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Seven Papers in Applied Mathematics

by

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Smooth Dynamical Systems. Introductory Article*

D. V. ANOSOV

1. Since about 1960, the theory of smooth dynamical systems has developed significantly. It has emerged as a branch of mathematics in its own right replete with its own concepts, methods, and problems, as well as a host of deep results. Of course, long before 1960, starting from Poincaré, remarkable results have appeared on smooth dynamical systems that are in no way inferior to modern achievements. However, these earlier results have not been combined into a single entity, due to the absence of unifying concepts and methods that would have distilled them from the corpus of works on differential equations.

The last ten years have seen a number of conferences and symposia in which the theory of dynamical systems was the main, or one of the main, themes. The results of these symposia are reflected in [1]–[4] and contain much valuable information. In addition, the total volume of publications on the subject which have appeared during the same period exceeds the combined size of these proceedings by several times.

Naturally, outside the scope of the present collection,** there lie whole lines of investigation which we can only mention:

1) Work of an analytical nature (including the investigation of neighborhoods of equilibrium positions and periodic solutions).

2) The theory of bifurcations. We refer the reader to the survey [5]. The collections [3] and [4] contain much recent foreign (i.e., non-Soviet) work in this area.

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* Translation of the editor's introduction to the book *Smooth dynamical systems*, Matematika: Novoe v Zarubezhnoĭ Nauke, vyp. 4, "Mir", Moscow, 1977, pp. 7–31; MR **58** #31208.

***Editor's note.* "The present collection" refers to the Russian translations of [A]–[F], [H] and [I], and the original paper [G], published in the Russian volume [J], to which this is the editor's introduction.

1980 *Mathematics Subject Classification*. Primary 58F15, 58F18, 34C35, 34C40; Secondary 70F10.

3) Questions involving ergodic theory and related subjects (for example, Markov partitions of hyperbolic sets). These are discussed in the recently published survey [6].

4) Isolating blocks (which are used to localize isolated invariant sets), their "continuation" in parameters, and the derivation of their properties. For want of a more recent survey, we refer the reader to an old one [7] and the detailed article [8]. The latter article does not, however, contain any applications. The recent paper [45] also deals with these topics.

We remark that many of the concepts and results associated with isolating blocks do not depend on smoothness and can be extended to more general situations [9]. They may, therefore, be viewed as belonging to so-called topological dynamics. However, in spirit, they are far removed from this discipline in its present form.

On the other hand, we shall dwell at somewhat greater length on several topics below. The various sections that follow do not depend on one another, and their order is more or less arbitrary.

2. Ruelle and Takens [10] have constructed an example in which the collapse of a quasiperiodic motion on a higher-dimensional torus is accompanied by the birth of an attracting hyperbolic set. By itself, this example does not stand out among the many other examples of bifurcations that have appeared in recent times. However, it has excited interest due to Ruelle and Taken's assertion that it can be used to model the onset of turbulence during the loss of stability of a laminar flow.

The value of [10] lies not so much in the concrete model proposed therein, as in the role it has played in fostering, among specialists in mathematical hydrodynamics, the understanding that the resources of contemporary mathematics are not exhausted by quasiperiodic motions and that, further, it makes sense to attempt to link turbulence with attracting hyperbolic sets. Hyperbolic sets are particularly attractive in this regard because they combine instability of individual trajectories with stability of the set as a whole (stability both with respect to perturbation of the initial data, as soon as the set is attracting, and with respect to perturbations of the dynamical system, as soon as it is hyperbolic). The attractiveness is heightened by the fact that hyperbolicity can, in principle, lead to emergence of statistical properties in systems. (It is not clear which of the many invariant measures to prefer in this case. For this reason, along with other Soviet mathematicians I prefer to use the words "quasirandomness" or "stochasticity". The words "random" or "statistical" are apparently reserved in contemporary Russian-language mathematical literature for situations in which some sort of measure ("probability") is involved.)

At present, many researchers are performing numerical experiments with systems of several (usually three) ordinary differential equations, which occur as Galerkin approximations for the partial differential equations of Navier-Stokes. Such systems may, therefore, be considered as simplified models of the latter.

