Graduate Series in Mathematics **2**

Wang Mingyi

Morita Equivalence and Its Generalizations

(Morita 等价理论及其推广)

Science Press

Responsible Editor: Lü Hong, Shan Jinghua

Published by Science Press
16 Donghuangchenggen North Street
Beijing 100717, China

Printed in Beijing

ISBN 7-03-008489-6/O·1240(Beijing)
ISBN 1-880132-71-0(New York)

Preface

The language and elementary results of category theory have now pervaded a substantial part of mathematics. Besides the use of these concepts and results everyday, we should note that categorical notions are fundamental in some of the most striking new developments in mathematics. One of these is the extension of algebraic geometry, which originated as the study of solutions in the field of complex numbers of systems of polynomial equations with complex coefficients, to the study of such equations over an arbitary commutative ring. The proper fundation of this study, due mainly to A. Grothendieck, is based on the categorical concept of scheme. Another deep application of category theory is K. Morita's equivalence theory for modules, which gives a new insight into the classical Wedderburn-Artin structure theorem for simple rings and plays an important role in the extension of a substantial part of the structure theory of algebras over fields to algebras over commutative rings.

From the conception of Wedderburn's theorem on the simple Artinian rings, it must has been intuitively clear that the category of left modules over the simple Artinian ring is equivalent to the category of left vector spaces over its associated division ring. Once the proper categorical notions were realized, Morita [M1,58] characterized category equivalences between the categories of left modules of two rings as those given by the covariant Hom and Tensor functors affored by progenerators (i.e. finitely generated projective generators).

Later, Fuller [Fu2,74] characterized the equivalences between certain subcategories of the category of modules over a ring (not necessarily with identity) and the category of all unital modules over another ring. The equivalences are determined by a certain kind of modules that we call quasi-progenerators (i.e. finitely generated quasi-projective and generate their own submodules).

Early in 1986, Y. Miyashita generalized the the concept of tilting modules over finite dimensional algebras to the case of finite projective dimension over any ring. He gave a local version to the tilting theorem. After that, Colby and Fuller [CF,90] presented a rather complete version of the Tilting theorem for an arbitrary ring R. In particular, if T_R is a classical tilting module (using the notion in [CT2,95]) and $S = \mathrm{End}(T_R)$, then the Brenner-Butler theorem holds, i.e., there are two category equivalences between $\ker(\mathrm{Tor}_1^S(-,T))$ and $\ker(\mathrm{Ext}_R^1(T,-)) = \mathrm{Gen}(T_R)$ induced by $-\otimes_S T$ and $\mathrm{Hom}_R(T,-)$, and between $\ker(-\otimes_S T)$ and $\ker(\mathrm{Hom}_R(T,-))$ induced by $\mathrm{Tor}_1^S(-,T)$ and $\mathrm{Ext}_R^1(T,-)$.

Consequently, the equivalences induced by quasi-progenerators and by classical tilting modules generalized Morita equivalences. Menini and Orsatti [MO2,89] gave a unified treatment to the two generalizations above mentioned. There they introduced a class of modules which are called $*$-modules later. The notion of $*$-modules seems to be the most natural generalization of the two important notions of contemporary module theory, classical tilting modules and quasiprogenerators.

Other generalizations for Morita equivalences were investigated by several authors, among them Sato [Sa1,78], [Sa2,79], Azumaya [Az1,75], [Az2,79], D'Este [D1,90], [D2,94], D'Este and Happel [DH,90].

The other direction of generalizing the Morita equivalence theory for rings is to discuss the equivalence of rings without the assumption that the rings have identities. Such equivalences are called Morita-like equivalences [XST,93].

On the other hand, I.P. Lin [L1,74] found some results for the equivalences of categories of comodules over coalgebras, which are similar to those of Morita; Takeuchi [Ta,77] used co-tensor and co-hom functors to investigate the equivalences of categories of comodules over coalgebras and obtained the Morita theorems for such equivalences, his strategy is quite different from that of Lin's. Meanwhile, Takeuchi obtained a well-known characterization of the categories of comodules in [Ta, 77], i.e., it is of the finite type.

