

THERMOELASTICITY

by

WITOLD NOWACKI

Translated by

HENRYK ZORSKI

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Revised and enlarged



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PREFACE TO THE FIRST EDITION

In the postwar years we have seen a rapid development of thermoelasticity, stimulated by various engineering sciences. A considerable progress in the field of aircraft and machine structures, mainly with gas and steam turbines, and the emergence of new topics in chemical and nuclear engineering have given rise to numerous problems in which thermal stresses play an important and frequently even a primary role.

Thermoelasticity embraces a wide field of phenomena. It contains the theory of heat conduction and the theory of strains and stresses due to the flow of heat, when coupling of temperature and strain fields occurs. Thermoelasticity makes it possible to determine the stresses produced by the temperature field, and, moreover, to calculate the distribution of temperature due to the action of internal forces which vary with time.

The particular case of thermoelasticity in which the influence of the coupling of temperature and strain fields is neglected is the basis of most quasi-static and dynamic problems dealt with in this monograph; the influence of the coupling is considered in detail in the Chapters IV and VIII.

We shall investigate mainly the thermoelastic problems which occur in isotropic and homogeneous bodies, under the assumption of small displacements and linear stress-strain laws. We assume, moreover, that the material constants are independent of temperature. These assumptions are peculiar to the classical theory of elasticity and obviously limit the applicability of the solutions obtained to certain ranges of temperature. They make it possible, however, to investigate a large class of engineering problems.

An attempt has been made in this monograph to unify the exposition. The solutions have been constructed consisting of two parts, the first representing the thermoelastic displacement potential while the second one is an additional solution expressed by the components of the Galerkin vector or the stress functions. Wherever it was convenient, Maysel's method was employed.

In many problems investigated in this monograph the principal point is the determination of the Green functions for stresses. Therefore the integral transforms of Fourier, Hankel and Laplace have consistently been used. The latter is particularly useful in solving non-stationary thermoelastic problems.

The problems have been divided into two large groups — the spatial and plane problems, each group containing both stationary and non-stationary solutions. The detailed examination of the problems is preceded by an extensive introduction to thermoelasticity (Chapter I) which contains the basic relations and equations, and the methods of solution.

In Chapter XI we deal with the non-classical problem of thermal stresses in viscoelastic bodies with linear material relations. Finally, the last chapter contains a review of new trends in thermoelasticity. We also consider briefly thermal stresses in non-homogeneous, isotropic and anisotropic bodies.

I wish here to express my gratitude to Doc. habil. Dr. H. Zorski, Dr. J. Ignaczak and Dr. W. Piechocki, for reading the manuscript and for many helpful suggestions.

W. Nowacki

Warsaw

PREFACE TO THE SECOND EDITION

The first edition of *Thermoelasticity* appeared 20 years ago, many changes, deletions and additions therefore have been made, first of all in Chapter I dealing with the fundamental relations and equations of thermoelasticity and in Chapter II devoted to the new trends and new developments in thermoelasticity.

It is assumed in the static elasticity that during a slow growth of the loadings and hence also the deformations, a complete exchange of heat with the surroundings takes place. Furthermore, we assume that the temperature in the whole body is equal to the temperature in the natural state. In classical dynamic elasticity the corresponding assumption is that the heat exchange occurring by means of the heat conduction, is very slow, the process therefore is adiabatic. The theory of thermal stresses is based on a different assumption, namely we take into account the action of heat sources and surface heating but we neglect the changes of the temperature field due to the deformations of the body.

The above three basic assumptions lead to three different sets of differential equations describing the fields of strain and temperature. The creation of the coupled thermoelasticity is due to the tendency to obtain one system of differential equations describing all thermodynamic processes. Chapter I treats thermoelasticity as a synthesis of the theory of elasticity and the theory of heat conduction; some particular cases of thermoelasticity are also investigated, e.g. stationary problems, the theory of thermal stresses, classical dynamic elasticity, etc.

