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in
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Azriel Levy

Basic Set Theory

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AZRIEL LEVY
Department of Mathematics, The Hebrew University
Jerusalem, Israel

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Preface to the Series

On Perspectives. *Mathematical logic arose from a concern with the nature and the limits of rational or mathematical thought, and from a desire to systematise the modes of its expression. The pioneering investigations were diverse and largely autonomous. As time passed, and more particularly in the last two decades, interconnections between different lines of research and links with other branches of mathematics proliferated. The subject is now both rich and varied. It is the aim of the series to provide, as it were, maps or guides to this complex terrain. We shall not aim at encyclopaedic coverage; nor do we wish to prescribe, like Euclid, a definitive version of the elements of the subject. We are not committed to any particular philosophical programme. Nevertheless we have tried by critical discussion to ensure that each book represents a coherent line of thought; and that, by developing certain* themes, *it will be of greater interest than a mere assemblage of results and techniques.*

The books in the series differ in level: some are introductory some highly specialised. They also differ in scope: some offer a wide view of an area, others present a single line of thought. Each book is, at its own level, reasonably self-contained. Although no book depends on another as prerequisite, we have encouraged authors to fit their book in with other planned volumes, sometimes deliberately seeking coverage of the same material from different points of view. We have tried to attain a reasonable degree of uniformity of notation and arrangement. However, the books in the series are written by individual authors, not by the group. Plans for books are discussed and argued about at length. Later, encouragement is given and revisions suggested. But it is the authors who do the work; if, as we hope, the series proves of value, the credit will be theirs.

History of the Ω-Group. *During 1968 the idea of an integrated series of monographs on mathematical logic was first mooted. Various discussions led to a meeting at Oberwolfach in the spring of 1969. Here the founding members of the group (R. O. Gandy, A. Levy, G. H. Müller, G. E. Sacks, D. S. Scott) discussed the project in earnest and decided to go ahead with it. Professor F. K. Schmidt and Professor Hans Hermes gave us encouragement and support. Later Hans Hermes joined the group. To begin with all was fluid. How ambitious should we be? Should we write the books ourselves? How long would it take? Plans for authorless books were promoted, savaged and scrapped. Gradually there emerged a form and a method. At the end of* *an infinite discussion we found our name, and that of the series. We established our centre in Heidelberg. We agreed to meet twice a year together with authors, consultants and assistants, generally in Oberwolfach. We soon found the value of collaboration: on the one hand the permanence of the founding group gave coherence to the over-all plans; on the other hand the stimulus of new contributors kept the project alive and flexible. Above all, we found how intensive discussion could modify the authors' ideas and our own. Often the battle ended with a detailed plan for a better book which the author was keen to write and which would indeed contribute a perspective.*

Acknowledgements. *The confidence and support of Professor Martin Barner of the Mathematisches Forschungsinstitut at Oberwolfach and of Dr. Klaus Peters of Springer-Verlag made possible the first meeting and the preparation of a provisional plan. Encouraged by the Deutsche Forschungsgemeinschaft and the Heidelberger Akademie der Wissenschaften we submitted this plan to the Stiftung Volkswagenwerk where Dipl. Ing. Penschuck vetted our proposal; after careful investigation he became our adviser and advocate. We thank the Stiftung Volkswagenwerk for a generous grant (1970–73) which made our existence and our meetings possible.*

Since 1974 the work of the group has been supported by funds from the Heidelberg Academy; this was made possible by a special grant from the Kultusministerium von Baden-Württemberg (where Regierungsdirektor R. Goll was our counsellor). The success of the negotiations for this was largely due to the enthusiastic support of the former President of the Academy, Professor Wilhelm Doerr. We thank all those concerned.

Finally we thank the Oberwolfach Institute, which provides just the right atmosphere for our meetings, Drs. Ulrich Felgner and Klaus Glöde for all their help, and our indefatigable secretary Elfriede Ihrig.

Oberwolfach
September 1975

R. O. Gandy *H. Hermes*
A. Levy *G. H. Müller*
G. E. Sacks *D. S. Scott*

Author's Preface

Almost all the recently-published books on set theory are of one of the following two kinds. Books of the first kind treat set theory on an elementary level which is, roughly, the level needed for studying point set topology and Steinitz's theorem on the existence of the algebraic closure of a general field. Books of the second kind are books which give a more or less detailed exposition of several areas of set theory that are subject to intensive current research, such as constructibility, forcing, large cardinals and determinacy. Books of the first kind may serve well as an introduction to the subject but are too elementary for the student or the mathematician who wants to gain a deeper understanding of set theory. The books of the second kind usually go hurriedly through the basic parts of set theory in their justified haste to get at the more advanced topics. One of the advantages of writing a book in a series such as the Perspectives in Mathematical Logic is that one is able to write a book on a rather advanced level covering the basic material in an unhurried pace. There is no need to reach the frontiers of the subject as one can leave this to other books in the series. This enables the author to pay close attention to interesting and important aspects of the subject which do not lie on the straight road to the very central topics of current research.

