O.C. ZIENKIEWICZ & R.L. TAYLOR

The FINITE ELEMENT METHOD

有限元法

Volume 2
SOLID MECHANICS

FIFTH EDITION

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Fifth edition

Volume 2: Solid Mechanics

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Dedication

This book is dedicated to our wives Helen and Mary Lou and our families for their support and patience during the preparation of this book, and also to all of our students and colleagues who over the years have contributed to our knowledge of the finite element method. In particular we would like to mention Professor Eugenio Oñate and his group at CIMNE for their help, encouragement and support during the preparation process.

Volume 1: The basis

Volume 3: Fluid dynamics

Preface to Volume 2

The first volume of this edition covered basic aspects of finite element approximation in the context of linear problems. Typical examples of two- and three-dimensional elasticity, heat conduction and electromagnetic problems in a steady state and transient state were dealt with and a finite element computer program structure was introduced. However, many aspects of formulation had to be relegated to the second and third volumes in which we hope the reader will find the answer to more advanced problems, most of which are of continuing practical and research interest.

In this volume we consider more advanced problems in solid mechanics while in Volume 3 we consider applications in fluid dynamics. It is our intent that Volume 2 can be used by investigators familiar with the finite element method in general terms and will introduce them here to the subject of specialized topics in solid mechanics. This volume can thus in many ways stand alone. Many of the general finite element procedures available in Volume 1 may not be familiar to a reader introduced to the finite element method through different texts. We therefore recommend that the present volume be used in conjunction with Volume 1 to which we make frequent reference.

Two main subject areas in solid mechanics are covered here:

1. *Non-linear problems* (Chapters 1–3 and 10–12) In these the special problems of solving non-linear equation systems are addressed. In the first part we restrict our attention to non-linear behaviour of materials while retaining the assumptions on small strain used in Volume 1 to study the linear elasticity problem. This serves as a bridge to more advanced studies later in which geometric effects from large displacements and deformations are presented. Indeed, non-linear applications are today of great importance and practical interest in most areas of engineering and physics. By starting our study first using a small strain approach we believe the reader can more easily comprehend the various aspects which need to be understood to master the subject matter. We cover in some detail problems in viscoelasticity, plasticity, and viscoplasticity which should serve as a basis for applications to other material models. In our study of finite deformation problems we present a series of approaches which may be used to solve problems including extensions for treatment of constraints (e.g. near incompressibility and rigid body motions) as well as those for buckling and large rotations.

2. *Plates and shells* (Chapters 4–9) This section is of course of most interest to those engaged in 'structural mechanics' and deals with a specific class of problems in which one dimension of the structure is small compared to the other two. This application is one of the first to which finite elements were directed and which still is a subject of continuing research. Those with interests in other areas of solid mechanics may well omit this part on first reading, though by analogy the methods exposed have quite wide applications outside structural mechanics.

Volume 2 concludes with a chapter on Computer Procedures, in which we describe application of the basic program presented in Volume 1 to solve non-linear problems. Clearly the variety of problems presented in the text does not permit a detailed treatment of all subjects discussed, but the 'skeletal' format presented and additional information available from the publisher's web site[1] will allow readers to make their own extensions.

We would like at this stage to thank once again our collaborators and friends for many helpful comments and suggestions. In this volume our particular gratitude goes to Professor Eric Kasper who made numerous constructive comments as well as contributing the section on the mixed–enhanced method in Chapter 10. We would also like to take this opportunity to thank our friends at CIMNE for providing a stimulating environment in which much of Volume 2 was conceived.

OCZ and RLT

[1] Complete source code for all programs in the three volumes may be obtained at no cost from the publisher's web page: http://www.bh.com/companions/fem

Contents

1

General problems in solid mechanics and non-linearity

1.1 Introduction

In the first volume we discussed quite generally linear problems of elasticity and of field equations. In many practical applications the limitation of linear elasticity or more generally of linear behaviour precludes obtaining an accurate assessment of the solution because of the presence of non-linear effects and/or because of the geometry having a 'thin' dimension in one or more directions. In this volume we describe extensions to the formulations previously introduced which permit solutions to both classes of problems.

Non-linear behaviour of solids takes two forms: material non-linearity and geometric non-linearity. The simplest form of a non-linear material behaviour is that of elasticity for which the stress is not linearly proportional to the strain. More general situations are those in which the loading and unloading response of the material is different. Typical here is the case of classical elasto-plastic behaviour.