Chapter XII has been completed by a brief examination of micropolar thermoelasticity, magnetothermoelasticity and thermopiezoelectricity. In order to reduce the volume of the book we omitted Section VI.8 concerning the application of the complex variable method to two-dimensional problems of thermoelasticity. Finally, we reduced the list of references to the papers which played a fundamental role in the development of thermoelasticity.

W. Nowacki

Warsaw, 1983

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CHAPTER I

BASIC RELATIONS AND EQUATIONS OF THERMOELASTICITY

1.1. Principle of energy conservation. Entropy balance

A deformation of a body is inseparably connected with a change of its heat content and therefore with a change of the temperature distribution in the body. A deformation of a body which varies in time leads to temperature changes, and conversely. The internal energy of the body depends on both the temperature and the deformation. The science which deals with the investigation of the above coupled processes, is called thermoelasticity.

It should be emphasized that investigations in the field of thermoelasticity were preceded by extensive investigations in the so-called theory of thermal stresses. The latter is a theory of the state of strain and stress in an elastic body, due to a heating, under the simplifying assumption that the influence of the deformation on the temperature field may be neglected.

In the theory of thermal stresses which goes back to the beginnings of the theory of elasticity, the classical heat conduction equation was used, which does not contain the term due to the deformation of the body. Knowing the temperature distribution from the solution of the heat conduction equation, the displacement equations of the theory of elasticity were solved, the latter containing the known terms proportional to the temperature gradient ⁽¹⁾.

At the same time the classical dynamic elasticity was being developed, also under the simplifying assumption that the heat exchange between parts of the body due to the heat conduction occurs very slowly and therefore the motion may be regarded as adiabatic.

Thermoelasticity deals with a wide class of phenomena. It contains the generalized theory of heat conduction, the generalized theory of thermal stresses, describes the temperature distribution produced by deformation and finally it contains a description of the phenomenon of thermoelastic dissipation. The cognitive merits of this theory are very large indeed. In spite of its mathematical complexity, the thermoelasticity enables us to examine deeper than before, the mechanism of the deformation and thermal processes occurring in elastic bodies.

⁽¹⁾ J. M. C. Duhamel, *Second mémoire sur les phénomènes thermo-mécaniques*, J. de l'Ecole Polytechnique, 1837, Vol. 15, p. 1.

Thermoelasticity is a new field. Although the coupling between the deformation and temperature fields was postulated already by J. M. C. Duhamel and the extended heat conduction equation was given by W. Voigt ⁽¹⁾ and H. Jeffreys ⁽²⁾, the real development of the theory occurred in the last twenty years. The starting point was the paper by M. A. Biot ⁽³⁾ in which on the basis of the thermodynamics of irreversible processes the basic relations and equations were derived and the variational theorems of thermoelasticity were formulated.

In this chapter we present in detail the derivation of the basic relations and equations of thermoelasticity and the methods of their solution, as well as the energy theorems and the integration methods they imply.

We begin our considerations by deducing the constitutive relations, i.e. the relations between the stress tensor and the entropy, and the strain tensor and the temperature.

We confine ourselves in principle to elastic isotropic homogeneous bodies. An elastic deformation is a state such that when the forces producing the deformation are removed, the body returns to its initial undeformed state. The isotropy means that the elastic properties of the body are independent of the direction and the homogeneity means the independence of the elastic properties of the position.

Assume that the body in its undeformed stress-free state (no external forces acting) has a certain constant temperature T_0 . This initial state is called the natural state of the body. Due to the action of external forces, i.e. body forces and surface tractions, and to the action of heat sources inside the body or a heating (or cooling) of its surface, the body undergoes a deformation. There arise in the body the displacements u and the temperature undergoes a change $T = \mathcal{T} - T_0$, where \mathcal{T} is the absolute temperature. The deformation of the body is accompanied by a production of the strain ε_{ij} and the stress σ_{ij} . These quantities depend on position and time. We assume that the change of temperature $T = \mathcal{T} - T_0$ accompanying the deformation is small and that the temperature increase T does not produce any appreciable changes of the material coefficients, both elastic and thermal. These coefficients will be regarded as quantities independent of T .