This book consists of two parts. One is to discuss Morita equivalences of the categories of modules over rings with identity, their generalizations, and Morita-like equivalence introduced by Xu-shum-

Turner in [XST,93]. The other is to discuss the Morita-Takeuchi equivalences of the categories of comodules over coalgebras and its generalizations.

Chapter 1 is the preliminaries for this book. First, we collect some basic categorical concepts, especially the concept of Grothendieck category. Secondly, it also contains some basic characterizations of Morita equivalences and many other properties preserved by Morita equivalences. This is the classical part of Morita theory. At last, we give some basic facts for coalgebras and comodules. We are not going to give the proofs for the results in this chapter.

In Chapter 2, we state Fuller's generalization for Morita equivalences, and the theory of quasi-progenerators. This is the beginning for generalizing Morita equivalence theory.

In Chapter 3, we state some results of Sato [Sa1,78], [Sa2,79], Azumaya [Az2,79], and Morita [M2,73] for subcategories equivalences. These results yield a further refinement and clarification of Fuller's characterization in Chapter 2.

In Chapter 4, we consider the equivalences induced by classical tilting modules and by tilting modules having finite projective dimension, which are basically due to Colby-Fuller [CF,90] and Miyashita [Mi,86], respectively. After these works, many ring-theorists began to study rings by tilting methods. We also give some characterizations of classical tilting modules in this chapter, these results are taken from Colpi's paper, but we avoid using the concept of $*$-modules here. Many further characterizations for classical tilting modules will be given in Chapter 6.

In Chapter 5, we consider the equivalences induced $*$-modules, which are the generalizations of the two very important concepts in contemporary module theory: quasi-progenerators and classical tilting modules. In this chapter, we state the surprising result obtained by Trlifaj [T3,94] that any $*$-module over an arbitrary ring is finitely generated. By using the newest result we give some characterizations for $*$-modules over any ring.

In Chapter 6, as an application for the theories of $*$-module, we first characterize classical tilting modules by the concept of $*$-modules. Then we give a representation theorem for equivalences between projective modules and injective modules over hereditary Noetherian rings. And we give many examples to show that the classes of progenerators, quasi-progenerators, classical tilting modules, and $*$-modules, do not coincide with each other in general. At last, we measure the gaps between the classes of $*$-modules, quasi-progenerators and classical tilting modules by a well-known theorem obtained by Trlifaj [T1,94]. These works are due to Colpi, Menini, Orsatti and Trlifaj. By the way, the authors in [CDT,97] introduce the concept of quasi-tilting modules, we can say, roughly speaking, that a module $_RV$ is quasi-tilting if and only if $_RV$ is "tilting in $\overline{\text{Gen}}(_RV)$". This situation is analogous to that of quasi-progenerators, which can be considered as "progenerators in $\overline{\text{Gen}}(_RV)$".

In Chapter 7, we give some results about Monta-like equivolence, which are due to professor Y. H. Xu, K. P. Shum and R. F. Turner-Smith [XST,93].

In Chapter 8, we first give some basic results about two functors: "$h-_C(-,-)$", "$-\square_C-$", and locally finite abelian categories. Then we give a very useful characterization of categories of comodules.

In Chapter 9, we present the equivalence theory for categories of comodules over coalgebras which now called Morita-Takeuchi equivalence [Ta,77]. Such an equivalence can be represented by some functors, for example, "$h-_C(-,-)$" and "$-\square_C-$". These results provide us with some effictive methods to study coalgebras.

In Chapter 10, we present the strongly equivalence theory of the categories of comodules, which is due to I. P. Lin [L1,74]. By using some results in Chapter 9, we can give some improvement to Lin's results.

In Chapter 11, we first study QcF-coalgebras which are dual to QF-rings. About such a class of coalgebras, we characterize it in many ways. We also generalize some results of co-Frobenius coalgebras to it. After that we dualize Noetherian algebras and obtain an interesting class of coalgebras, i.e., conoetherian coalgebras. After which we obtain some characterizations which are dual to those of Noetherian algebras.