I started writing this book in 1970. During the long period since that time I have been helped by so many people that I cannot name them all here. Several of my colleagues advised me on the material in the book, read parts of the manuscript and made very useful remarks, and taught me new theorems and better proofs of theorems I knew. Many typists typed the numerous versions of the manuscript and bore with admirable patience all my inconsistent instructions. I shall mention in name only Klaus Glöde and Uri Avraham to whom I am most grateful for diligently reading the galley proofs, correcting many misprints and mistakes. This book would not have been written without the initiative and encouragement of my colleagues in the Ω group. I enjoyed very much their company and collaboration.

I shall be most grateful to any reader who will point out misprints, mistakes and omissions and who will supply me with additional bibliographical references. This will hopefully be incorporated in later printings of this book.

Most of this book was written while I stayed as a visitor at Yale University in the academic year 1971–72 and at UCLA in 1976–77. I extend my thanks to the

National Science Foundation of the United States for partially supporting me during those years.

May 12, 1978
Jerusalem

A. Levy

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Part A

Pure Set Theory

Chapter I
The Basic Notions

All branches of mathematics are developed, consciously or unconsciously, in set theory or in some part of it. This gives the mathematician a very handy apparatus right from the beginning. The most he usually has to do in order to have his basic language ready is to describe the set theoretical notation he uses. In developing set theory itself we have no such advantage and we must go through the labor of setting up our set theoretical apparatus. This is a relatively long task. Even the question as to which objects to consider, only sets or also classes, is by no means trivial, and its implications will be discussed here in detail. In addition, we shall formulate the axioms of set theory; we shall show how the concepts of ordered pair, relation and function, which are so basic in mathematics, can be developed within set theory, and we shall study their most basic properties. By the end of this chapter we shall be just about ready to begin our real mathematical investigation of the universe of sets.

1. The Basic Language of Set Theory

In the present section and in Sections 3 and 4 we shall thoroughly discuss the language we are going to use for set theory. Usually when one studies a branch of mathematics one is not concerned much with the question as to which exactly is the language used in that branch. The reason why here we must look carefully at the language lies in the difference between set theory and most other branches of mathematics. Most mathematical fields use a relatively "small" fragment of set theory as their underlying theory, and rely on that fragment for the language, as well as for the set theoretical facts. The source of the difficulty we have in set theory with the language is the fact that not every collection of objects is a set (something which will be discussed in detail in the next few sections), but we still have to refer often to these collections, and we have to arrange the language so that we shall be able to do it handily. This difficulty does not come up in the fragments of set theory used for most mathematical theories, since those fragments deal only with a very restricted family of collections of objects, and all these collections are indeed sets.

Our present aim is to obtain for set theory a language which is sufficiently rich and flexible for the practical development of set theory, and yet sufficiently simple so as not to stand in the way of metamathematical investigation of set theory. For this purpose we start by choosing for set theory a very simple basic language. The simplicity of this language will be a great advantage when we wish to discuss set theory from a metamathematical point of view. The only objects of our set theory will be sets. One could also consider atoms, i.e., objects which are not sets and which serve as building blocks for sets, but they are not essential to what we shall do and, therefore, will not be considered in the present book. As a consequence of this decision, we view the sets as follows. We start with the null set 0, from it we obtain the set $\{0\}$, from the two sets 0 and $\{0\}$ we obtain the sets $\{0, \{0\}\}$ and $\{\{0\}\}$, and so on. Much of set theory is concerned with what is meant by this "and so on".

The language which we shall use for set theory will be the first-order predicate calculus with equality. Why first-order? Because a second-order or a higher-order theory admits already a part of the set theory in using its higher order variables. To see, for example, that second-order variables are essentially set variables let us consider the following axiom of second-order logic: $\exists A \forall x(x \in A \leftrightarrow \Phi(x))$, where $\Phi(x)$ is, essentially, any formula. This axiom is read: there exists a *set* A such that for every x, x is a member of A if and only if $\Phi(x)$. It would, of course, change nothing if we would choose another term instead of "set", since "set" is what we mean anyway. When we develop a formal system of set theory it does not seem right to handle sets in two or more parts of the language, i.e., by considering some sets as first-order objects, while having around also second-order objects which are sets. As a consequence of our decision we shall have, in principle, just one kind of variable, lower case letters, which will vary over sets.