The assumption $|T/T_0| \ll 1$ is now completed by another one concerning the magnitude of the deformation; we assume that the squares and products of the components of the strain ε_{ij} can be neglected as compared with ε_{ij} . Thus, we confine ourselves to the geometrically linear thermoelasticity. The relation between the strain and displacement has then the linear form

$$(1) \quad \varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}).$$

It is known that the strain components cannot be arbitrary and must satisfy six so-called compatibility relations ⁽⁴⁾

$$(2) \quad \varepsilon_{i,j,kl} + \varepsilon_{kl,ij} - \varepsilon_{jl,ik} - \varepsilon_{ik,jl} = 0, \quad i, j, k, l = 1, 2, 3.$$

⁽¹⁾ W. Voigt, *Lehrbuch der Kristallphysik*, Teubner, Leipzig, 1910.

⁽²⁾ H. Jeffreys, *The thermodynamics of an elastic solid* in Proc. Cambr. Phill. Soc., 1930.

⁽³⁾ M. A. Biot, *Thermoelasticity and irreversible thermodynamics*, J. Appl. Phys., 1956, Vol. 27, p. 240.

⁽⁴⁾ J. S. Sokolnikoff, *Mathematical theory of elasticity*, New York, 1956.

The basic problem consists in deducing the constitutive relations, i.e. the relations connecting the components of the stress tensor σ_{ij} , the entropy S , the components of the strain tensor ε_{ij} and the temperature \mathcal{T} .

Observe that the mechanical and thermal state of the medium at a certain instant of time, is completely described by the distribution of the deformations ε_{ij} and the temperature \mathcal{T} . Thus, in an isothermal change of state we are faced with a process both elastically and thermally reversible. However, in processes in which the temperature varies in time we deal with two coupled processes, namely the reversible elastic process and the irreversible thermodynamic process. The latter is due to a spontaneous and hence irreversible process of heat transfer by means of heat conduction.

The thermoelastic changes cannot be described by means of the classical thermodynamics valid for equilibrium states; we must use the relations of the thermodynamics of irreversible processes.^{(1) (2)}

The point of departure of our considerations is the first and second law of thermodynamics. The first law, the law of energy conservation, has the form

$$(3) \quad \frac{d}{dt} (\mathcal{U} + \mathcal{K}) = \mathcal{L} + \frac{dQ}{dt}.$$

Here \mathcal{U} is the internal energy, \mathcal{K} the kinetic energy, \mathcal{L} the power of external forces and Q is the increment in time of the quantity of heat absorbed by the body. Equation (3) is the energy balance. It states that the increment in time of the sum of the internal and kinetic energies is equal to the sum of the power of the external forces and the heat absorbed by the body.

The power of the external forces is given by the formula

$$(4) \quad \mathcal{L} = \int_V \mathbf{X}_i v_i dV + \int_A p_i v_i dA,$$

where \mathbf{X} is the vector of the body forces, \mathbf{p} is the vector of surface tractions occurring on the surface A bounding the body, $\mathbf{v} = d\mathbf{u}/dt$ is the vector of the displacement velocity. The components p_i of the vector \mathbf{p} are connected with the stress vector by the relations

$$(5) \quad p_i = \sigma_{ji} n_j.$$

Here n_i is the component of normal — of the surface A . We assume that the normal \mathbf{n} is directed towards the exterior of the body.

The kinetic energy appearing in (3) is expressed by the integral

$$(6) \quad \mathcal{K} = \frac{\rho}{2} \int_V v_i v_i dV,$$

(1) S. R. de Groot, *Thermodynamics of irreversible processes*, Amsterdam, 1952.

(2) L. Prigogine, *Etude thermodynamique des phénomènes irréversibles*, Liège, 1947.

and the non-mechanical power \dot{Q} has the form

$$(7) \quad \dot{Q} = - \int_A q_i n_i dA + \int_V W dV.$$

Here \mathbf{q} is the vector of the heat flux and W the quantity of heat generated in unit time and unit volume.

