the grounds of inconsistency since curved shocks are of non-uniform strength and the flow behind is therefore not isentropic. The explanation is that the isentropic equations of motion are used not because it is assumed that the flow is exactly isentropic, but because it is thought that they will give a good approximation to the correct (non-isentropic) one since the shocks concerned are weak and the entropy changes at a shock are of the third order in its strength. In order to clarify the position and put this extremely general and valuable approach on a firm basis, Lighthill [2] has investigated the accuracy of Friedrichs' theory and applied a comprehensive check on its results. He finds that the theory is correct to the second order (as expected) with one important exception: the position of the rear shock is

[2]It would not be so for a body of revolution satisfying these conditions since even on linearized theory, the flow behind the body is not uniform (the "tail" of a cylindrical sound pulse).

correct only to a first order. This is due to a wide third order pressure wave, spread out behind the main disturbance, which interacts with the rear shock over a large distance to modify its position The pressure wave is determined and the results for the rear shock are corrected. The knowledge of the validity of the approach in Friedrichs' work justifies its use in other problems, since a similar behaviour is expected. Thus in the flow past the projectile strictly similar effects may be sketched in but in this paper, since only the first order flow pattern is obtained they will not appear in the analysis.

For the axisymmetrical flow past a projectile no exact solution of the equations of motion is available, so that the first step in this theory is to provide a valid description of the non-linear flow. The existing linearized theory is now well-known (see, for example, Lighthill [3]), but it is easily seen to be inadequate, as it stands, for a detailed description of the flow outlined above. In it the disturbance is propagated diagonally downstream along straight parallel characteristics $x - r \sqrt{(M^2 - 1)} =$ constant, where x is the distance along the axis from the nose, r the distance from the axis and M the Mach number of the main stream. This is obviously incorrect since, in fact, the disturbed region spreads out with curved characteristics which ultimately diverge (see Figure 1). Moreover, the shocks, whose presence in the correct theory is most important, are entirely absent since they are a non-linear phenomenon; for example, in the linearized solution, the flow is uniform ahead of the leading characteristic $x - r \sqrt{(M^2 - 1)} = 0$ whereas it is known that the front shock is there. The same failures occur in Broderick's further approximations [4], since his reduction of the equations to a series of linear ones avoids the essential non-linearity of the problem. However, in spite of this criticism, these theories are extremely valuable because (i) they do give valid approximations to the pressure forces acting on the body, and (ii) (much more important from the present point of view) *the failure of the linear theory as a description of the flow can be remedied.* The modified linear theory forms the basis of the work. It is the solution, in this problem, corresponding to the "simple wave" used in Friedrichs' work; in fact, when the method is applied to the two-dimensional flow (see Appendix) it does give the first approximation to the "simple wave." From it the theory is developed as outlined in the summary.

The ideas which have culminated in this theory arose from the author's previous work on the problem [5]. In this work, a direct attack was made on the exact equations of motion but to make the task less formidable, the discussion was limited to the behaviour at large distances. A solution was found as a series in descending powers of r and it was noticed that, for the case of a slender body when the disturbance could be assumed to be small from the outset and hence certain terms neglected, the solution had the same *form* as the expansion of the linearized one except that the approximate characteristic variable $x - r \sqrt{(M^2 - 1)}$ was replaced therein by the exact one $y(x, r)$ such that $y =$ constant is an exact characteristic curve. Hence it was deduced that the only failure of linearized theory at large distances is that the characteristics in it are incorrect. The only real use made of this interesting fact was that, by comparison, the arbitrary function and constants appearing in the general theory were obtained in terms of the body shape; the full meaning and the possibility of its application in a general way were not appreciated. Now it becomes the starting point of the whole theory; the fundamental hypothesis is made that linearized theory gives a valid first approximation to the flow *everywhere* provided that in it the approximate characteristics are replaced by the exact ones, or at least by a sufficiently good approximation to the exact ones. (A more precise statement is given with the mathematical details in Part 2.)

