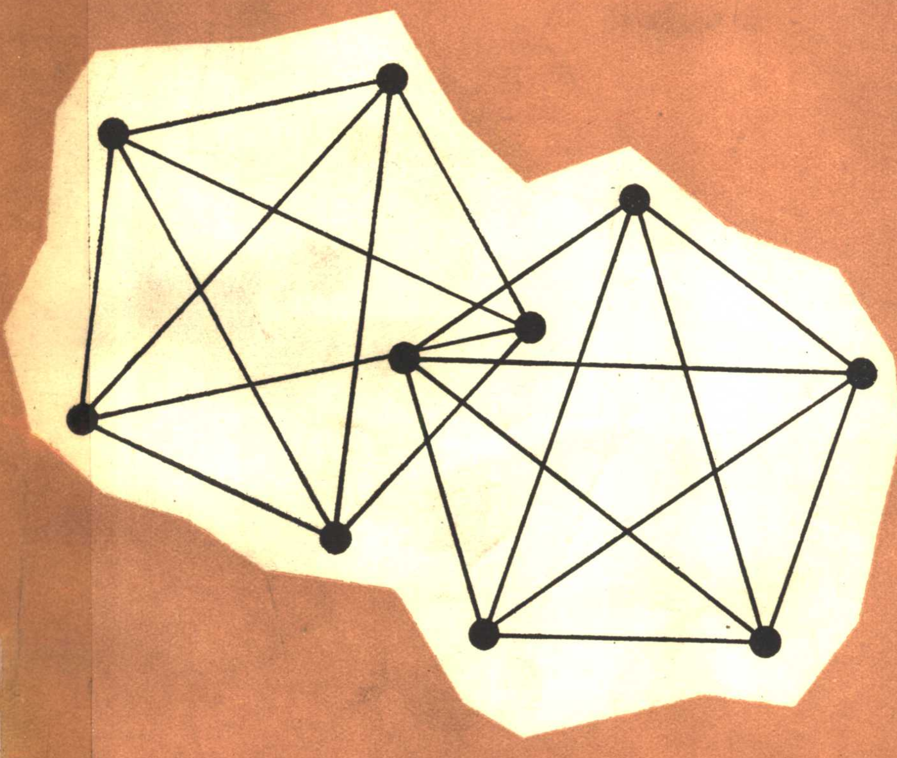
Recent Developments in Network Theory



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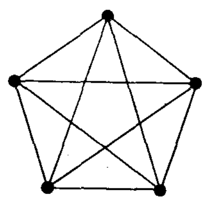
Edited by

S. R. DEARDS



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PREFACE

ELECTRICAL network theory has its origin in a paper published by Kirchhoff at the age of twenty-one. From an investigation of the distribution of steady current in a system of interconnected conductors, he deduced the basic laws and established the combinatorial principles. Maxwell developed the general theory of electromagnetic networks within the framework of Lagrangian dynamics and Larmor gave an interpretation of Maxwell's theory based on a correspondence between Kirchhoff's interconnexion constraints and Euler's theory of polyhedra. He also observed the reciprocal relation between the current and voltage representations of network behaviour which Sire de Vilar subsequently identified with Gergonne's geometrical principle of duality. With the introduction of Heaviside's algebra of transients and the development of electrical technology at the beginning of the present century, the dynamical theory gave way to the specification of electrical problems in terms of the terminal behaviour of idealized elements. Modern electrical network theory has developed from this simpler and more direct approach to the study of electrical systems in which the associated electromagnetic field is not the predominant feature.

With the invention of the telephone and the introduction of alternating current as a source of commercial power, the centre of activity in network theory moved to America. Kennelly and Steinmetz applied the algebra of complex numbers to the study of linear networks in the sinusoidal steady state and Campbell and Zobel developed the theory of the wave-filter. Foster adapted Campbell's reactance theorem to the exact design of reactive one-ports according to prescribed reactance functions and Cauer in Germany extended Foster's work to the design of two-element one-ports containing resistance. Brune showed that the impedance function of a reciprocal one-port is a "positive real" function and that any such function can be realized by a reciprocal one-port. He also gave a procedure for the design of one-ports containing three kinds of elements and thus initiated the theory of network synthesis.

During recent years, network analysis and synthesis have been the subjects of extensive study and network theory has developed into a unified discipline. It has reached a level of abstraction comparable with that of pure science. The modern theory proceeds from a set of definitions (the elements) and axioms (the constraints) and is concerned with the logical outcome according to the methods of strict mathematical reasoning. By venturing beyond the bounds of physical realizability, it has indicated the way to worthwhile technological research.

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LINEAR PASSIVE NETWORK THEORY

COMPOUND MATRICES IN NETWORK THEORY

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1. Introduction

AN OUTSTANDING problem in electric network theory is the realization of multi-port networks containing only resistors, self-inductors and capacitors (i.e. RLC multi-ports) from their admittance or impedance matrices. Even the realization problem of resistor-only multi-ports is still unsolved. In 1952 Professor Weber made the following remark[14] "Synthesis is like the roof of a building it has come to last. We must first have a strong structure of analysis before we can proceed to the much more intricate problem of synthesis." This paper is essentially an analysis paper, but it is believed that it may have interest and utility in the field of network synthesis.