In Chapter 12, we first introduce a new class of coalgebras, i.e., conoetherian coalgebras which are invariant under Morita-Takeuchi equivalence. Secondly, we prove the tilting theorem for the situation of classical tilting comodules over conoetherian semiperfect coalgebras. So we obtain two equivalences of subcategories of comodules. It is a generalization of Takeuchi equivalences [Ta,77].

In Chapter 13, we give some important properties which are preserved by equivalence of the categories of comodules.

In the final chapter, we list some interesting open problems in ring theory and module theory, some of them are very well-known.

I express my sincere thanks to my Ph.D adviser, Professor Yonghua Xu (Fudan University), for his encouragement. I also wish to express my gratitude to Professor Weimin Xue, Professor Huiling Li, Professor Daoji Meng, Professor Fuchang Cheng and Professor Zhong Yi, for their helpful suggestions.

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January 25, 2000

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Chapter 1
Preliminaries

This chapter contains some basic concepts and the basic characterizations of Morita equivalences. In section 1.1 we collect some concepts of categories, functors, natural transformations, the equivalences of categories. In section 1.2, we state the concept of Grothendieck category, which plays an important role in the theory of rings of quotients. Section 1.2 is a basic introduction to Morita equivalence and most results in this section are taken from the standard ring theory book, Anderson-Fuller ([AF, 92] section 22). In the last section, we collect some basic concepts of coalgebras and comodules which come from any standard book for Hopf algebras.

1.1 Some Basic Concepts about Equivalences

Definition 1.1.1 A category $\mathbf{C}$ consists of

1. A class ob$\mathbf{C}$ of objects (usually denoted by A, B, C, etc.).

2. For each ordered pair of objects (A, B), a set $\hom_{\mathbf{C}}(A, B)$ (or simply, $\hom(A, B)$ if $\mathbf{C}$ is clear) whose elements are called morphisms with domain A and codomain B (or from A to B).

3. For each ordered triple of (A, B, C), a map $(f, g) \to gf$ of the product set $\hom(A, B) \times \hom(B, C)$ into $\hom(A, C)$.

It is assumed that the objects and morphisms satisfy the following conditions:

C1. If $(A, B) \neq (C, D)$, then $\hom(A, B)$ and $\hom(C, D)$ are disjoint.

C2. (Associativity) If $f \in \hom(A, B), g \in \hom(B, C)$, and $h \in$

hom(C, D), then $(hg)f = h(gf)$.(As usual, we simplify this to hgf.)

C3. (Unit) For every object A we have an element $1_A \in \hom(A, A)$ such that $f1_A = f$ for every $f \in \hom(A, B)$ and $1_A g = g$ for every $g \in \hom(B, A)(1_A$ is unique).

A category **D** is called a subcategory of the category **C** if ob**D** is a subclass of ob**C** and for any $A, B \in$ ob**D**, $\hom_{\mathrm{D}}(A, B) \subseteq \hom_{\mathrm{C}}(A, B)$. It is required also (as part of the definition) that 1_A for $A \in$ ob**D** and the product of morphisms for **D** is the same as for **C**. The subcategory **D** is called full if $\hom_{\mathrm{D}}(A, B) = \hom_{\mathrm{C}}(A, B)$ for every $A, B \in$ **D**.

Let us give a list of examples of categories:

1. **Set**, the category of sets.
2. **Grp**, the category of groups, the morphisms are homomorphisms (mapping 1 into 1).
3. **Ring**, the category of (associative) rings (with unit for the multiplication composition), the morphisms are homomorphisms (mapping 1 into 1).
4. $_{\mathbf{R}}\mathbf{M}$, the category of left modules for a fixed ring R; $_{\mathbf{R}}\mathbf{FM}$, the category of finitely generated left modules for a fixed ring R.
5. $\mathbf{Cog_k}$, the category of coalgebras over a fixed field k, the morphisms are the co-linear maps; $\mathbf{Ccog_k}$, the category of cocommutative coalgebras over a fixed field k, the morphisms are the co-linear maps.
6. $\mathcal{M}^{\mathrm{C}}$, the category of right comodules over a coalgebra C, the morphisms are the co-linear maps of comodules.