An examination of the underlying physical ideas reveals the reasons for making this hypothesis. Linearized theory is essentially an acoustic one in that disturbances are propagated at a *constant* speed equal to the speed of sound in the main stream; it does not take account of the variation in the local speed of sound or of the convection of sound with the moving fluid. It is permissible to use such a theory to describe the propagation of disturbances in very small regions; but if they are to be combined into a complete picture, the appropriate local speed of propagation which is equal to the local speed of sound plus the local fluid velocity, must be used in each region, otherwise the error accumulates. Thus it is expected that the linearized theory has the correct variation of physical quantities along the characteristics, which trace the paths of the wave fronts, but has the wrong curves for the characteristics. The hypothesis is designed to adjust this. Apart from the physical interpretation, the theory is amply substantiated by the checks that can be made in several places of its correct prediction of certain results that are already known by other methods. Among these checks, there is the complete reproduction of the results at large distances which were found previously, and others will be remarked as they arise. Finally, exactly the same procedure in the problems which are discussed by Friedrichs yields the first approximation to his results. Thus there can be little doubt of the validity of the hypothesis.

It is assumed in the theory that the body is slender and pointed at the nose, with the front shock attached, but even if these conditions are not satisfied it may still be used to deduce the behaviour of the flow at large distances. For, certainly at a sufficient distance from the body the disturbance will be small. Therefore, consider a stream tube with radius so large that its deviation from a cylinder of constant radius is very small. Then the problem of the flow outside is that of the flow past a quasi-cylindrical duct and it may be described by the theory of this paper, and hence the similar results at large distances obtained. In this case, of course, the arbitrary function occurring in the solution remains undetermined since it depends on the shape of the stream tube which is unknown unless the flow near the body is solved. This is in agreement with the previous work [5] since then the arbitrariness could only be resolved when the body was slender and the general theory linked up with the shape of the body by means of linear theory. Thus the present theory entirely replaces the previous paper; it is much fuller, it includes all the "large distance" results and is obtained much more quickly and easily since the exact equations of motion are now avoided. The other restrictive condition is that of axial symmetry. Suppose that this condition is relaxed but that the body is slender and pointed at the nose (with attached shock). The linearized theory of this problem has been given by Ward [6], and it is found that the flow becomes axisymmetrical when $r \sqrt{(M^2 - 1)}/\{x - r\sqrt{(M^2 - 1)}\}$ is not small. The quantity $x - r\sqrt{(M^2 - 1)}$ is the linearized form of the characteristic variable and measures the distance from the nose at which the characteristic starts; hence the condition is, correctly interpreted, that at any point the distance from the axis divided by the distance from the nose at which the characteristic surface (on which the point lies) started from the body, should not be small. This is clearly satisfied at large distances, but it is also true at points on the front shock; they are effectively at large distances because the appropriate characteristic surfaces arise so very close to the nose. (Considerable use is also made of this in the axisymmetrical case; it is discussed in more detail in Part 4.) Hence the results for the front shock and all the theory at large distances apply unchanged to the non-axisymmetrical slender body. From these extensions, it seems reasonable to suggest that the results for large distances apply to the supersonic flow past *any* finite body. These results are (see Parts 4 and 6) that there are two main shocks whose equations are approximately $x = r\sqrt{(M^2 - 1)} - br^{1/4}$ and $x = r\sqrt{(M^2 - 1)} + b_1 r^{1/4}$, where b and b_1 are constants depending on the body shape. The strengths fall off like $r^{-3/4}$ and at points between the shocks the pressure falls linearly with time at a rate $0.2\sqrt{(1 - M^{-2})}\, r^{-1}$atmospheres/millisec (where r is measured in metres) which is independent of the body shape.

Before proceeding to the detailed theory, it is of interest to consider the applications of the method presented here to other problems. In principle at least, the method is extremely simple, the main reason for this being that the non-linear equations of motion are avoided, although the geometrical treatment of the shocks by the "angle property" adds much to the simplicity of its application. Moreover, since the means by which the linearized theory is rectified is of a general nature, it is hoped that this new approach will prove to be of value in other problems of research. It is immediately applicable to the connected problems of fluid flow in which there are only two independent variables. The one-dimensional unsteady waves and the two-dimensional steady supersonic flow (already discussed by Friedrichs) have been mentioned. Others which are easily solved are the problems of unsteady waves with cylindrical or spherical symmetry. The latter of these is of practical interest in explosions but since in explosions a very large disturbance of the air is desired (projectiles are designed to have the opposite effect), only the behaviour the theory gives at large distances would be of value. Hence there is little of practical importance to be added to the author's paper [7] on the subject, although the work could now be considerably shortened, and for scientific interest a very weak explosion could be described completely. The author hopes to solve other problems involving two independent variables, where the only difficulty is the application of the method, and also to develop a similar technique for problems involving three independent variables.