When the deformation of a solid reaches a state for which the undeformed and deformed shapes are substantially different a state of *finite deformation* occurs. In this case it is no longer possible to write linear strain-displacement or equilibrium equations on the undeformed geometry. Even before finite deformation exists it is possible to observe *buckling* or *load bifurcations* in some solids and non-linear equilibrium effects need to be considered. The classical Euler column where the equilibrium equation for buckling includes the effect of axial loading is an example of this class of problem.

Structures in which one dimension is very small compared with the other two define plate and shell problems. A *plate* is a flat structure with one thin direction which is called the thickness, and a *shell* is a curved structure in space with one such small thickness direction. Structures with two small dimensions are called *beams*, *frames*, or *rods*. Generally the accurate solution of linear elastic problems with one (or more) small dimension(s) cannot be achieved efficiently by using the three-dimensional finite element formulations described in Chapter 6 of Volume 1[1] and conventionally in the past separate theories have been introduced. A primary reason is the numerical ill-conditioning which results in the algebraic equations making their accurate solution difficult to achieve. In this book we depart from past tradition and build a much stronger link to the full three-dimensional theory.

This volume will consider each of the above types of problems and formulations which make practical finite element solutions feasible. We establish in the present chapter the general formulation for both static and transient problems of a non-linear kind. Here we show how the linear problems of steady state behaviour and transient behaviour discussed in Volume 1 become non-linear. Some general discussion of transient non-linearity will be given here, and in the remainder of this volume we shall primarily confine our remarks to quasi-static (i.e. no inertia effects) and static problems only.

In Chapter 2 we describe various possible methods for solving non-linear algebraic equations. This is followed in Chapter 3 by consideration of material non-linear behaviour and the development of a general formulation from which a finite element computation can proceed.

We then describe the solution of plate problems, considering first the problem of thin plates (Chapter 4) in which only bending deformations are included and, second, the problem in which both bending and shearing deformations are present (Chapter 5).

The problem of shell behaviour adds in-plane membrane deformations and curved surface modelling. Here we split the problem into three separate parts. The first, combines simple flat elements which include bending and membrane behaviour to form a faceted approximation to the curved shell surface (Chapter 6). Next we involve the addition of shearing deformation and use of curved elements to solve axisymmetric shell problems (Chapter 7). We conclude the presentation of shells with a general form using curved isoparametric element shapes which include the effects of bending, shearing, and membrane deformations (Chapter 8). Here a very close link with the full three-dimensional analysis of Volume 1 will be readily recognized.

In Chapter 9 we address a class of problems in which the solution in one coordinate direction is expressed as a series, for example a Fourier series. Here, for linear material behavior, very efficient solutions can be achieved for many problems. Some extensions to non-linear behaviour are also presented.

In the last part of this volume we address the general problem of finite deformation as well as specializations which permit large displacements but have small strains. In Chapter 10 we present a summary for the finite deformation of solids. Basic relations for defining deformation are presented and used to write variational forms related to the undeformed configuration of the body and also to the deformed configuration. It is shown that by relating the formulation to the deformed body a result is obtain which is nearly identical to that for the small deformation problem we considered in Volume 1 and which we expand upon in the early chapters of this volume. Essential differences arise only in the constitutive equations (stress–strain laws) and the addition of a new stiffness term commonly called the *geometric* or *initial stress* stiffness. For constitutive modelling we summarize alternative forms for elastic and inelastic materials. In this chapter contact problems are also discussed.

In Chapter 11 we specialize the geometric behaviour to that which results in large displacements but small strains. This class of problems permits use of all the constitutive equations discussed for small deformation problems and can address classical problems of instability. It also permits the construction of non-linear extensions to plate and shell problems discussed in Chapters 4–8 of this volume.

In Chapter 12 we discuss specialization of the finite deformation problem to address situations in which a large number of small bodies interact (multiparticle or granular bodies) or individual parts of the problem are treated as rigid bodies.

In the final chapter we discuss extensions to the computer program described in Chapter 20 of Volume 1 necessary to address the non-linear material, the plate and shell, and the finite deformation problems presented in this volume. Here the discussion is directed primarily to the manner in which non-linear problems are solved. We also briefly discuss the manner in which elements are developed to permit analysis of either quasi-static (no inertia effects) or transient applications.

1.2 Small deformation non-linear solid mechanics problems

1.2.1 Introduction and notation

In this general section we shall discuss how the various equations which we have derived for linear problems in Volume 1 can become non-linear under certain circumstances. In particular this will occur for structural problems when non-linear stress–strain relationships are used. But the chapter in essence recalls here the notation and the methodology which we shall adopt throughout this volume. This repeats matters which we have already dealt with in some detail. The reader will note how simply the transition between linear and non-linear problems occurs.