Definition 1.1.2 A morphism $f : A \to B$ is called an isomorphism if there exists a $g : B \to A$ such that $fg = i_B$ and $gf = 1_A$. If $f : A \to B, g : B \to A$ and $gf = 1_A$, then f is called a section of g and g is called a retraction of f.

A morphism $f : A \to B$ is called monic (epic) if it is left (right) cancellable in **C**.

Proposition 1.1.1 A morphism in $_{\mathbf{R}}\mathbf{M}$ or **Grp** is monic (epic) if and only if the map of the underlying set is injective (surjective).

A morphism in $\mathbf{Cog_k}$ or in $\mathbf{Ccog_k}$ is epic if and only if it is surjective for the vector space.

Proposition 1.1.2 A morphism in **Ring** is monic if and only if it is injective. However, there exist epics in **Ring** that are not surjective.

Definition 1.1.3 If **C** and **D** are categories, a covariant (contravariant) functor F from **C** to **D** consists of

1. A map $A \to FA$ of ob**C** into ob**D**.

2. For every pair of objects (A, B) of **C**, a map $f \to F(f)$ of $\hom_{\mathbf{C}}(A, B)$ into $\hom_{\mathbf{D}}(FA, FB)$.

We require that these satisfy the following conditions:

F1. If gf is defined in **C**, then $F(gf) = F(g)F(f)(F(f)F(g))$.

F2. $F(1_A) = 1_{FA}$.

A functor F is called a faithful functor (full) if for every pair of objects (A, B) in **C** the map $f \to F(f)$ of $\hom_{\mathbf{C}}(A, B)$ into $\hom_{\mathbf{D}}(FA, FB)$ is injective (surjective).

Definition 1.1.4 Let F and G be functors from **C** to **D**. We define a natural transformation η from F to G to be a map that assigns to every object A in **C** a morphism $\eta_A \in \hom_{\mathbf{D}}(FA, GA)$ such that for any objects A, B of **C** and any $f \in \hom_{\mathbf{C}}(A, B)$ the rectangle in

$$\begin{array}{ccc} FA & \xrightarrow{\eta_A} & GA \\ \downarrow F(f) & & \downarrow G(f) \\ FB & \xrightarrow{\eta_B} & GB \end{array}$$

is commutative. Moreover, if every η_A is an isomorphism then η is called a natural isomorphism.

Definition 1.1.5 We say that the categories **C** and **D** are isomorphic if there exist functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ such that $GF = 1_{\mathbf{C}}$ and $FG = 1_{\mathbf{D}}$. More generally, we say that the categories **C** and **D** are equivalent if there exist functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ such that $GF \simeq 1_{\mathbf{C}}$ and $FG \simeq 1_{\mathbf{D}}$, where $\simeq$ denotes the natural isomorphism of functors.

Proposition 1.1.3 Let F be a functor from **C** to **D**. Then there exists a functor $G : \mathbf{D} \to \mathbf{C}$ such that (F, G) is an equivalence if and only if F is faithful and full and for every object A' of **D** there exists an object A of $\mathbf{C}$ such that FA and A' are isomorphic in $\mathbf{D}$, that is, there is an isomorphism contained in $\hom_{\mathbf{D}}(FA, A')$.

Definition 1.1.6 Let $\{A_\alpha | \alpha \in I\}$ be an indexed set of objects in a category $\mathbf{C}$. We define a product $\prod A_\alpha$ of the A_α to be a set $\{A, p_\alpha | \alpha \in I\}$ where $A \in \mathrm{ob}\mathbf{C}, p_\alpha \in \hom_{\mathbf{C}}(A, A_\alpha)$ such that if $B \in \mathrm{ob}\mathbf{C}$ and $f_\alpha \in \hom_{\mathbf{C}}(B, A_\alpha), \alpha \in I$, then there exists a unique $f \in \hom_{\mathbf{C}}(B, A)$ such that $f_\alpha = p_\alpha f$.