In the account of the work many figures are necessary; they are of two types: First there are sketches which form part of the explanation of the text; in order to make the essential details clear they are not drawn in correct proportion (for example, distances in the Mach direction are very large and therefore sketches of the flow plane are contracted in this direction). These figures are numbered and are inserted in the text at the appropriate points. Secondly, there are graphs which show the results of numerical calculations, and may be referred to throughout the paper; because of their different nature these are collected separately at the end of the paper.

2. *Improvement of Linearized Theory*

Let the steady stream have velocity U in the x-direction, and at a general point (x, r) let the velocity be $(U + Uu, Uv)$. The flow is assumed to be irrotational hence the perturbation velocities u and v may be deduced from a potential ϕ which, on the linearized theory, satisfies the equation

$$\phi_{rr} + \frac{1}{r}\phi_r - \alpha^2\phi_{xx} = 0, \tag{1}$$

where $\alpha = \sqrt{(M^2 - 1)}$ and suffixes denote partial differentiation. The solution of (1) which represents a disturbance propagated downstream from a body is

$$\phi = -\int_0^{x-\alpha r} \frac{f(t)\,dt}{\sqrt{(x-t)^2 - \alpha^2 r^2}}, \tag{2}$$

giving

$$u = -\int_0^{x-\alpha r} \frac{f'(t)\,dt}{\sqrt{(x-t)^2 - \alpha^2 r^2}}, \tag{3}$$

$$v = \frac{1}{r}\int_0^{x-\alpha r} \frac{(x-t)f'(t)\,dt}{\sqrt{(x-t)^2 - \alpha^2 r^2}}; \tag{4}$$

the downstream characteristics of the equation are $x - \alpha r =$ constant. The arbitrary function $f(x)$ is determined from the boundary condition on the body and will be dealt with in detail in the next paragraph.

The method by which linearized theory must be modified has been described in the introduction and is embodied in the following precise hypothesis. *Linearized theory gives a correct first approximation everywhere provided that the value which it predicts for any physical quantity, at a given distance r from the axis on the approximate characteristic $x - \alpha r =$ constant, pointing downstream from a given point on the body surface, is interpreted as the value, at that distance from the axis, on the exact characteristic which points downstream from the said point.* Carrying out the modification, u and v become, replacing $x - \alpha r$ by $y(x, r)$,

$$u = -\int_0^y \frac{f'(t)\,dt}{\sqrt{(y-t)(y-t+2\alpha r)}}, \tag{5}$$

$$v = \frac{1}{r}\int_0^y \frac{(y-t+\alpha r)f'(t)\,dt}{\sqrt{(y-t)(y-t+2\alpha r)}}; \tag{6}$$

y is now determined from the condition that $y(x, r) =$ constant is a characteristic curve, that is, along it $dx/dr = \cot(\mu + \theta)$, where μ is the local Mach angle and θ is the local direction of flow. The local velocity of sound, a, is determined from Bernoulli's equation,

$$\frac{a^2}{\gamma - 1} + \tfrac{1}{2} U^2\{(1+u)^2 + v^2\} = \frac{a_0^2}{\gamma - 1} + \tfrac{1}{2} U^2, \tag{7}$$

where suffix 0 refers to the value in the undisturbed stream and γ is the ratio of the specific heats, hence

$$\mu = \sin^{-1}\frac{a}{q} = \mu_0 - \left(1 + \frac{\gamma - 1}{2} M^2\right)\alpha^{-1} u + O(u^2 + v^2), \tag{8}$$

where q is the magnitude of the velocity. The stream direction θ is given by $\tan^{-1}\{v/(1+u)\} = v + O(u^2 + v^2)$. Therefore, on $y \doteq$ constant,

$$\frac{dx}{dr} = \alpha + \frac{(\gamma+1)M^4}{2\alpha} u - M^2(v + \alpha u) + O(u^2 + v^2). \tag{9}$$