The field equations for solid mechanics are given by equilibrium (balance of momentum), strain-displacement relations, constitutive equations, boundary conditions, and initial conditions.[2–7]

In the treatment given here we will use two notational forms. The first is a cartesian tensor indicial form (e.g. see Appendix B, Volume 1) and the second is a matrix form as used extensively in Volume 1.[1] In general, we shall find that both are useful to describe particular parts of formulations. For example, when we describe large strain problems the development of the so-called 'geometric' or 'initial stress' stiffness is most easily described by using an indicial form. However, in much of the remainder, we shall find that it is convenient to use the matrix form. In order to make steps clear we shall here review the equations for small strain in both the indicial and the matrix forms. The requirements for transformations between the two will also be again indicated.

For the small strain applications and fixed cartesian systems we denote coordinates as x, y, z or in index form as x_1, x_2, x_3. Similarly, the displacements will be denoted as u, v, w or u_1, u_2, u_3. Where possible the coordinates and displacements will be denoted as x_i and u_i, respectively, where the range of the index i is 1, 2, 3 for three-dimensional applications (or 1, 2 for two-dimensional problems). In matrix form we write the coordinates as

$$\mathbf{x} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \tag{1.1}$$

and displacements as

$$\mathbf{u} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{1.2}$$

1.2.2 Weak form for equilibrium – finite element discretization

The equilibrium equations (balance of linear momentum) are given in index form as

$$\sigma_{ji,j} + b_i = \rho \ddot{u}_i, \qquad i,j = 1,2,3 \tag{1.3}$$

where σ_{ij} are components of (Cauchy) stress, ρ is mass density, b_i are body force components and $(\dot{\ })$ denotes partial differentiation with respect to time. In the above, and in the sequel, we always use the convention that repeated indices in a term are summed over the range of the index. In addition, a partial derivative with respect to the coordinate x_i is indicated by a comma, and a superposed dot denotes partial differentiation with respect to time. Similarly, moment equilibrium (balance of angular momentum) yields symmetry of stress given indicially as

$$\sigma_{ij} = \sigma_{ji} \tag{1.4}$$

Equations (1.3) and (1.4) hold at all points x_i in the domain of the problem Ω. Stress boundary conditions are given by the traction condition

$$t_i = \sigma_{ji} n_j = \bar{t}_i \tag{1.5}$$

for all points which lie on the part of the boundary denoted as Γ_t.

A variational (weak) form of the equations may be written by using the procedures described in Chapter 3 of Volume 1 and yield the virtual work equations given by[1,8,9]

$$\int_\Omega \delta u_i \rho \ddot{u}_i \, d\Omega + \int_\Omega \delta\varepsilon_{ij}\sigma_{ij} \, d\Omega - \int_\Omega \delta u_i b_i \, d\Omega - \int_{\Gamma_t} \delta u_i \bar{t}_i \, d\Omega = 0 \tag{1.6}$$

In the above cartesian tensor form, virtual strains are related to virtual displacements as

$$\delta\varepsilon_{ij} = \tfrac{1}{2}(\delta u_{i,j} + \delta u_{j,i}) \tag{1.7}$$

In this book we will often use a transformation to matrix form where stresses are given in the order

$$\begin{aligned} \boldsymbol{\sigma} &= \begin{bmatrix} \sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{12} & \sigma_{23} & \sigma_{31} \end{bmatrix}^{\mathrm{T}} \\ &= \begin{bmatrix} \sigma_{xx} & \sigma_{yy} & \sigma_{zz} & \sigma_{xy} & \sigma_{yz} & \sigma_{zx} \end{bmatrix}^{\mathrm{T}} \end{aligned} \tag{1.8}$$

and strains by

$$\begin{aligned} \boldsymbol{\varepsilon} &= \begin{bmatrix} \varepsilon_{11} & \varepsilon_{22} & \varepsilon_{33} & \gamma_{12} & \gamma_{23} & \gamma_{31} \end{bmatrix}^{\mathrm{T}} \\ &= \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{zz} & \gamma_{xy} & \gamma_{yz} & \gamma_{zx} \end{bmatrix}^{\mathrm{T}} \end{aligned} \tag{1.9}$$

where symmetry of the tensors is assumed and 'engineering' shear strains are introduced as*

$$\gamma_{ij} = 2\varepsilon_{ij} \tag{1.10}$$

to make writing of subsequent matrix relations in a consistent manner.

The transformation to the six independent components of stress and strain is performed by using the index order given in Table 1.1. This ordering will apply to

* This form is necessary to allow the internal work always to be written as $\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\varepsilon}$.