Experiments of this type were first conducted more than a decade ago by Lorenz, who observed that for some values of the parameters of the system the trajectories behave stochastically. Namely, under restricted accuracy of the initial conditions the motion is more or less determined over only a comparatively short time span, while later various possibilities may occur depending on initial data unknown to us. In a trajectory there exist segments which repeat rather precisely, but these repetitions occur very irregularly; immediately following the approximate repetition of a segment, the segment immediately following the repeated one may or may not repeat. Initially, Lorenz's work did not attract a great amount of attention (owing, perhaps, to the lack of concepts which could have helped to theoretically interpret his observations?). However, Ruelle has currently expressed the opinion that Lorenz observed motions in a hyperbolic attracting set [11]. We can only hope that soon new numerical experiments combined with attempts at theoretical comprehension of their results will allow us to judge (at least convincingly enough, if not absolutely strictly) whether in fact hyperbolic sets have something to do with the behavior of trajectories of those model systems of ordinary differential equations. It is a different question whether the conclusions obtained will ever be carried over to the genuine Navier-Stokes equations with any degree of persuasiveness.

With regard to the model itself in [10], the authors themselves note that it is not understood how a higher-dimensional torus can arise in the hydrodynamical situation. It is well known that, under sufficiently general conditions, the loss of stability of a periodic solution is accompanied by the genesis of a two-dimensional torus (Hopf bifurcation). However, [10] requires a higher-dimensional torus. One can, of course, imagine that there exist bifurcations of the torus which lead to the increase of its dimension, but this process has not been studied, and it is possible that such bifurcations occur only under some unlikely confluence of circumstances. (In fact, the genesis "à la Hopf" of a two-dimensional torus may be followed soon after by its "disintegration".)

Apart from the above, there are some other considerations which make it questionable whether the example in [10] could, in fact, serve as an appropriate model for the onset of turbulence. In this example the limit set generated is close to the collapsed torus, whereas the turbulent flow can hardly be regarded as close to the laminar. Moreover, experimental data suggest that the turbulence originates when the laminar flow is still stable but its domain of stability is very small.

Up to this point we have been concerned with nonconservative dynamical systems. The theory of smooth dynamical systems (both the Kolmogorov–Arnol'd–Moser theory and the "hyperbolic" idealogy) has been applied even earlier to conservative (Hamiltonian) dynamical systems (with a similar degree of rigor by using numerical experiments) by a group of Novosibirsk physicists—Chirikov, Zaslavskiĭ, et al. [12], [13].

3. Schweitzer [14] has proved that *every homotopy class of nonsingular vector fields on a three-dimensional manifold contains a* C^1 *vector field which does not have*

a periodic trajectory. An analogous result holds for three-dimensional manifolds with boundary if we consider vector fields transverse to the boundary (and if, of course, the given manifold admits nonsingular vector fields transverse to the boundary). Higher-dimensional analogues of the above results hold even for C^∞ fields.

Special cases of the above theorems settle two well-known problems. The first was posed by Seifert [15] and pertains to the three-dimensional sphere. The second, which I ascribe to Smale (although the folklore tradition, justified or not, connects it with Poincaré), pertains to the solid torus (that is, the direct product $S^1 \times D^2$ of the circle and the disk). Earlier, Fuller [16] had constructed a simple example of a nonsingular vector field on the solid torus which was transverse to the boundary and which, while it possessed a periodic trajectory, had the property that this trajectory, contrary to naive expectations, did not orbit along S^1 but, rather, lay in a disk $x_0 \times D^2$.

The possibility that nonsingular vector fields on the three-sphere may not possess periodic trajectories sharpens the significance of results which guarantee the existence of a periodic trajectory under various special conditions [15]. (See also [19], [17] and [18].)

It is not known whether or not the analogue of Schweitzer's result for C^r ($r > 1$) vector fields on three-dimensional manifolds is true. It is also natural, although somewhat less general, to ask the same question about vector fields which define flows with a smooth invariant metric. Finally, Wilson [20] has considered analogous problems in which the question is about minimal sets of codimension greater than or equal to two, instead of periodic trajectories.