Making use of the Gauss-Ostrogradski theorem we obtain from (7)

$$(8) \quad \dot{Q} = - \int_V (q_{i,i} - W) dV.$$

We introduce now the internal energy U per unit volume

$$(9) \quad \mathcal{U} = \int_V U dV.$$

Substituting (4)–(9) into the energy balance (3) we obtain

$$(10) \quad \int_V (\rho v_i \dot{v}_i + \dot{U}) dV = \int_V X_i v_i dV + \int_A p_i v_i dA - \int_V (q_{i,i} - W) dV.$$

Since

$$p_i = \sigma_{ji} n_j, \quad \int_A \sigma_{ji} n_j v_i dA = \int_V (\sigma_{ji} v_i)_{,j} dV,$$

Equation (10) takes the form

$$(11) \quad \int_V [\dot{U} - (\sigma_{ji,j} + X_i - \rho \dot{v}_i) v_i - \sigma_{ji} v_{i,j} + q_{i,i} - W] dV.$$

Equation (11) should be valid for every part of the body; thus we arrive at the local principle of energy conservation

$$(11') \quad \dot{U} = (\sigma_{ji,j} + X_i - \rho \dot{v}_i) v_i + \sigma_{ji} v_{i,j} - q_{i,i} + W.$$

We now demand that the expression (11') be invariant with respect to a rigid motion of the body.⁽¹⁾

Consider first the rigid displacement

$$(12) \quad v_i \rightarrow v_i + b_i,$$

where b_i is an arbitrary constant vector. We assume that during a rigid displacement the quantities ρ , σ_{ij} , X_i , q_i , W remain constant. Introducing (12) into (11') we obtain

$$(13) \quad \dot{U} = (v_i + b_i) (\sigma_{ji,j} + X_i - \rho \dot{v}_i) + \sigma_{ji} v_{i,j} - q_{i,i} + W.$$

Subtracting (11') from (13) we have

$$b_i (\sigma_{ji,j} + X_i - \rho \dot{v}_i) = 0.$$

(1) A. E. Green, R. S. Rivlin, Arch. Rat. Mech. Analysis, 17 (1964), 113.

This equations should be satisfied for an arbitrary value of b_i . Thus, we arrive at the equations of motion

$$(14) \quad \sigma_{ji,j} + X_i - \rho \dot{v}_i = 0.$$

Taking into account equation (14) we obtain a considerably simplified law of energy balance (11'), namely

$$(15) \quad \dot{U} = \sigma_{ji} v_{i,j} - q_{i,i} + W.$$

This expression should be invariant with respect to a rigid rotation of the body. We assume that

$$(16) \quad v_i \rightarrow v_i - \varepsilon_{ikl} x_k \Omega_l, \quad v_{i,j} \rightarrow v_{i,j} - \varepsilon_{ijl} \Omega_l, \quad \Omega_l = \text{const.}$$

Substituting from (16) to (15) we have

$$(17) \quad \dot{U} = \sigma_{ji}(v_{i,j} - \varepsilon_{ijl} \Omega_l) - q_{i,i} + W.$$

Subtracting (15) from (17) and taking into account the invariance of the quantities U , σ_{ji} , q_i , W we obtain

$$\Omega_l \varepsilon_{ijl} \sigma_{ji} = 0, \quad \Omega_l \neq 0.$$

This result proves that the stress tensor is symmetric, i.e.

$$(18) \quad \sigma_{ji} = \sigma_{ij}.$$

Observe that

$$v_{i,j} = \frac{1}{2} (\dot{u}_{i,j} + \dot{u}_{j,i}) + \frac{1}{2} (\dot{u}_{i,j} - \dot{u}_{j,i}) = \dot{\varepsilon}_{ij} + \dot{\omega}_{ij}.$$

Here $\dot{\varepsilon}_{ij} = \partial \varepsilon_{ij} / \partial t$, ε_{ij} is defined by the relation (1), and $\dot{\omega}_{ij} = \partial \omega_{ij} / \partial t$ is the time derivative of the rotation tensor ω_{ij} . Since the stress tensor σ_{ji} is symmetric and the tensor ω_{ji} is antisymmetric, we have $\sigma_{ji} \dot{\omega}_{ji} = 0$.