The reason why we take up first-order predicate calculus *with equality* is a matter of convenience; by this we save the labor of defining equality and proving all its properties; this burden is now assumed by the logic.

Our basic language consists now of all the expressions obtained from $x = y$ and $x \in y$, where x and y are any variables, by the sentential connectives $\neg$(not), $\rightarrow$ (if...then...), $\vee$ (or), $\wedge$ (and), $\leftrightarrow$ (if and only if), and the quantifiers $\exists x$ (there exists an x) and $\forall x$ (for all x). These expressions will be called *formulas*. For metamathematical purposes we can consider the connectives $\neg$ and $\vee$ as the only primitive connectives, and the other connectives will be considered as obtained from the primitive connectives in the well known way (e.g., $\phi \wedge \psi$ is $\neg(\neg\phi \vee \neg\psi)$, $\phi \rightarrow \psi$ is $\neg\phi \vee \psi$, etc.). For the same reason, we can consider $\exists$ as the only primitive quantifier and the quantifier $\forall$ as defined by means of $\exists$ by taking $\forall x\phi$ to be an abbreviation of $\neg\exists x \neg\phi$. We shall also use the abbreviations $x \neq y$ and $x \notin y$ for $\neg x = y$ and $\neg x \in y$. We shall write $\exists! x\phi$, and read: there is exactly one x such that ϕ, for the formula $\exists y \forall x(x = y \leftrightarrow \phi)$, where y is any variable which is not free in ϕ. Finally, we shall write $(\exists x \in y)\phi$ and $(\forall x \in y)\phi$ for $\exists x(x \in y \wedge \phi)$ and $\forall x(x \in y \rightarrow \phi)$, respectively, and read: "there is an x in y such that ϕ", and "for all x in y, ϕ".

By a *free variable* of a formula we mean, informally, a variable occurring in that formula so that it can be given different values and the formula says something concerning the values of the variable. E.g., x is a free variable in each of the following formulas (which are not necessarily taken from set theory): $x < 3x$,

$x^2 = y$, x is a real number, $\sin x > \frac{1}{2}$, $\forall y(z < y \to x < y)$. x is not a free variable in the formulas $\forall x(x^2 \geqslant 0)$, $\neg \exists x(x \in y)$, $\int_0^y \sin x\, dx < \frac{1}{2}$. In the latter three formulas x is an auxilliary variable which cannot be given a definite value and which can be replaced throughout each formula by another variable, say z, without changing the meaning of the formula. In these examples x is used as a *bound variable*. Note that $\forall x(x^2 \geqslant 0)$ says exactly what $\forall z(z^2 \geqslant 0)$ says, while $\sin x > \frac{1}{2}$ does not say the same thing as $\sin z > \frac{1}{2}$; in fact, for appropriate values of x and z $\sin x > \frac{1}{2}$ may be true, while $\sin z > \frac{1}{2}$ may be false. A variable may have both free and bound occurrences in the same formula, even though one would usually try to avoid it; e.g., in $7 < z \wedge \exists z(z > x)$, the occurrences of z in $\exists z(z > x)$ are bound, while the occurrence of z in $7 < z$ is free (since the quantifier $\exists z$ applies only to $z > x$).

A formula with free variables says something about the values of its free variables. A formula without free variables makes a statement not about the value of some particular variable, but about the universe which the language describes. A formula of the latter kind is called a *sentence*. We shall also refer, informally, to formulas and sentences as *statements*.

Whenever we use a formula with free variables as an axiom or as a theorem we mean to say that the formula holds for all possible values given to its free variables. Thus, if we state a theorem $\exists z(z = x \cup y)$ we mean the same thing as $\forall x \forall y \exists z(z = x \cup y)$.

By a *theory* we mean a set of formulas, which are called the *axioms* of the theory. If T is a theory, we shall write $T \vdash \phi$ for "ϕ is provable from T".