The value of y on a characteristic has not been defined uniquely, although on the body it must be approximately equal to $x - \alpha r$ (which it replaces in linearized theory); it is now made quite definite by taking it *equal* to the value of $x - \alpha r$ at the point where the characteristic meets the body surface.[3] Then, substitution of (5) and (6) in (9) gives, on performing the integration,

$$\begin{aligned} x = \alpha r &- \frac{(\gamma+1)M^4}{2\alpha^2}\int_0^y \left\{\frac{\sqrt{y-t+2\alpha r} - \sqrt{y-t+2\alpha R(y)}}{\sqrt{y-t}}\right\} f'(t)\,dt \\ &- 2M^2 \int_0^y \log\left\{\frac{\sqrt{y-t+2\alpha r} - \sqrt{y-t}}{\sqrt{y-t+2\alpha r} + \sqrt{y-t}} \cdot \frac{\sqrt{y-t+2\alpha R(y)} + \sqrt{y-t}}{\sqrt{y-t+2\alpha R(y)} - \sqrt{y-t}}\right\} f'(t)\,dt + y \\ = \alpha r &- c(y, r) + y, \end{aligned} \tag{10}$$

say, where $R(x)$ is the radius of the body. The expression (10) determines $y(x, r)$ only approximately, since in (9) terms $O(u^2 + v^2)$ have been neglected

[3]It is convenient, however, to think of y as being approximately the distance x from the nose, where the characteristic produced meets the axis.

and the first order approximations (5), (6) substituted. But, unlike linearized theory which takes $y = x - \alpha r$, thus neglecting a term which becomes infinitely large compared to $x - \alpha r$ both at large distances and near the leading Mach cone (see (12)), this is a valid approximation since the terms neglected really are small compared to those retained. Equations (5), (6) and (10) now describe the solution.

This equation for the characteristics, which determines $y(x, r)$, is extremely complicated and fortunately it is not necessary to make use of it in this form. In general only the approximation for $\alpha r/y$ large is required. Then the expressions (5), (6) and (10) become

$$u = -\frac{F(y)}{\sqrt{2\alpha}} r^{-1/2}, \qquad v = -\alpha u, \tag{11}$$

$$x = \alpha r - kF(y)r^{1/2} + y, \tag{12}$$

where

$$F(y) = \int_0^y \frac{f'(t)\,dt}{\sqrt{y - t}} \tag{13}$$

and

$$k = 2^{-1/2}(\gamma + 1)M^4\alpha^{-3/2}.$$

Taking $\gamma = 1.4$, a graph of the variation of k with Mach number M is given in Graph A. The function $F(y)$ is fundamental to the whole theory and is the most important function associated with flow past a body of revolution, as will be seen. The next paragraph is devoted to its determination from the given body shape, and to a consideration of its properties.

It should be noted that, as far as the value of the physical quantities u and v are concerned, nothing is gained by preferring (10) to (12), since when $\alpha r/y$ is not large, both $c(y, r)$ and $kF(y)r^{1/2}$ are small and the differences they make in (5) and (6) are of the same order as terms already neglected there. Hence for the physical quantities, the value of y determined by (12) may be used everywhere. However (10) still provides the correct approximation to the characteristic curves near the body; this will be used later.

It may be mentioned as additional support for the theory that (11) and (12) are in agreement with the general principle, deduced by Lighthill in §6 of his paper [8].

3. *The Function F(y)*

The arbitrary function $f(x)$ is determined by application of the boundary condition that the normal velocity at the body surface is zero. This condition when linearized becomes $v = R'(x)$ on $r = R(x)$, where $R(x)$ is the body radius at a distance x from the nose. In the most well known form of the linearized theory, it is assumed that the body is sufficiently smooth ($S'(x)$ continuous at least) for the expression (4) for v to be approximated as $f(x)/r$ when r is small, hence the boundary condition gives

$$f(x) = R(x)R'(x) = S'(x)/2\pi,$$

so that from (13),

$$F(y) = \frac{1}{2\pi}\int_0^y \frac{S''(t)\,dt}{\sqrt{y-t}}. \tag{14}$$

If $S'(x)$ and $S''(x)$ are continuous and $O(\delta^2)$ then the error in using this value of $f(x)$ and hence the error in $F(y)$ is $O(\delta^4 \log 1/\delta)$ (see [3]), where δ is the thickness ratio; under these conditions the body will be said to be "smooth" and the simple form (14) will be used. If $S''(x)$ has discontinuities, (14) still gives some approximation and its accuracy and limitations may be investigated directly as in the previous case but this will not be done here. It is easier and more instructive to see when the general expression, which must be found in any case to apply when $S'(x)$ is discontinuous, may be reduced to (14).