The algebraic relationship between the admittance and impedance matrices y and z of any multi-port network, and the matrices of two "parent" networks has been given by Cederbaum.[7] This relation may be expressed in the form

$$y \text{ or } z = [\text{principal submatrix of } M^{-1}]^{-1} \tag{1}$$

where M is an admittance or impedance matrix of an associated "parent" network.

The relationship (1) is common to many physical situations. Thus, suppose a linear system is specified by an $n \times n$ matrix M, so that a set of n "forcing functions" represented by a column vector $\mathbf{y}$, gives a set of n "response functions" represented by a vector $\mathbf{x}$, where

$$M\mathbf{x} = \mathbf{y}. \tag{2}$$

Now suppose that access to this physical system from the outside world is such that only k of the n forcing functions y_i can be activated, and only the corresponding k response functions x_i can be observed. For convenience, let us suppose it is the last k of the y_i which are available, denoted by the vector $\mathbf{y}_{(k)}$; then the first $(n-k)$ of the y_i are zero. Denote the last k of the x_i by $\mathbf{x}_{(k)}$, and let us ask for the relation between $\mathbf{y}_{(k)}$ and $\mathbf{x}_{(k)}$; i.e. if we write

$$M_{(k)}\mathbf{x}_{(k)} = \mathbf{y}_{(k)} \tag{3}$$

then we ask for $M_{(k)}$ in terms of M.

Before we assume that the first $(n - k)$ of the y_i are zero, let us rewrite eqn. (2) in the following form:

$$\begin{bmatrix} M_{11} & M_{22} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} \tag{4}$$

where $\mathbf{x}_2$ and $\mathbf{y}_2$ are $k \times 1$ vectors, and M is conformably partitioned as shown. Let us denote the inverse of M by N, then if we also conformably partition N, we obtain

$$\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}. \tag{5}$$

If now we assume $\mathbf{y}_1 \equiv 0$, so that $\mathbf{y}_2$ becomes $\mathbf{y}_{(k)}$ and $\mathbf{x}_2$ becomes the required $\mathbf{x}_{(k)}$, then we obtain from eqn. (5)

$$\mathbf{x}_{(k)} = N_{22}\mathbf{y}_{(k)},$$

which gives, assuming N_{22} to be non-singular,

$$N_{22}^{-1}\mathbf{x}_{(k)} = \mathbf{x}_{(k)}. \tag{6}$$

Thus we see that

$$M_{(k)} \equiv N_{22}^{-1} \tag{7}$$

i.e.

$$M_{(k)} \equiv \{\text{Last principal submatrix of order } k \text{ of } M^{-1}\}^{-1}. \tag{8}$$

This is, of course, obvious and well known.

Another well known[11] form for $M_{(k)}$ derives as follows. From equation (4)

$$M_{11}\mathbf{x}_1 + M_{12}\mathbf{x}_2 = \mathbf{y}_1 \tag{9a}$$

$$M_{21}\mathbf{x}_1 + M_{22}\mathbf{x}_2 = \mathbf{y}_2 \tag{9b}$$

If now $\mathbf{y}_1 = \mathbf{0}$, and $\mathbf{x}_2 = \mathbf{x}_{(k)}$, $\mathbf{y}_2 = \mathbf{y}_{(k)}$, we obtain from 9(a), assuming M_{11} to be non-singular,

$$\mathbf{x}_1 = -M_{11}^{-1}M_{12}\mathbf{x}_{(k)} \tag{10}$$

and on substituting into 9(b) we obtain

$$[M_{22} - M_{21}M_{11}^{-1}M_{12}]\mathbf{x}_{(k)} = \mathbf{y}_{(k)} \tag{11}$$

which gives

$$M_{(k)} \equiv [M_{22} - M_{21}M_{11}^{-1}M_{12}]. \tag{12}$$

It may be shown that the non-singularity of N_{22} in (5) and (6), and the non-singularity of M_{11} in (9) and (10) are equivalent conditions, and in fact are necessary and sufficient for the existence of $M_{(k)}$.

In this paper we obtain yet another form for $M_{(k)}$, in terms of compounds of the matrix M.

This expression for $M_{(k)}$ was originally obtained by the author in his doctoral thesis[5] in 1959. The author has recently discovered that an equivalent result had previously been given by Campbell[6] in 1922, although Campbell's expression is not in terms of compound matrices (see Section 3).

The compound expressions obtained are closely allied to the Gaussian method of elimination by pivotal condensation, and indeed in the English translation of his book, Gantmacher[10] gives a mechanical interpretation of Gauss's algorithm which is essentially identical with the physical interpretation of $M_{(k)}$ given above.