When we refer to a formula as $\phi(x)$ this does not mean that x is necessarily a free variable of $\phi(x)$ nor does it mean that $\phi(x)$ has no free variables other than x; it means that the interesting cases of what we shall say are those where x is indeed a free variable of $\phi(x)$. When we shall mention $\phi(z)$ after we have first mentioned $\phi(x)$, then $\phi(z)$ denotes the formula obtained from $\phi(x)$ by substituting the variable z for the free occurrences of x. (z may also be a bound variable of $\phi(x)$, and then before we substitute z for the free occurrences of x we may have to replace the bound occurrences of z by some other variable.)

2. *The Axioms of Extensionality and Comprehension*

By a set we mean a completely structure-free set, and therefore a set is determined solely by its members. This leads us to the first axiom of set theory.

2.1 Axiom of Extensionality (Frege 1893). $\forall x(x \in y \leftrightarrow x \in z) \to y = z$.
In words: if y and z have the same members they are equal. The converse, that equal objects have the same members, is a fogical truth.

2.2 The Existence of Sets. Now we face the question of finding or constructing the sets. We want any collection whatsoever of objects, i.e., sets, to be a set. This is not a precise idea and therefore we cannot translate it into our language. We must therefore be satisfied with a somewhat weaker stipulation. We shall require that every collection of sets which is "specifiable" in our language is a set; i.e., for every statement of our language the collection of all objects which satisfy it is a set. We shall by no means assume that it is necessarily true that *all* sets are specifiable; moreover, by introducing the axiom of choice we shall require the existence of sets which are not necessarily specifiable. The requirement that all specifiable collections are indeed sets is the following one.

2.3 Axiom of Comprehension (Frege 1893). $\exists y \forall x(x \in y \leftrightarrow \phi(x))$,
where $\phi(x)$ is any formula (of the language of set theory) in which the variable y is not free (since if y were free in $\phi(x)$ this would cause a confusion of the y free in $\phi(x)$ with the y whose existence is claimed by the axiom). Our only reason in writing $\phi(x)$ instead of just ϕ is to draw attention to the fact that the "interesting" cases of this axiom schema are those for which the formula ϕ does actually contain free occurrences of the variable x.

The axiom of comprehension is an *axiom schema*, i.e., it is not a single sentence but an infinite set of sentences obtained by letting ϕ vary over all formulas. Any single sentence obtained from 2.3 by choosing a particular formula for ϕ in 2.3 is said to be an *instance* of the axiom schema, and is also called "an axiom of comprehension." □

Those readers who were convinced by the axiom schema of comprehension are now in for a shock; the axiom schema of comprehension is not consistent—Theorem 2.4 below is the negation of one of its instances.

2.4 Theorem (Russell's antinomy—Russell 1903). $\neg \exists y \forall x(x \in y \leftrightarrow x \notin x)$.

Proof. Notice that this theorem is not just a theorem of set theory; it is a theorem of logic, since we do not use in its proof any axiom of set theory. We prove it by contradiction. Suppose y is a set such that $\forall x(x \in y \leftrightarrow x \notin x)$, then, since what holds for every x holds in particular for y, we have $y \in y \leftrightarrow y \notin y$, which is a contradiction. □

Russell's antinomy is the simplest possible refutation of an instance of the comprehension schema. We refer to a refutation of such an instance as an *antinomy*. The first antinomy to be discovered is the Burali–Forti paradox discovered by Cantor and by Burali–Forti in the 1890's; it is given in II.3.6 and II.3.15. Some variants of Russell's antinomy are given in 2.5.

2.5 Exercise. Prove the negation of the instance of the axiom of comprehension where $\phi(x)$ is one of the following formulas:

(a) $\neg \exists u(x \in u \wedge u \in x)$,

(b) $\neg \exists u_1 \ldots \exists u_n(x \in u_1 \wedge u_1 \in u_2 \wedge \cdots \wedge u_{n-1} \in u_n \wedge u_n \in x)$. □