The linearized theory for a body with discontinuities of slope has been given by Lighthill [9] and in fact the author is indebted to Professor Lighthill for suggesting the expression for $F(y)$ which must be used in this case. The "discontinuity theory" makes use of the powerful Heaviside calculus in which the operation $\int_0^x dx$ is represented symbolically by p^{-1}. Then (2), (3) and (4) become

$$\begin{aligned} \phi &= -K_0(\alpha pr)f(x), \\ u &= -pK_0(\alpha pr)f(x), \\ v &= -\alpha pK_0'(\alpha pr)f(x) = \alpha pK_1(\alpha pr)f(x), \end{aligned} \tag{15}$$

where K_0 and K_1 are the Bessel functions as defined by Watson. Application of the boundary condition yields

$$R'(x) = [\alpha pK_1(\alpha pr)f(x)]_{r=R(x)}\,, \tag{16}$$

and although this gives a formal relation for $f(x)$ its interpretation is difficult in general because $[K_1(\alpha pr)]_{r=R(x)}$ cannot simply be replaced by $K_1(\alpha pR(x))$, (the latter would allow the operators to act on $R(x)$ and this is not intended). If the body is *smooth* it is permissible in (16) to use the expansion $K_1(\alpha pr) \sim (\alpha pr)^{-1}$ for αpr small since a formal rule of Heaviside calculus is that in such an expression αpr behaves like $\alpha r/x$ (which *formally* it represents) in that if $\alpha r/x$ is small then the expression in question may be expanded for αpr small, and $\alpha r/x$ is small on the body. Hence $\alpha pK_1(\alpha pr)f(x) \sim f(x)/r$ and (16) gives $f(x) = R(x)R'(x)$. But, if the body has a discontinuity of slope at $x = t_1$, αpr behaves in the expression like $\alpha r/(x - t_1)$ which is small only away from the discontinuity; hence some other approach is required. If the problem were that of flow outside a quasi-cylindrical duct with $R(x)$ approximately constant there would be no difficulty; (16) would give immediately

$$f(x) = \frac{1}{\alpha pRK_1(\alpha pR)} \frac{S'(x)}{2\pi}. \tag{17}$$

This suggests that if in the case of the projectile the body is divided into small intervals $x = t_i$, $i = 1, 2, 3, \cdots$, the contribution to $f(x)$ of the increment $\Delta S'(t_i)$ in $S'(x)$ in the i-th interval, is

$$f_i(x) = \frac{1}{\alpha pR_iK_1(\alpha pR_i)} \frac{\Delta S'(t_i)}{2\pi} H(x - t_i), \tag{18}$$

where $R(t_i) \equiv R_i$ and $H(x)$ is the Heaviside unit step function. Now $f(x)$ is taken to be $\sum f_i(x)$ and it may be verified that the boundary condition (16) is satisfied. For at $x = t_n$, say, the contribution of $f_i(x)$ to the right hand side of (16) is

$$\frac{K_1(\alpha pR_n)}{R_iK_1(\alpha pR_i)} \frac{\Delta S'(t_i)}{2\pi} H(x - t_i). \tag{19}$$