Thus, we have the local relation

$$(19) \quad \dot{U} = \sigma_{ij} \varepsilon_{ij} - q_{i,i} + W.$$

Consider now the local entropy balance

$$(20) \quad \mathcal{F} \dot{S} = -q_{i,i} + W,$$

where S is the entropy per unit volume and unit time. Let us integrate (20) over the volume of the body. Then

$$\int_V \dot{S} dV = - \int_V \left(\frac{q_i}{\mathcal{F}} \right)_{,i} dV - \int_V \frac{q_i \mathcal{F}_{,i}}{\mathcal{F}^2} dV + \int_V \frac{W}{\mathcal{F}} dV,$$

or

$$(21) \quad \int_V \dot{S} dV = - \int_A \frac{q_i n_i}{\mathcal{F}} dA - \int_V \frac{q_i \mathcal{F}_{,i}}{\mathcal{F}^2} dV + \int_V \frac{W}{\mathcal{F}} dV.$$

The increment of the entropy in time consists of two basic parts. The first is described by the surface integral constituting the increase (or decrease) of the entropy due to the heat flux through the surface A . Thus, this integral describes the heat exchange with the surroundings. The volume integrals are due to the entropy production in the volume V . The first integral is the entropy produced in V by the heat exchange while the second is the entropy produced by the action of the heat sources.

Let us now return to the relation (20) written in the form

$$(22) \quad \frac{dS}{dt} = - \left(\frac{q_i}{\mathcal{F}} \right)_{,i} - \frac{q_i \mathcal{F}_{,i}}{\mathcal{F}^2} + \frac{W}{\mathcal{F}}.$$

Here, too, the first term refers to the heat exchange with the surroundings while the two remaining ones describe the entropy production in an elementary volume of the body.

The local statement of the second law of thermodynamics of irreversible processes leads to the Clausius-Duhem inequality

$$(23) \quad - \frac{q_i \mathcal{F}_{,i}}{\mathcal{F}^2} \geq 0,$$

or

$$(24) \quad \frac{dS}{dt} + \left(\frac{q_i}{\mathcal{F}} \right)_{,i} - \frac{W}{\mathcal{F}} \geq 0.$$

In what follows it will be more convenient to introduce the Helmholtz free energy $F = U - S\mathcal{F}$ which depends on the variables ε_{ij} and \mathcal{F} . Thus, we are led to the equation

$$(25) \quad \dot{F} = \dot{U} - S\dot{\mathcal{F}} - \dot{S}\mathcal{F} = \sigma_{ij} \dot{\varepsilon}_{ij} - S\dot{\mathcal{F}} - \dot{S}\mathcal{F} - q_{i,i} + W.$$

Eliminating the heat sources from the inequality (24) and equation (25) we obtain

$$(26) \quad -(\dot{F} + S\dot{\mathcal{F}}) + \sigma_{ij} \dot{\varepsilon}_{ij} - \frac{q_i \mathcal{F}_{,i}}{\mathcal{F}} \geq 0.$$

We have

$$(27) \quad \dot{F} = \frac{\partial F}{\partial \varepsilon_{ij}} \dot{\varepsilon}_{ij} + \frac{\partial F}{\partial \mathcal{F}} \dot{\mathcal{F}} + \frac{\partial F}{\partial \mathcal{F}_{,i}} \dot{\mathcal{F}}_{,i},$$

where we assumed that $F \equiv F(\varepsilon_{ij}, \mathcal{F}, \mathcal{F}_{,i})$. Introducing (27) into (26) we arrive at the inequality

$$(28) \quad \left(\sigma_{ij} - \frac{\partial F}{\partial \varepsilon_{ij}} \right) \dot{\varepsilon}_{ij} - \left(S + \frac{\partial F}{\partial \mathcal{F}} \right) \dot{\mathcal{F}} + \frac{\partial F}{\partial \mathcal{F}_{,i}} \dot{\mathcal{F}}_{,i} - \frac{q_i \mathcal{F}_{,i}}{\mathcal{F}} \geq 0.$$