In a broader vein, we remark that this is not the only instance in which the existence of a system with well-defined qualitative behavior has been established only in the C^1 case. This is also the case for the celebrated closing lemma (see Takens [H]) and in the problems considered in [H] and [I]. In most cases, including the Seifert problem, the corresponding question in the case of greater smoothness remains open. An exception, to which [I] is devoted, is the problem about invariant curves for mappings of the annulus which preserve area elements. Here, due to the efforts of a number of mathematicians (Kolmogorov, Arnol'd, Moser, and Rüssmann), it can be proved that the situation is different for greater smoothness. Although there are "lacunae" between Rüssmann's result and Taken's example, this example is of great interest because it exhibits an effect principally connected with less differentiability (it is true, however, that here the question is not about the smoothness of the transformation, but about the smallness of the perturbation in the C^r-topology).

Along the same lines we can also cite the question about the metric characteristics of Anosov systems. Here, we do not acutally have C^1 counterexamples to the natural conjectures which have been established for C^r mappings with $r > 1$. However, the premises on which the proofs rest are apparently not true in the C^1 situation (see [40]). In a related question, concerning the measure of nowhere dense hyperbolic sets, there is a counterexample, due to Bowen [41], which shows that the situation for C^1 diffeomorphisms differs from the C^r case, $r > 1$ (see [42]).

4. Mather [21] has shown that a fundamentally new type of singularity occurs in the n-body problem with a Newtonian potential when $n \geq 4$, in comparison with $n = 2$ or 3. At the end of the last century Painlevé completely characterized the singularities in the three-body problem: they were either double or triple collisions (in which, by definition, the distance between the colliding bodies approaches zero). Moreover, in a finite time, only finitely many such collisions could occur. (For a survey of the classical research undertaken around the turn of the century concerning the analytic characteristics of the solutions of the three-body problem, together with references to the literature, see the lectures of Alekseev [48].) We now know that the situation in the four-body problem is different. Namely, *there exist trajectories for which infinitely many double collisions occur in finite time* t_0, *and where, as* $t \to t_0$, *three of the bodies leave to infinity—one in one direction and two of the others in the opposite direction. The latter two bodies unboundedly approach one another* (*which gives the energy of the whole process*). *The fourth body oscillates between two of the bodies.* Here, the most interesting feature is, of course, the approach to infinity at finite time t_0. At t_0 we have an essentially new type of singularity. It is entirely possible that the approach to infinity could happen without the preceding collisions in the strict sense of the word. However, in the example under consideration all four bodies move along a straight line, and it is clear that as long as this is the case we cannot manage without the collisions. It is natural to ask whether a small neighborhood of this motion in the three-dimensional case contains motions which approach infinity in finite time without the collisions. This has not yet been clarified.

Mather's research builds on the work of McGehee [22] concerning the regularization of singularities in the collinear three-body problem (whence the joint authorship of [21]). When the "oscillating" body draws near the two "approaching" bodies, we almost get a triple collision. Mather exploits McGehee's analysis of the motion in a neighborhood of the triple collision. The techniques employed are themselves a combination of Alekseev's techniques for analyzing "quasirandom motions" and the technique of regularizing singularities which was developed by Moser to investigate simultaneous regularization of all singularities on a constant energy surface in the two-body problem.

5. Recently, Herman has substantially advanced the investigation of properties of cascades generated by diffeomorphisms of the unit cirlce. (*Translator's note*: Anosov uses the term "cascade" to refer to flows in the context of discrete dynamical systems.) See Herman's articles [23] and [46] and also Deligne's report of his work [24].

Given a map φ: $S^1 \to S^1$ of the circle $\{x \bmod 1\}$, it is convenient to work in terms of the covering map f: $\mathbf{R} \to \mathbf{R}$, which we call the *angular function* of the map φ. (Two different angular functions of a single map φ differ by an integral constant.) For example, the statement that φ is an orientation-preserving diffeomorphism of class C^n can be formulated in terms of the angular function as follows: $f \in C^n$ and, for all x,

$$df(x)/dx > 0, \qquad f(x+1) = f(x). \tag{1}$$

For any orientation-preserving homeomorphism φ the limit

$$\alpha(f) = \lim_{n\to\infty} (1/n) f^n(x)$$

(where f^n, as usual, denotes the nth iterate of f) exists and does not depend on x. It is called the *rotation number* (of the homeomorphism). More precisely, $\alpha(f) \bmod 1$ is the appropriate invariant of φ, since $\alpha(f+k) = \alpha(f) + k$ for $k \in \mathbf{Z}$.