2.6 How to Avoid the Antinomies. One can react to Russell's antinomy in two different ways. One way is to think again of what led us to the axiom of comprehension, and to decide that since a set is something like 0, $\{0\}$, $\{0, \{0\}\}$, etc., we should not have come up with anything like the axiom of comprehension anyway. According to this view, the axiom of comprehension is basically false, since it represents a mental act of "collecting" all sets which satisfy $\phi(x)$, and this cannot be done since we can "collect" only those sets which have been "obtained" at an "earlier" stage of the game. This point of view was suggested first by Russell 1903 as one of the ingredients of his theory of types. The other possible reaction to Russell's antinomy is to continue believing in the essential truth of the axiom schema of comprehension, viewing the Russell antinomy as a mere practical joke played on mankind by the goddess of wisdom. According to this point of view the axiom schema of comprehension is only in need of some tinkering to avoid the antinomies; the guide on how to do it will be the *doctrine of limitation of size*. The doctrine says that we should use the axiom schema of comprehension only in order to obtain new sets which are not too "large" compared to the sets whose existence is assumed in the construction. Also this doctrine, which is already implicit in Cantor 1899, was formulated first by Russell 1906. In our framework of set theory both approaches lead to the same result, and therefore there is no mathematical need to go through the arguments in favor of each one of them. Motivations for the choice of the axioms, from both points of view, are presented in the literature (see, e.g., Fraenkel, Bar–Hillel and Levy 1973 and Scott 1974) and will hopefully be presented in a later book in this series devoted to the axiomatics of set theory. Here we shall mostly rely on the acceptance by the reader of the axioms which we shall introduce as intuitively reasonable axioms. □

Let us still notice one feature of the axiom of comprehension. After the failure of the full axiom of comprehension, we cannot be sure that, given a formula ϕ, there is a set y such that $\forall x(x \in y \leftrightarrow \phi(x))$. However, if there is such a y it is unique, as stated in the next theorem.

2.7 Proposition. *If there is a y such that*

$$\forall x(x \in y \leftrightarrow \phi(x))$$

then this y is unique.

Proof. If y' is also such, i.e., $\forall x(x \in y' \leftrightarrow \phi(x))$, then we have, obviously, $\forall x(x \in y' \leftrightarrow x \in y)$, and by the axiom of extensionality, $y' = y$. □

3. *Classes, Why and How*

As we shall come to see, the main act of generation of set theory is that objects are collected to become a set, which is again an object which can be collected into a new set. We saw above that, because of the antinomies, not every collection of objects which can be specified in our language can be collected to become a "new" object. This is by no means disastrous for mathematics, since, by means of appropriate axioms which we shall introduce, we shall be able to show that sufficiently many of the intuitive collections can indeed be taken as sets to satisfy the mathematical needs. This will enable us to obtain sets such as the set of all real numbers, the set of all countable ordinals, the set of all measures on some given set, etc.

There are many things we can say about an intuitive collection of objects without assuming that the collection is an object itself. Let us see an example. Suppose we want to say

(1) Every non-void subset u of the collection of all sets x such that $x \notin x$ has a member y such that y has no common member with the collection.

This can also be said as

(2) $$\forall u(u \neq 0 \wedge (\forall x \in u) x \notin x \rightarrow (\exists y \in u)\ (\forall x \in y) x \in x).$$

Notice that (2) does not mention the collection mentioned in (1). We could decide never to use (1) and always to use (2) instead; but, as we shall point out now, and as will become even clearer to the reader as he goes on reading this book, this would have required a considerable sacrifice of convenience. Sometimes we want to say about many or all specifiable collections what we said in (1) about one particular collection. We can proceed as in (2) but this requires using an infinite family of formulas. This is illustrated by the following example. Suppose we want to say that for every specifiable collection A

(3) $$\forall u(u \neq 0 \wedge u \subseteq A \rightarrow (\exists y \in u) y \notin A) \rightarrow A \textit{ has at most 10 members.}$$

We can say the same thing also by asserting that for all formulas $\phi(x, x_1, \ldots, x_n)$

(4) $$\forall x_1 \ldots \forall x_n[\forall u(u \neq 0 \wedge (\forall x \in u)\phi(x, x_1, \ldots, x_n) \rightarrow (\exists y \in u) \neg \phi(y, x_1, \ldots, x_n)) \rightarrow \textit{there are at most 10 objects } x \textit{ such that } \phi(x, x_1, \ldots, x_n)].$$

To see that (3) and (4) say the same thing notice that the specifiable collections A are exactly those given as the collection of all objects x such that $\phi(x, x_1, \ldots, x_n)$ holds, for some formula $\phi(x, x_1, \ldots, x_n)$ and for some fixed values of $x_1, \ldots, x_n$. Comparing (3) with (4) shows that (3) is not only shorter but also much easier to comprehend than (4). Thus we see that it is a great advantage to be able to talk about collections as if they were sets even though we know, as a result of Russell's antinomy, that not all of them are sets. A uniform way of talking about sets and collections has also the following advantage. We often come across collections which, at a certain point in the discussion, we do not know whether they are sets or not. Speaking about them in the same way as we speak about sets puts what we say about them in a form which retains its convenience even after the collections turn out to be sets.