This inequality should be satisfied for all rates $\dot{\varepsilon}_{ij}$, $\dot{\mathcal{F}}$, $\dot{\mathcal{F}}_{,i}$. Hence the coefficients of the above variables must vanish:

$$(29) \quad \sigma_{ij} = \frac{\partial F}{\partial \varepsilon_{ij}}, \quad S = - \frac{\partial F}{\partial \mathcal{F}}, \quad \frac{\partial F}{\partial \mathcal{F}_{,i}} = 0.$$

The relation (29)₃ implies that the free energy is independent of the temperature gradient. We are therefore left with the inequality

$$(30) \quad -\frac{q_i \mathcal{T}_{,i}}{\mathcal{T}} \geq 0,$$

which is satisfied if we assume that

$$(31) \quad q_i = -\lambda_{ij} \mathcal{T}_{,j}.$$

The above is the Fourier law of heat conduction for an anisotropic body. The quantity $\Omega = -q_i \mathcal{T}_{,i}$ should be a positive definite quadratic form

$$(32) \quad \Omega = -q_i \mathcal{T}_{,i} = \lambda_{ij} \mathcal{T}_{,i} \mathcal{T}_{,j}.$$

In view of Silvester's theorem, the considered inequality imposes additional restrictions on the symmetric tensor of heat conduction λ_{ij} . For an isotropic body we have

$$(33) \quad q_i = -\lambda_0 \mathcal{T}_{,i}, \quad \lambda_0 > 0.$$

Let us now expand the free energy $F(\varepsilon_{ij}, \mathcal{T})$ into the Taylor series in the vicinity of the natural state ($\varepsilon_{ij} = 0, \mathcal{T} = T_0$)

$$(34) \quad F(\varepsilon_{ij}, \mathcal{T}) = F(0, T_0) + \frac{\partial F(0, T_0)}{\partial \varepsilon_{ij}} \varepsilon_{ij} + \frac{\partial F(0, T_0)}{\partial \mathcal{T}} (\mathcal{T} - T_0) + \\ + \frac{1}{2} \left(\frac{\partial^2 F(0, T_0)}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \varepsilon_{ij} \varepsilon_{kl} + 2 \frac{\partial^2 F(0, T_0)}{\partial \varepsilon_{ij} \partial \mathcal{T}} \varepsilon_{ij} (\mathcal{T} - T_0) + \frac{\partial^2 F(0, T_0)}{\partial \mathcal{T}^2} (\mathcal{T} - T_0)^2 \right) + \dots$$

Here $F(0, T_0)$ is the energy of the natural state, which we assume to be equal to zero. Moreover, we assume that the quantities $\partial F(0, T_0)/\partial \varepsilon_{ij}$ and $\partial F(0, T_0)/\partial \mathcal{T}$ vanish; they constitute the stress and the entropy in the natural state, respectively. Thus

$$(35) \quad F(\varepsilon_{ij}, \mathcal{T}) = \frac{1}{2} c_{ijkl} \varepsilon_{ij} \varepsilon_{kl} - \beta_{ij} \varepsilon_{ij} T - \frac{1}{2} m T^2 + \dots,$$

where

$$T = \mathcal{T} - T_0, \quad c_{ijkl} = \frac{\partial^2 F(0, T_0)}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}}, \quad \beta_{ij} = -\frac{\partial^2 F(0, T_0)}{\partial \varepsilon_{ij} \partial \mathcal{T}}, \quad m = -\frac{\partial^2 F(0, T_0)}{\partial \mathcal{T}^2}.$$

Retaining in the expansion (35) the quadratic terms only we obtain from (29)

$$(36) \quad \sigma_{ij} = \frac{\partial F}{\partial \varepsilon_{ij}} = c_{ijkl} \varepsilon_{kl} - \beta_{ij} T.$$

Observe that

$$(37) \quad \left(\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \right)_T = c_{ijkl}, \quad \left(\frac{\partial \sigma_{ij}}{\partial \mathcal{T}} \right)_\varepsilon = -\beta_{ij}.$$

Equation (36) is the law of elasticity, the so-called Duhamel–Neumann relation for an anisotropic body. The constants c_{ijkl} and β_{ij} refer here to the natural state of the body,