A classical theorem of Denjoy asserts that *if an orientation-preserving diffeomorphism φ has an irrational rotation number, and if the derivative df/dx of the angular function f is a function of bounded variation (in finite intervals), then there exists a homeomorphism $\chi: S^1 \to S^1$ such that φ is conjugate to a rotation ψ via χ*:

$$\psi: x \mapsto x + \alpha(f) \quad \bmod 1.$$

The homeomorphism is uniquely defined to within a rotation of the circle, a fact which immediately raises the question of its smoothness.

It is easy to see that the (conjugating) homeomorphism in the above case is either absolutely continuous or singular (it carries a set of measure zero to a set of full measure). The latter case actually occurs even when f is analytic ([25] and, somewhat less neatly, [26] and [27]). It is not difficult to show that this is the case for a diffeomorphism with angular function

$$f_\lambda(x) = x + \lambda + (1/4\pi)\sin 2\pi x \tag{2}$$

for some λ. In fact, the following is proved in [25]–[27]. *Let $f_\lambda(x)$ be a function of two variables which is defined and analytic for all $x \in \mathbf{R}$ and $\lambda \in \{\beta, \gamma\}$. Further, suppose that, for each fixed λ, $f_\lambda(x)$ satisfies (1) for all x, and thus defines a diffeomorphism φ_λ of the circle. Suppose $\alpha(f_\beta) \neq \alpha(f_\gamma)$ and, whenever $\alpha(f_\lambda)$ is rational, φ_λ is not topologically conjugate to the rotation $x \mapsto x + \alpha(f_\lambda)$; the latter condition is equivalent to the condition that for each n and λ the identity*

$$f_\lambda^n(x) = x + \text{const} \quad (\text{for all } x) \tag{3}$$

does not hold. Then there exists λ for which $\alpha(f_\lambda)$ is irrational and the conjugating homeomorphism is singular.

To see that (3) does not hold for the family (2), note that $f_\lambda(x)$ is an entire function of x, and so, if (3) were to hold for all $x \in \mathbf{R}$, it would also hold for all complex x. Differentiating, we obtain

$$\prod_{i=1}^{n} \frac{df_\lambda(f_\lambda^{i-1}(x))}{dx} = 1.$$

The expression on the left side is a product of entire functions. It must have zeros, because when $i = 1$ we obtain

$$df_\lambda(x)/dx = 1 + \tfrac{1}{2}\cos 2\pi x,$$

and this function has zeros in the complex domain. This gives a contradiction. (More generally, no nonlinear entire function f can satisfy the identity $f^n(x) = x + \text{const}$, for this identity implies that $f\colon \mathbf{C} \to \mathbf{C}$ is injective.)

The argument used to construct the λ for which φ_λ is singular shows that $\alpha(f_\lambda)$ is approximated exceedingly well by rational numbers. Experiments connected with "small denominators" suggest that such anomalously rapid approximations could be the source of the "pathology". Therefore, Arnol'd [25] conjectured that there exists a set M of full measure such that, for each $\mu \in M$ and each orientation-preserving analytic diffeomorphism $\varphi\colon S^1 \to S^1$ with rotation number μ, the homeomorphism χ under which φ is conjugate to a rotation is analytic.

In [25] this conjecture was shown to be true for diffeomorphisms sufficiently close to the rotation $x \to x + \mu$. An analogue of this result was later established in the finitely smooth case. In [26] Finzi asserted that if $\varphi \in C^3$ and its rotation number is approximated sufficiently slowly by rational numbers, then $\chi \in C^1$. However, Glimm discovered that Finzi had incorrectly estimated a sum on p. 269.

Herman has proved Arnol'd's conjecture [46]. We present a weaker theorem, also due to Herman [23], [24]. It pertains to those μ which are approximated as slowly as possible by rational numbers: namely, *those μ which are such that if the a_i are coefficients of the continued fraction expansion of μ, then*

$$\sup_n \frac{1}{n}\sum_1^n a_i < \infty.$$

(The set of all such μ is a set of measure 0.) The theorem asserts that *if the rotation number of the diffeomorphism φ satisfies the condition stated above, and if $\varphi \in C^n$, where $n \geq 3$, then $\chi \in C^{n-2}$ (and, in addition, the $(n-2)$th derivative of the corresponding angular function satisfies the Hölder condition with any exponent smaller than* 1). *Also, if φ is analytic, so is χ.*

6. In this section and the next we discuss in more detail some special questions which are closely related to the problems considered in [A]–[D].