If t_i is not near t_n so that $\alpha R_n/(t_n - t_i)$ is small the expansions of the Bessel functions for small argument can be used as for the smooth body, and (19) is approximately $\Delta S'(t_i)/R_n$; if t_i is near t_n so that $(R_n - R_i)/R_n$ is small, (19) is again approximately $\Delta S'(t_i)/R_n$. Now for all the t_i, one of these conditions holds, because if $(t_n - t_i) = O(\alpha R_n)$ then $R(t_n) = R(t_i + O(\alpha R_n)) = R(t_i) + O(\alpha R'(t_i)R_n)$, that is $(R_n - R_i)/R_n = O(\alpha R_i')$ which is small by definition of a slender body, hence (19) is $\Delta S'(t_i)/R_n$ for all i and clearly (16) is satisfied by the sum. Essentially the method is a combination of the methods used for the smooth body and the duct; away from the discontinuities the former can be used with its expansions for small αpR, whilst near a discontinuity $R(x)$ is approximately constant and the duct expressions apply. Thus $f(x)$ is the sum over i of the terms given by (18). If $g(x)$ represents $\{pK_1(p)\}^{-1}H(x)$ then $f_i(x) = g\{(x - t_i)/\alpha R_i\}\Delta S'(t_i)/2\pi$, hence summing over i and taking the limit,

$$f(x) = \int_0^\infty g\left(\frac{x - t}{\alpha R(t)}\right) \frac{dS'(t)}{2\pi}. \tag{20}$$

The properties of $g(x)$ and $f(x)$ can be discussed in detail by the methods used below but it is unnecessary in this work since the important function is $F(x) = \pi^{1/2}p^{1/2}f(x)$. The value of F is found similarly to f by summing the contributions from (18) and taking the limit, and is

$$F(y) = \int_0^\infty \left(\frac{2}{\alpha R(t)}\right)^{1/2} h\left(\frac{y - t}{\alpha R(t)}\right) \frac{dS'(t)}{2\pi}, \tag{21}$$

where

$$h(x) = \sqrt{\frac{\pi}{2p}} \frac{1}{K_1(p)} H(x). \tag{22}$$

Expressions (20) and (21) are Stieltjes integrals and apply whether $S'(x)$ is discontinuous or not.

In order to discuss the properties of $F(y)$, it is necessary to know the function $h(x)$. The expression (22) for $h(x)$ may be interpreted by the usual methods of the Heaviside calculus and calculated numèrically, but in addition its behaviour for small and large values of x may be deduced from the corresponding behaviour of its representation (22) for large and small values of p, respectively. For large p, $K_1(p) \sim \pi^{1/2}e^{-p}/(2p)^{1/2}$ hence for small x, $h(x) \sim e^pH(x) = H(x+1)$; for small p, $K_1(p) \sim p^{-1}$ hence for large x, $h(x) \sim (2x)^{-1/2}$. Therefore $h(x)$ is zero until $x = -1$ where it jumps to the value 1, and it ultimately tends to zero like $(2x)^{-1/2}$; the graph of $h(x)$ obtained from the numerical work is shown in Graph B, together with that of $(2x)^{-1/2}$. It is observed that $h(x)$ attains its asymptotic value very quickly, the two curves being indistinguishable for $x > 4$, and since $p^{-1}\{h(x) - (2x)^{-1/2}\} = p^{-1}\pi^{1/2}(2p)^{-1/2}\{(K_1(p))^{-1} - p\} \to 0$ as $p \to 0$, the areas under the curves are equal.

The upper limit in the integral for $F(y)$ may be replaced by $T(y)$, where $y = T - \alpha R(T)$, because $h(x) = 0$ for $x < -1$. This value T is the distance from the nose at which the characteristic y = constant leaves the body surface, thus $F(y)$ depends upon the shape of the body up to this point as would be expected from the nature of supersonic flow. However, for the smooth body, expression (14) for $F(y)$ includes values of $S(t)$ only for the shorter range $0 \leq t \leq y$. Of course, this deviation from the expected range of dependence occurs in the first place in the ordinary linearized theory, and it is a remarkable result of that theory (see [3]) that although on general grounds it would be expected to introduce an error, for a *smooth* body it actually improves the accuracy. It is of interest to consider the connection between (14) and (21) further. Assuming that $S'(t)$ is continuous, it is observed that (14) is obtained from (21) by replacing the integrand in the latter by its asymptotic value $(y - t)^{-1/2}$, and replacing the upper limit $T(y)$ by y. The first step is true for the part of the range for which $t < \tau$ where $y = \tau + 4\alpha R(\tau)$ since $h(x)$ approximately attains its asymptotic form at $x = 4$; hence to obtain (14)

$$(23) \qquad \int_\tau^T \left(\frac{2}{\alpha R(t)}\right)^{1/2} h\left(\frac{y-t}{\alpha R(t)}\right) \frac{dS'(t)}{2\pi}$$