Definition 1.1.7 Let $\{A_\alpha | \alpha \in I\}$ be an indexed set of objects in a category $\mathbf{C}$. We define a coproduct $\coprod A_\alpha$ of the A_α to be a set $\{A, i_\alpha | \alpha \in I\}$ where $A \in \mathrm{ob}\mathbf{C}, i_\alpha \in \hom_{\mathbf{C}}(A_\alpha, A)$ such that if $B \in \mathrm{ob}\mathbf{C}$ and $g_\alpha \in \hom_{\mathbf{C}}(A_\alpha, B), \alpha \in I$, then there exists a unique $g \in \hom_{\mathbf{C}}(A, B)$ such that $g_\alpha = g i_\alpha f$.

Definition 1.1.8 Let $f_i : A_i \to B, i = 1, 2$, in a category $\mathbf{C}$. Define a pullback diagram of $\{f_1, f_2\}$ to be a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{g_1} & A_1 \\ \downarrow g_2 & & \downarrow f_1 \\ A_2 & \xrightarrow{f_2} & B \end{array}$$

such that if

$$\begin{array}{ccc} D & \xrightarrow{h_1} & A_1 \\ h_2 \downarrow & & \downarrow f_1 \\ A_2 & \xrightarrow{f_2} & B \end{array}$$

is any commutative rectangle containing f_1 and f_2, then there exists a unique $k : D \to C$ such that $g_1 k = h_1, g_2 k = h_2$. Dually, we can define the concept of pushout diagram.

Definition 1.1.9 We define the functor $\hom : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Set}$ by specifying that this maps the object (A, B) into the set $\hom(A, B)$ (which is an object of $\mathbf{Set}$) and the morphism $(f, g) : (A, B) \to (A', B')$ into the map of $\hom(A, B)$ into $\hom(A', B')$ defined by

$$\hom(f, g) : k \to gkf.$$

In this way, we can get a covariant functor $\hom(A, -)$ determined by A and a contravariant functor $\hom(-, B)$ determined by B.

Proposition 1.1.4 (Yoneda's lemma) Let F be a functor from $\mathbf{C}$ to $\mathbf{Set}$, A an object of $\mathbf{C}$, a an element of the set FA. For any $B \in \text{ob}\mathbf{C}$ let a_B be the map of $\hom_{\mathbf{C}}(A,B)$ into FB such that $k \to F(k)(a)$. Then $B \to a_B$ is a natural transformation $\eta(a)$ of $\hom_{\mathbf{C}}(A,-)$ into F. Moreover, $a \to \eta(a)$ is a bijection of the set FA onto the class of natural transformations of $\hom_{\mathbf{C}}(A,-)$ to F. The inverse of $a \to \eta(a)$ is the map $\eta \to \eta_A(1_A) \in FA$.

1.2 Grothendieck Categories

Definition 1.2.1 A category $\mathbf{C}$ is preadditive if each set $\hom_{\mathbf{C}}$ (A,B) is an abelian group and the composition maps $\hom(B,E) \times \hom(A,B) \to \hom(A,E)$ are bilinear.

If $\mathbf{C}$ and $\mathbf{D}$ are preadditive, then a functor $F : \mathbf{C} \to \mathbf{D}$ is additive if it satisfies

$F(f+g) = F(f) + F(g)$ for $f, g \in \hom_{\mathbf{C}}(A,B)$.

Definition 1.2.2 Let $\mathbf{C}$ be a preadditive category. Akernel of a morphism $f : A \to B$ is a morphism $k : K \to A$ such that

(i) $fk = 0$;

(ii) for every $g : X \to A$ with $fg = 0$, there exists a unique $h : X \to K$ such that $g = kh$.

Dually, we can define the concept of cokernel.

Definition 1.2.3 A category $\mathbf{C}$ is abelian if

A1. $\mathbf{C}$ is preadditive.

A2. Every finite family of objects has a product (and coproduct).

A3. Every morphism has a kernel and a cokernel.

A4. $\overline{\alpha}$:Coker(ker$f$)$\to$ Ker(coker$f$) is an isomorphism for every morphism f.