The articles in the present collection, which deal with Anosov systems* of codimension one (that is, Anosov systems for which, in standard notation, the leaves W^u and W^s have codimension one) use, as well as "dynamical" concepts, concepts from the theory of foliations due to Haefliger, Novikov, and Sacksteder. It is interesting to see how much can be obtained using only foliation theory.

Let us first recall some definitions. For simplicity, we assume that all foliations are smooth (since this is the case for those foliations which most interest us, namely, codimension one foliations arising from Anosov systems of class C^2).

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* *Translator's note.* Anosov uses the term *У-system* (*У-flow*, *У-cascade*, etc.) instead of *Anosov system* (*Anosov flow*, *Anosov cascade*, resp.).

Let M be an n-dimensional manifold with a distinguished foliation, and let $W(x)$ denote the leaf passing through the point x. Suppose that γ: $[0,1] \to W(x_0)$ is a closed path in the leaf which begins and ends at x_0. At each point $\gamma(t)$ we can construct a small "plane" $\Pi(t)$, transverse to the foliation, in such a way that $\Pi(t)$ depends continuously on t, its tangent space depends continuously on t, and $\Pi(0) = \Pi(1)$. More formally, let the leaf be k-dimensional and set $l = n - k$. Choose a map f: $D^l \times [0,1] \to M^n$ (D^l is the l-dimensional disk) such that $f(0,t) = \gamma(t)$, $f(x,0) = f(x,1)$, and $f|D^l \times \{t\}$ is a smooth imbedding, transverse to the leaves, whose derivative with respect to x at the point (x,t) is continuous in (x,t). Then $\Pi(t) = f(D^l \times \{t\})$. Given any point x in $\Pi(0)$ lying in a sufficiently small neighborhood U of x_0, there exists exactly one continuous path γ_x: $[0,1] \to M$ such that $\gamma_x(t) \in \Pi(t) \cap W(x)$. Thus, we can define the *succession map* along the path γ to be the map $U \to \Pi(0)$ which sends x to $\gamma_x(1)$. (It is also called the *monodromy map*.) A closed loop is called a *limiting cycle* (of the given foliation) if the succession map is not the identity in any neighborhood of x_0 (that is, there are points arbitrarily close to x_0 which are displaced). This definition does not depend on the choice of framing (that is, on the choice of $\Pi(t)$). Moreover, two loops in the same leaf which are freely homotopic in the leaf are either both limit cycles, or else, neither is a limit cycle.

A foliation will be called *coorientable* if the normals to its tangent field $T_xW(x)$ (constructed using some auxiliary Riemannian metric) can be oriented consistently on all points $x \in M$ (here, "consistently" means that the orientation depends continuously on x). Equivalently, a foliation is coorientable if the bundle over M whose fiber over x is the quotient space $T_xM/T_xW(x)$ is orientable. The reader should beware that Novikov [28] and Brakhman [29] (whose results are cited below) use "coorientable" to mean "orientable", whereas in this collection a foliation being orientable means that its tangent field $T_xW(x)$ is orientable.

We now suppose that we have a foliation which has codimension one. In this case the transversals $\Pi(t)$ are arcs, each of which is divided by the point $\gamma(t)$ into two halves which locally lie on different sides of W (that is, on some neighborhood U of $\gamma(t)$ they lie on different sides of the connected component of $U \cap W$ containing $\gamma(t)$). If the foliation is coorientable, then for all paths in a leaf W we can consistently declare one of the semiarcs to be "right" and the other to be "left". Moreover, by restricting the succession map to the right (left) semiarcs, we can define *right* (*left*) *limit cycles*. Like limit cycles they are independent of the choice of "framing" and well defined up to homotopy on the leaf. A *one-sided limit cycle* is a right (left) limit cycle which is not a left (right) limit cycle.