is replaced by

$$(24) \qquad \frac{1}{2\pi}\int_\tau^y \frac{S''(t)\,dt}{\sqrt{y-t}}.$$

For a smooth body, $S''(t)$ is continuous. Hence using the fact that the areas under the curves of $h(x)$ and $(2x)^{-1/2}$ are equal, the difference of (24) and (23) is certainly of smaller order than the error $O(\delta^3)$ in (21). (The error in (21) is a factor $1 + O(\delta)$ and for the smooth body $F(y)$ is $O(\delta^2)$.) Thus (14) and (21) are equivalent in this case and it is interesting that in fact (14) is more accurate. Near a discontinuity of $S''(t)$ it may be shown that the error in replacing (23) by (24) is $O(\delta^{5/2})$. Hence (14) gives a poor approximation to (21) and moreover the value of $F'(y)$ given by (14) becomes infinite at the point whereas it may be shown from (21) that it should really be $O(\delta^{3/2})$; for these reasons (14) will not be used except for the smooth body. It may be remarked in connection with this that even when $S'(t)$ is discontinuous, away from the discontinuities (there must be no discontinuity of $S'(t)$ or $S''(t)$ in $\tau \leq t \leq T$) the general expression for $F(y)$ may be approximated by the Stieltjes integral

$$\frac{1}{2\pi}\int_0^y \frac{dS'(t)}{\sqrt{y-t}} \tag{25}$$

which might have been expected to provide the necessary extension of the smooth body theory to the discontinuous case.

Now a discontinuity of $S'(t)$ at $t = t_1$, say, causes a discontinuity in $F(y)$ of magnitude

$$\left(\frac{2}{\alpha R(t_1)}\right)^{1/2} \frac{\Delta S'(t_1)}{2\pi}. \tag{26}$$

Such a jump in $F(y)$ means that for the corresponding value of y, F can take a whole range of values and hence there will be a fan of characteristics $x = \alpha r - kF(y)r^{1/2} + y$ through that point on the body. If the discontinuity in slope is a decrease, $\Delta F < 0$ and the flow expands round the corner in direct analogy with the Prandtl-Meyer expansion of two dimensions; if the discontinuity in slope is an increase, the fan is reversed so that there is a fold in the flow plane and an attached shock must intervene. These occurrences are discussed in detail in Part 4 but it may be noted here that the range of slopes of the characteristics in the fan is in exact agreement with the two-dimensional result.

The behaviour of $F(y)$ for small and large y will be required in the course of the work. For y sufficiently small, there are no discontinuities of $S'(t)$ (for the bodies under consideration) and the expression (14) is applicable. Near the nose $S''(t) \doteq 2\pi\epsilon^2$ where ϵ is the initial slope $R'(0)$, i.e., the nose semi-angle, hence

$$F(y) \sim 2\epsilon^2 y^{1/2} \quad \text{as} \quad y \to 0. \tag{27}$$

For y sufficiently large, there will be no discontinuities near $t = y$, hence (25) may be used, i.e. $F(y) = \pi^{1/2}p^{1/2}S'(y)/2\pi = \pi^{1/2}p^{3/2}S(y)/2\pi$. The behaviour of $F(y)$ for large y is deduced from the behaviour of its operational representation, and for a body whose ultimate radius is finite, the Heaviside representation of $S(y)$ is $S(\infty) + O(p)$ for small p; therefore

$$F(y) \sim \pi^{1/2}p^{3/2}S(\infty)/2\pi = y^{-3/2}S(\infty)/4\pi \quad \text{as} \quad y \to \infty. \tag{28}$$

Finally, the result that $\int_0^\infty F(y)\,dy = 0$ will be required later. Since $F(y) = \pi^{1/2}p^{1/2}f(y)$, $\int_0^y F(y')\,dy' = p^{-1}F(y) = \pi^{1/2}p^{-1/2}f(y)$, and this integral tends to zero as $y \to \infty$ if $\pi^{1/2}p^{-1/2}f(y) \to 0$ as $p \to 0$. This is the case provided that the representation of f is like $p^{1/2+\beta}$, where $\beta > 0$, as $p \to 0$, i.e. $f(y) = O(y^{-1/2-\beta})$