Other writers who have given results closely allied to those obtained here are Adams,[1,2] Boxall,[4] Cederbaum,[8] Shipley and Coleman.[12]

It is felt that the expressions and results we obtain in this paper may provide the network theorist with a new tool to help him in his search for network realization criteria and techniques.

2. Compound Matrices

The definitions and results of this section may all be found in Chapter 5 of Aitken.[3]

DEFINITION 1. The *k-th compound matrix* $M^{(k)}$ of the $n \times n$ matrix M, $(1 \leqslant k \leqslant n)$, has as its elements all the kth order minors of M; all the minors which come from the same group of k rows (or columns) of M are placed in the same row (or column) of $M^{(k)}$, and arranged in "lexical order".

Notes: (i) By "lexical order", it is meant that the priority of minors in rows and columns of $M^{(k)}$ is decided on the same basis by which words are ordered in a dictionary or lexicon. For example, taking the case of $k = 2$, the second order minors formed from the first two rows of M form the first row of $M^{(2)}$, the minor formed from columns (1, 2) coming first ($M_{11}^{(2)}$), followed by those formed by columns (1, 3), (1, 4), ... (1, n), (2, 3), ... (2, n), ... , $(n - 1, n)$, in that order.

(ii) $M^{(k)}$ will be square and of order $\binom{n}{k}$, which we denote by n_k.

(iii) $M^{(1)} \equiv M$; $M^{(0)}$ is defined to be 1 (scalar unity).

(iv) $M^{(n)}$ is a 1×1 matrix whose single element $M_{11}^{(n)}$ is det M.

(v) If Adj M is the adjugate matrix of M, then

$$(\text{Adj } M)_{ij} = (-)^{i+j} M_{n-j+1,n-i+1}^{(n-1)} \qquad (i, j = 1, 2, \dots, n).$$

(vi) Since, when M is non-singular,

$$M^{-1} = [\text{Adj } M]/\det M \tag{13}$$

we have

$$(M^{-1})_{ij} = \frac{(-)^{i+j} M_{n-j+1,n-i+1}^{(n-1)}}{M_{11}^{(n)}} \qquad (i, j = 1, 2, \dots, n). \tag{14}$$

(vii) Definition 1 may be extended to rectangular matrices in the obvious way. If M is of order $m \times n$, then we must have $1 \leqslant k \leqslant \text{Min}(m, n)$, and then $M^{(k)}$ is of order $m_k \times n_k$.

DEFINITION 2. To form the *k-th adjugate compound matrix* $Adj^{(k)}M$ of the $n \times n$ matrix M, take the kth compound matrix $M^{(k)}$ and replace every element in it by its cofactor, or signed minor, of order $(n - k)$ which is associated with it in the Laplacian expansion of det M. The transpose of the resultant matrix we call the kth adjugate compound of M, and denote it by $\text{Adj}^{(k)}M$.

Notes: (viii) $\text{Adj}^{(1)}M = \text{Adj}\, M$.

(ix) Since $\text{Adj}^{(n-k)}M$ and $M^{(k)}$ have identical elements, though transposed, written in complete reversed order and with the sign added, we have

$$M_{ij}^{(k)} = (-)^{i+j}[\text{Adj}^{(n-k)}M]_{n_k-j+1,n_k-i+1} \qquad (15)$$

$$(i, j = 1, 2, \ldots, n_k).$$

The following two important theorems are proved by Aitken:[3]

THEOREM 1 (Binet–Cauchy theorem). *If M and N are two rectangular matrices of order* $m \times n$ *and* $n \times p$ *respectively, and if* $1 \leqslant k \leqslant \text{Min}(m, n, p)$ *then*

$$(MN)^{(k)} = M^{(k)}N^{(k)}$$

THEOREM 2 (Jacobi's theorem). *If M is square, of order* $n \times n$, *and* $1 \leqslant k \leqslant n$, *then* $(\text{Adj}\, M)^{(k)} = (\det M)^{k-1}\, \text{Adj}^{(k)}M$.

3. Two New Theorems

In this section we obtain an expression for $M_{(k)}$ in terms of compounds of M, and also derive an iterative relation involving $M_{(k)}$. We recall first that in Section 1 we defined $M_{(k)}$ as follows:

$$M_{(k)} = \{\text{Last principal submatrix of order } k \text{ of } M^{-1}\}^{-1}.$$

We now prove the following theorem:

THEOREM 3. *If M is non-singular, and the first principal minor of order* $(n - k)$ *of M is also non-zero, then* $M_{(k)}$ *exists and is given by*

$$[M_{(k)}]_{ij} = \frac{M_{ij}^{(n-k+1)}}{M_{11}^{(n-k)}} \qquad (i, j = 1, 2, \ldots, k).$$

Proof. Throughout this proof, great attention must be paid to the ranges of the indices which are indicated.

We have

$$M^{-1} = (\text{Adj}\, M)/\det M. \qquad (16)$$

Denote the last principal submatrices of order k of Adj M and M^{-1} by V and U respectively; then

$$U = V/\det M \qquad (17)$$