Suppose that a loop γ in $W(x_0)$ is not a right limit cycle. Then it is possible to move along the right transversal semiarcs to a nearby leaf $W(x_\varepsilon)$ and thereby obtain a loop γ_ε. It may happen that while the loop γ is not contractible in $W(x_0)$ to a point, the displaced loops γ_ε are contractible in $W(x_\varepsilon)$ to a point for all sufficiently small $\varepsilon > 0$. In this case γ is called a *right vanishing cycle* (the term was proposed by Haefliger; Novikov speaks about "cycles, limitwise right-homotopic to zero"). As usual, this property is independent of the specific choice of transversal arcs and invariant under free homotopy. In particular, the word "cycle" is often used instead of "loop", since the choice of initial point on the curve (which is understood to have a fixed orientation) is immaterial.

We shall be particularly interested in the following propositions due to Novikov [28]. They concern coorientable foliations on a closed manifold M^n.

1) *If a foliation does not have vanishing cycles, then the universal cover $\hat{M}$ of the manifold M is diffeomorphic*([1]) *to the direct product $\hat{W} \times \mathbf{R}$ of the universal cover $\hat{W}$ of any leaf W with the line $\mathbf{R}$. The diffeomorphism carries each leaf of the foliation covering the original foliation on M to $\hat{W} \times \{t\}$.*

2) *Under these conditions the inclusion $i: W \to M$ induces a monomorphism $i_*: \pi_1(W) \to \pi_1(M)$ of fundamental groups, the image of which is a normal subgroup. The quotient group $\pi_1(M)/i_*\pi_1(W)$ is a free abelian group with a finite number of generators.*

3) *If a foliation does not have vanishing cycles, then either there exists a leaf W for which the homotopy group $\pi_2(W) \neq 0$, or else $\pi_2(M) = 0$.*

4) *Under the same condition no closed transversal contracts to a point.*

Statements 1) and 2) essentially comprise Theorem 5.1. Statements 3) and 4) constitute a part of Theorem 6.1; by passing to a suitable double cover it is easy to see that they do not depend on coorientability.

In Anosov cascades the foliations by W^u and W^s cannot have limit cycles (since all leaves are contractible), and for Anosov flows of codimension one the corresponding foliation does not have vanishing cycles (because incontractible closed curves exist only on leaves containing periodic trajectories, and in this case they are two-sided limit cycles). These observations allow us to draw the following conclusions.

a) *If a closed manifold M admits an Anosov flow of codimension one and the foliation is coorientable, then M has the homotopy type of a torus.*

In addition, we obtain information about the covering foliation in the universal cover; see 1). (This is the case even if the foliation is not coorientable, since we can first pass to the two-sheeted cover.)

b) *If there exists an Anosov flow of codimension one on a closed manifold M, then $\pi_2(M) = 0$ and no closed transversal is contractible.*

The above statement about closed transversals is used (and proved) by Plante and Thurston [D] (see also the footnotes inserted in the Russian translation of [D]). The result concerning π_2 is apparently new. In the three-dimensional case it easily implies that M is contractible. Since there exists a contractible three-dimensional manifold which is not homeomorphic to $\mathbf{R}^3$, this is somewhat weaker than the following result due to Margulis [30], which is proved by using both the foliations by W^u and W^s: *the universal cover of a closed three-dimensional manifold M on which there exists an Anosov flow is homeomorphic to Euclidean*

([1]) Recall that we have restricted ourselves to smooth foliations—were this not the case we would be obliged to replace the word "diffeomorphic" by "homeomorphic".

space. (Margulis obtains this as a corollary of his proof that $\pi_1(M)$ has exponential growth, a result which is now subsumed by the work of Plante and Thurston.)

For Anosov cascades proposition a) can be sharpened by using "dynamical" considerations (see Franks [A] and Newhouse [B]). Franks essentially uses the properties of $\hat{M}$ and the foliation $\{\hat{W}\}$ formulated above. However, the proof (which, as Franks notes, is due to Novikov) requires the assumption that all points of M are nonwandering. It is possible to avoid this assumption in two ways. On the one hand, Newhouse has shown that for codimension one Anosov cascades the assumption is always satisfied. On the other hand, it is clear from the derivations of the results cited above that the theory of foliations yields the requisite conclusions about $\hat{M}$ and $\{\hat{W}\}$ given only the absence of limit cycles.