Definition 1.2.4 A limit (or projective limit) of the functor $F : I \to C$ is an object $\varprojlim F$ in $\mathbf{C}$ together with a compatible family of morphisms $f_i : \varprojlim F \to F(i)$, such that for each other compatible family $g_i : X \to F(i)$ there exists a unique $h : X \to \varprojlim F(i)$ with $f_i h = g_i$.

Dually, we can define the colimit (or inductive limit of a functor F).

Definition 1.2.5 A category $\mathbf{C}$ is called complete if the limit exists for every functor $F : \mathbf{I} \to \mathbf{C}$ when $\mathbf{I}$ is small.

Dually, a category $\mathbf{C}$ is called cocomplete if the colimit exists for every functor $F : \mathbf{I} \to \mathbf{C}$ when $\mathbf{I}$ is small.

Definition 1.2.6 An object G in $\mathbf{C}$ is a generator for $\mathbf{C}$ if $\hom(G, -)$ is faithful and E is a cogenerator for $\mathbf{C}$ if $\hom(-, E)$ is faithful.

Definition 1.2.7 A cocomplete abelian category $\mathbf{C}$ is called a Grothendieck category if direct limits are exact in $\mathbf{C}$ and $\mathbf{C}$ has a generator.

Example The comodules category $\mathcal{M}^C$ is a Grothendieck category.

1.3 The Morita Theory of Equivalences

The prototype of (Morita) equivalence is provided by a ring R and the ring $M_n(R)$ of $n \times n$ matrices over R. Indeed, the Wedderburn simple Artinian rings may be viewed as one of the earlist treatments of the theory of equivalence of rings. In this section we describe the complete characterizations of equivalences, that are due to Morita [M1, 58].

First, we present a list of properties preserved by equivalence.

Proposition 1.3.1 Let $F : {}_{\mathbf{R}}\mathbf{M} \to {}_{\mathbf{S}}\mathbf{M}$ be a category equivalence. Then

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is (split) exact in ${}_{\mathbf{R}}\mathbf{M}$ if and only if the sequence

$$0 \to F(M') \xrightarrow{F(f)} M \xrightarrow{F(g)} \to 0$$

is (split) exact in ${}_{\mathbf{S}}\mathbf{M}$.

Proposition 1.3.2 Let R and S be equivalence rings via an equivalence $F : {}_R\mathbf{M} \to {}_S\mathbf{M}$. Let M, M' and U be left R-modules. Then

(i) U is M-projective (M-injective) if and only if $F(U)$ is $F(M)$-projective ($F(M)$-injective);

(ii) U is projective (injective) if and only if $F(U)$ is projective (injective);

(iii) U generates (cogenerates) M if and only if $F(U)$ generates (cogenerates) $F(M)$;

(iv) U is a generator (a cogenerator) (faithful) if and only if $F(U)$ is a generator (a cogenerator) (faithful);

(v) A monomorphism (epimorphism) $f : M \to M'$ is essential (superfluous) if and only if $F(f) : F(M) \to F(M')$ is essential (superfluous);

(vi) $f : M \to M'$ is an injective envelop (projective cover) if and only if $F(f) : F(M) \to F(M')$ is an injective envelop (projective cover);

(vii) M has projective dimension k iff $F(M)$ has projective dimension k;

(viii) M has uniform dimension k iff $F(M)$ has uniform dimension k.

Proposition 1.3.3 Let R and S be equivalence rings via an equivalence $F : {}_R\mathbf{M} \to {}_S\mathbf{M}$. Then for each left R-module M, the mapping defined by

$$\gamma_M : K \to \mathrm{Im}F(i_{K \leq M})$$

is a lattice isomorphism from the lattice of submodules of M onto the lattice of submodules of $F(M)$.

Proposition 1.3.4 Let R and S be equivalence rings via an equivalence $F : {}_R\mathbf{M} \to {}_S\mathbf{M}$, and let M and M' be left R-modules. Then

(i) M is simple (semisimple) if and only if $F(M)$ is simple (semisimple);

(ii) M is finitely generated (finitely cogenerated, finitely presented) if and only if $F(M)$ is finitely generated (finitely cogenerated, finitely presented);