At this juncture a rather complex situation has arisen. It springs from the fact that the proof of proposition 1) in [28] is incorrect. The error is due to the insufficient attention paid by the author to his own Figure 8. As a result of this error, the arguments on Russian p. 261 (English pp. 283–284) stand in need of serious revision. Corrections were advanced first by Novikov himself and later by Brakhman [29]. These, however, did not suffice to establish the proposition in full generality. It was Novikov's argument which Franks cited.

Suppose that a closed manifold M admits a coorientable codimension one foliation without a limit cycle. We can construct a smooth vector field which is everywhere transverse to the leaves. The flow defined by this field is naturally called the *transversal flow* (in [29] it is called the "normal flow"). In [29] it is shown that proposition 1) holds if the following condition holds: each trajectory of a transversal flow intersects all the leaves. However, no one seems to have noticed that this additional condition is automatically satisfied for any coorientable codimension one foliation on a closed manifold. In particular, it follows that 1) *is valid in the generality with which it is formulated above.*

The result we require is easily deduced from Theorem 4 and Proposition 3.4 of Sacksteder and Schwartz's paper [31]. Comparing these two results immediately yields the following corollary. *Suppose that a closed manifold M admits a coorientable codimension one foliation. If some closed transversal does not intersect all the leaves, then the foliation has a limit cycle.* (In other, better known, work of Sacksteder and Schwartz, C^2 smoothness is required and, indeed, essential. Thus, we stress that in [31], as well as [29], the smoothness requirements are minimal. C^1 smoothness and, in fact, even weaker requirements, such as smoothness of the leaves and continuity of the tangent space, suffice.) Thus, it remains to consider nonclosed trajectories $x(t)$ of the transversal flow. Let x_0 be an ω-limit point. Choose a neighborhood U of x_0 and coordinates $u_1, \ldots, u_n$ on U, each varying between $-\varepsilon$ and ε, such that the leaves (or, more precisely, the connected components of the intersection of the leaves with U) are given by the equations $u_1 = \text{const}$, and the trajectories of the transversal flow by the equations $u_2 = \text{const}, \ldots, u_n = \text{const}$. Now, some interval on the trajectory $x(t)$ which passes through U will have equations $u_2 = a_2, \ldots, u_n = a_n$. Choose the point $x(t_1)$ which has the coordinates $(\varepsilon/2, a_2, \ldots, a_n)$ and follow the trajectory until it again intersects U in a segment given by equations $u_2 = b_2, \ldots, u_n = b_n$. Let $x(t_2)$ be the point with coordinates $(-\varepsilon/2, b_2, \ldots, b_n)$. Choose a smooth function φ satisfying $0 \leqslant \varphi(t) \leqslant 1$ for all t such that $\varphi(t) = 0$ for $t \leqslant -\varepsilon/2$ and $\varphi(t) = 1$ for $t \geqslant \varepsilon/2$. We can now "close the loop" between $x(t_1)$ and $x(t_2)$ by joining them by means of a smooth arc lying entirely in U and having equations

$$u_i = a_i + (b_i - a_i)\varphi(u_1), \qquad |u_1| \leqslant \varepsilon/2,\ i = 2, \ldots, n.$$

This yields a closed transversal to the foliation (which is not, of course, a trajectory of our flow) which does not intersect any leaves not meeting $x(t)$. Thus, it follows from the above corollary of the results of [31] that if $x(t)$ does not meet some leaf, then the foliation might have limit cycles.

7. In this section we set forth several diverse results about Anosov systems.

a) Anosov systems are usually considered on closed manifolds. In this case compactness ensures that the choice of Riemannian metric used in the definition does not matter. However, the definition of Anosov systems may be carried over verbatim to open manifolds (that is, manifolds without boundary which are not compact). In this case however, the definition depends in an essential way on the choice of Riemannian metric. (It is natural to use a complete metric. Moreover, it seems advisable to require that some sort of uniformity condition be satisfied. The requirement that the derivative of the mapping or vector field be bounded immediately comes to mind. However, it might possibly be worthwhile to require more, perhaps that the metric have bounded curvature or that the derivative of the mapping or vector field be uniformly continuous.)

At present, open manifolds, as well as closed manifolds, are considered in the theory of geodesic flows on manifolds with negative curvature. A number of substantial results have been obtained for open manifolds (see [32] and [33]). These by themselves justify the consideration of Anosov systems on open manifolds. However, even the simplest (or so it seems) such systems can give rise to sudden surprises. An example of such has been discovered by White [34]. He constructed an Anosov diffeomorphism on the plane (with a nonstandard metric) which has very different properties than the usual hyperbolic automorphism (with the standard Euclidean metric). In particular, this diffeomorphism has no fixed point and each unstable leaf intersects only part of the stable leaves (and conversely). Under such circumstances doubt must arise as to whether the generalization of the notion of an Anosov system to an open manifold is productive. (It may be better to modify the definition by including some sort of additional conditions. In White's example it is clear that the derivatives are bounded, but it is not clear what the situation is with respect to different variants of uniformity. White does not discuss this, but it is obvious that the question can be clarified without undue effort.)

We present the formulas describing White's example and leave it to the reader to decipher their geometrical meaning. Let $\psi(x)$ be a C^∞ function which equals 0 for $x \leqslant 0$, increases from 0 to $\pi/2$ for $0 \leqslant x \leqslant 1/4$, and equals $\pi/2$ for $x \geqslant 1/4$. Define a periodic function $\varphi(x)$ with period 2 by setting (for $0 \leqslant x \leqslant 2$)

$$\varphi(x) = \psi(x - \tfrac{1}{4}) - \psi(x - \tfrac{3}{4}) - \psi(x - \tfrac{5}{4}) + \psi(x - \tfrac{7}{4}).$$

Let $e(\varphi)$ denote the vector

$$\cos\varphi \frac{\partial}{\partial x} + \sin\varphi \frac{\partial}{\partial y}$$

and set $e_1(x, y) = e(\varphi)$ and $e_2(x, y) = e(\varphi + \pi/2)$. Suppose that $X, Y \in T_{(x,y)}\mathbf{R}^2$ have components (X_1, X_2) and (Y_1, Y_2), respectively, with respect to the basis $e_1(x, y)$, $e_2(x, y)$. Introduce a Riemannian metric by setting

$$\langle X, Y \rangle = \lambda^{2x} X_1 Y_1 + \lambda^{-2x} X_2 Y_2,$$

where $\lambda > 1$ is a fixed number. With respect to this metric the map $(x, y) \mapsto (x + 1, -y)$ is an Anosov diffeomorphism. The integral curves of the field e_1 are the unstable leaves, while those of e_2 are the stable leaves.

b) We return to Anosov systems on closed manifolds. It is desirable to have a variety of examples of such systems. For discrete time systems an extensive stock of examples is furnished by hyperbolic automorphisms of infranilmanifolds which are obtained algebraically. This subject is dealt with in this collection. To construct Anosov flows, algebraic methods can also be used. This is well known in the case of geodesic flows on manifolds of constant negative curvature (and on a number of other manifolds), but they can also be used to obtain new examples. See Tomter [35], [36].

c) In addition to properties common to all Anosov flows, geodesic flows on manifolds of negative curvature have a number of special properties of a geometric nature. Klingenberg [37] has shown that all geodesic flows satisfying the Anosov condition enjoy a string of such properties—in fact, all the basic properties which are generally studied. (See also [38] regarding irreversible Finsler metrics.)

d) In the translation of Franks's article [A] some attention is alloted to Anosov coverings. The selection of this class of objects is motivated by the desire to unify the study of Anosov diffeomorphisms and expanding mappings. It turns out, however, that the unification thus obtained is rather "relative". Namely, Anosov diffeomorphisms and expanding mappings occupy a distinguished position among all Anosov coverings; it is only in these two cases that Anosov coverings are structurally stable (see [43] and [44]). (Apparently, the results in these papers allow one to deduce that in these two cases the covering is a π_1-covering. This does not, of course, mean that it is inappropriate to study Anosov coverings. The investigation of their ergodic properties is, for example, a completely reasonable undertaking.) We explain the situation in general terms below. We will see that the definition of an Anosov covering must be altered somewhat if we demand that a small (in the C^1 sense) perturbation of an Anosov covering again yield an Anosov covering.